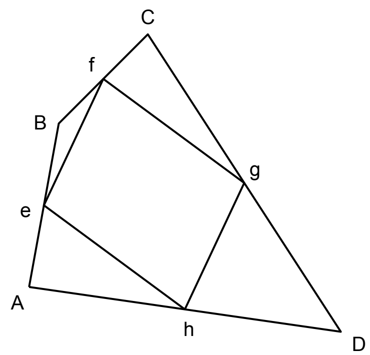
INTRODUCTION

Those who assert that the mathematical
sciences say nothing of the beautiful or the good
are in error.

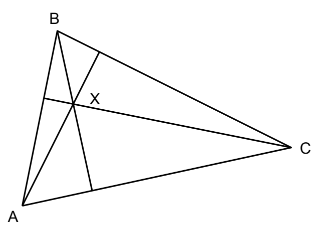
– Aristotle

There are at least four reasons to study geometry.

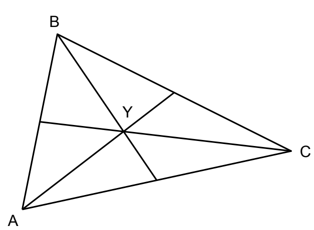
(1) GEOMETRY IS FULL OF WONDERS. At every level of this science, from the most elementary to the most advanced, we are confronted with the unexpected. Often, the seemingly possible proves impossible, and conversely what at first seemed impossible turns out to be possible. An example from the most elementary level is the possibility of one rectangle having *more* area than another one, and yet *less* total length around its sides. At first, it sounds impossible to enclose a bigger yard with less fence, or a smaller yard with a longer fence. (Incidentally, that is why swindlers in ancient times used to sell land by perimeter instead of by area.)



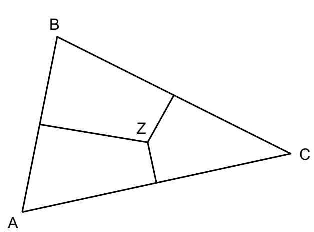
Now a more advanced example. Take any quadrilateral *ABCD*, as ugly and random-looking as you please. Cut each side in half at *e*, *f*, *g*, *h*, and – surprise!– the quadrilateral *e f g h* is a perfect parallelogram.



And a slightly more advanced example. If you take any triangle *ABC* and drop perpendiculars from each vertex to its opposite side, these three perpendiculars all meet at one point, *X* ...



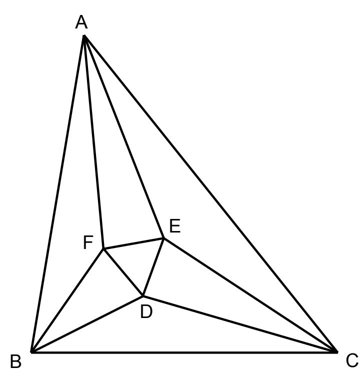
And if you join each vertex of triangle *ABC* to the midpoint of the opposite side, these three lines all meet at one point, *Y* ...



Finally, if you draw perpendiculars from the midpoints of the sides of *ABC*, they also all meet at one point, *Z*. Now the surprise: in any triangle *ABC*, these three points *X*, *Y*, *Z* are in a perfectly straight line. (Not only that, but *XY* is exactly double the length of *YZ*!)

There are many more surprises than these in geometry, but to get into them would take us out of this introduction, and into the science itself.

(2) GEOMETRY IS BEAUTIFUL. There are at least three sources of beauty in geometry. First, there are the figures themselves: perfect things of their kind, without bump or wrinkle, such as a perfect circle, or the five Platonic Solids. Symmetry and proportion, which are universal principles of beauty in nature, architecture, poetry and music, abound in geometric diagrams. There is also a beauty in the truths of geometry themselves. For example, if you take any triangle you like (Δ ABC), and cut each of its angles into three equal parts, the six trisecting lines you have drawn will meet each other inside the triangle at three points (D, E, F). The beautiful thing is that the three points D, E, F will be the vertices of a perfectly equilateral triangle. Such revelations are not only surprising, but pleasing in their simplicity and symmetry. In geometry, order pops up unlooked for; a beauty that we do not make, but only discover.



The proofs of geometry can also be beautiful. The best geometric proofs are adorned with a brilliance all their own in virtue of their ingenuity, clarity, universality, and rigor. Geometry, properly presented, yields an experience of intelligible beauty, introducing minds to the special pleasures attending insight and understanding.

(3) GEOMETRY IS FULL OF FUNDAMENTALS. Over the entrance to Plato's Academy there hung a sign which read *Let no one ignorant of mathematics enter here*. Why did a school of philosophy designate mathematics as a prerequisite for admission? Plato saw that many universal principles are most readily accessible to us through mathematics.

The geometrical science of proportion, for example, shows in a concrete way how some things can be known by proportions or analogies. We can come to know an unknown quantity x if we see it in proportion to other terms already known to us, say if x has to 4 the same ratio that 3 has to 2. Knowing 4, 3, and 2, and knowing the relationship between 3 and 2, we can come to know the mysterious x. This is a useful way of getting at something to which we have no direct access, say if x were a length we could not measure directly, like the height of an Egyptian pyramid. There is no way to drop a plumb line from the peak of a pyramid straight down to its base, but three other lengths that we can measure might form a proportion with the inaccessible height. Philosophers and scientists, too, must sometimes find ways to investigate things not directly observable or imaginable, and one tool for this purpose is proportion or analogy, the most fundamental use of which we find in geometry.

Geometry is also fundamental in another way. It is the science most easily acquired by the human mind with rigor and exactness. In geometry, one can settle disagreements. One can draw inescapable conclusions. This makes geometry an ideal entryway into the whole life of the mind.

(4) GEOMETRY EXERCISES THE MIND. People exercise their bodies to maintain their strength and health, and also because it feels

good. There is such a thing as mental exercise, too, which both strengthens and exhilarates the mind. Studying geometry is among the best of mental workouts, simultaneously exercising one's imagination, memory, and reason. In the course of a proof, the imagination must follow a line of reasoning from one part of a diagram to another; it must flip, rotate, and otherwise manipulate geometrical objects; it must interpret two-dimensional diagrams of three-dimensional things; it must picture how the other parts of a diagram are affected if one part is moved or changed. Memory also gets a workout, since geometry is cumulative. Each conclusion must be understood, and then used to establish later results, which in turn help to establish still more advanced results. And geometry obviously exercises reason. There is no reasoning more exact than a mathematical argument. Geometrical objects are perfect subject matter for forming definitions and proofs, proposing difficulties and finding resolutions, drawing distinctions, finding examples … in short, for doing all the best things that human reason can do. Thus geometry builds people's confidence that reason can find satisfying answers to serious questions.

For the above reasons geometry is justly recognized as an essential element in the formation of every educated person and is worthy of lifelong study. Current books written on the premise that geometry is interesting in itself are largely intended for advanced students or professional mathematicians. They presuppose a mastery of elementary theorems. On the other hand, geometry books which begin at the very beginning are generally not written for enthusiastic readers, but for students who need to pass an exam. Such introductions gloss over proofs (or skip them entirely), emphasizing instead various formulas, exercises, and problem-solving techniques.

This course is written for anyone motivated to study geometry for the wonder and beauty of it, for readers disposed to contemplate theorems as if they were works of art. And yet it begins at the very beginning. To master it, you need no prior training in mathematics. In consequence, this course represents a unique introduction to geometry. Readers interested in learning mathematics will find it better suited to their needs than study manuals or high school geometry books because of its scope, its purity, and its rigor.

THE SCOPE OF THIS COURSE by far surpasses that of the typical introduction. This course covers most of the content of the thirteen books of Euclid's *Elements*, whereas typical introductions do not cover material much beyond the first three or four books of Euclid. Written most often for the high school level, they do not go deep enough into geometry to reach the most beautiful and exciting material accessible to recreational mathematicians. Yet this course is not longer than the average high school textbook, but actually shorter, since it does not multiply exercises.

THE PURITY OF THIS COURSE should be refreshing to anyone who loves geometry. Other introductions to the science, written so readers can "get the right answer," employ algebra, trigonometry, number lines, a system of coordinate axes, and a host of other devices. Such devices and techniques, though useful (elsewhere) and important to study (elsewhere), have no place in a formal introduction to geometry intended for those who wish to begin at the beginning and understand the reasons for things. The impression is given that there is no geometry without these extras. The truth is that geometrical things can be known geometrically, without recourse to algebra or trigonometry.

The proof of the Pythagorean Theorem given in this book, for example, makes no use of algebraic operations. The theorem is demonstrable on purely geometrical grounds. The proof given for this theorem in many introductory books is an algebraic one that quickly leaves behind the diagram altogether. The result is a very abstract and unmemorable proof, the steps of which are not explicitly correlated with the right triangle and the squares that the geometrical theorem is about. The purpose of teaching the Pythagorean Theorem algebraically is to encourage proficiency in applying it to problems. This denies students any real understanding of the theorem, however, and reinforces the idea that geometry has no intrinsic worth or beauty.

RIGOR. Many introductory books use theorems they do not prove, such as the theorem that if a cone and a cylinder stand on the same circle and have the same height, the volume of the cone is one third that of the cylinder. Current high school textbooks including this theorem or a formula based on it do not attempt even a sketchy proof for it. In this course a complete proof is given for this theorem and for every other theorem covered. Once again, the implicit message of the textbook is that understanding the theorem is not important, but only the use of a formula which one should be willing to take on faith. This presumes an audience uninterested in the reasons for things, or incapable of understanding them.

Like a novel, it is essential to read this book in the order in which it is written, but unlike a novel, you can stop after any chapter or theorem and come away with something completely understood. But enough of introductions. On to the adventure of geometry.

SYMBOLS

Here are the symbols and abbreviations I will use:

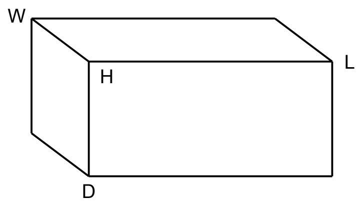
$A = B$	A is equal to B
$A > B$	A is greater than B
$A < B$	A is less than B
$AB \perp CD$	AB is perpendicular to CD
$AB \parallel CD$	AB is parallel to CD
$60°$	sixty degrees
$\angle ABC$	angle ABC
$\triangle ABC$	triangle ABC
$\triangle ABC \cong \triangle DEF$	triangle ABC is congruent to triangle DEF
$\square AB$	the square on line AB
$\square ABCD$	the square with corners A, B, C, D
$AB \cdot CD$	the rectangle with sides of length AB and length CD
$3A$	three times A
$A : B = C : D$	A has to B the same ratio that C has to D
$A : B > C : D$	A has to B a greater ratio than C has to D
Q.E.D.	Short for *Quod Erat Demonstrandum*, a Latin expression meaning "that which was to be demonstrated," a customary way of marking the end of a demonstrative theorem.

Q.E.F. Short for *Quod Erat Faciendum*, a Latin expression meaning “that which was to be done,” and a customary way of marking the end of a construction or “how to” theorem.

Chapter One

Triangles
and
Parallelograms

DEFINITIONS

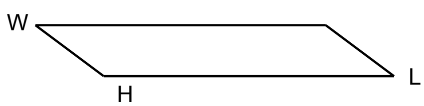


1. A SOLID is whatever has length, width, and depth.

A gold brick is a solid, having a length HL, a width HW, and a depth HD.

2. A solid stops at its SURFACE (or surfaces); so a surface has length and width, but no depth.

The top face of a gold brick is a surface, having a length HL, and a width HW. But it has no depth, since it is only the face of the brick; if it had any depth, it would not be the top face of a brick, but it would be a brick itself, even if a very slim one.



3. When a surface comes to an end, it stops at a LINE (or lines); so a line has length, but no width or depth.

One edge of the brick's top surface, such as HL, is a line, having a certain length. But it has no width or depth. It has no depth, since it is an edge of a surface, which has no depth. It has no width, since it is only the edge of the surface; if it had any width, it would not be only the edge of a surface, but it would be a surface itself, even if a very narrow one.

Today a finite line is often called a "line segment." But since infinite lines don't come up too frequently, and it is tedious to say "line segment" every time a finite line is meant, I will call a finite line simply a "line."

4. When a line comes to an end, it stops at a POINT; so a point has no length, no width, no depth.

One end of the edge of the brick's top surface, such as H, is a point, having no length, width, or depth. It has no width or depth, since it is the end of a line, which itself has no width or depth. It has no length, since it is only the end of a line; if it had any length, it would not be only the end of the line, but it would itself be a line, even if a very short one. Although it has no shape or size, a point does have one positive feature: its location.

5. A STRAIGHT LINE is a perfectly uniform line. Every part of it is the same "shape" as every other part, regardless of length, and different straight lines differ only in length, location, and orientation. Every other kind of line is called a CURVED LINE.

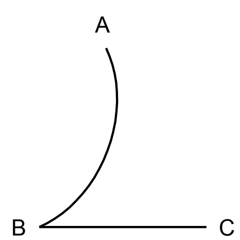
One imaginative way to express the uniformity of a straight line is with the following "thought experiment." If you look at a straight line on end, it will look like a point; that is, none of the line's length will be visible. Another way to express the uniformity of straight lines is like this: it is impossible for both endpoints of one straight line to coincide with those of another, without the straight lines themselves completely coinciding.

A straight line is sometimes defined as the shortest distance between two points. We will see more about the "shortness" of straight lines in Theorem 17. In the meantime, it is enough to note that the reason a straight line is so short and direct compared to other lines is because of its uniformity.

NOTE: Since this book is only about straight lines and circles, and no other kinds of lines, when I say "line" I will mean one of these, and which one I mean will be clear by context.

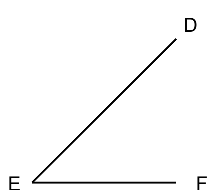
6. A FLAT SURFACE is a perfectly uniform surface. Every part of it, regardless of size, is the same "shape" all over and on both sides, with no difference of "terrain" between one part and another. A flat surface can also be called a PLANE.

One imaginative way to express the uniform terrain of a plane is with the following thought experiment. If you look at a plane on edge, it will look like a straight line; that is, none of the plane's width will be visible.



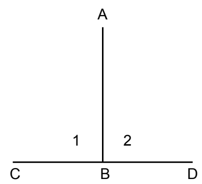
7. When two distinct lines in the same plane meet at a point, the inclination of one to the other is a PLANE ANGLE. The point at which the lines meet is called the VERTEX of the angle.

Angle ABC is an example. Since the line AB has an inclination to line BC on one side of itself, 1*, but also another inclination to (or away from) line BC on the other side of itself,* 2, *we may speak of an "interior" angle* 1, *and an "exterior" angle* 2. *By "angle ABC," I will mean inclination* 1.



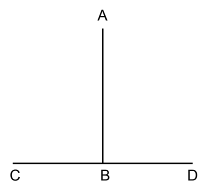
8. A RECTILINEAL ANGLE is an angle formed by two different straight lines.

Angle DEF is an example.



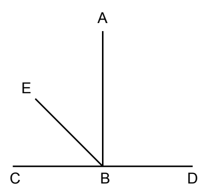
9. When one straight line stands on another in such a way that the two adjacent angles formed are equal to each other, each angle is called a RIGHT ANGLE.

Thus if AB stands on CD making angle 1 equal to angle 2, then each of these angles is a "right" angle.



10. A straight line standing on another at right angles is said to be PERPENDICULAR to the line on which it stands.

AB is perpendicular to CD, for example.



11. An OBTUSE ANGLE is a rectilineal angle greater than a right angle. An ACUTE ANGLE is a rectilineal angle less than a right angle.

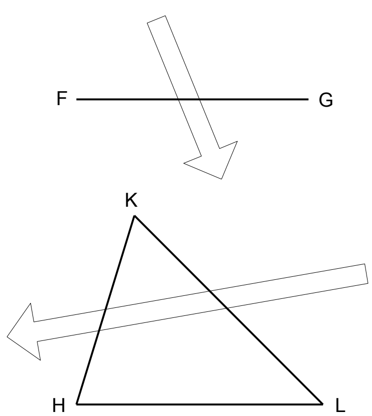
Angle CBE is acute, being less than angle CBA, which is right. Angle EBD is obtuse, being greater than angle ABD, which is right.

12. An obtuse angle and an acute angle are called SUPPLEMENTARY when they add up to two right angles. Two acute angles are called COMPLEMENTARY when they add up to one right angle.

Angles CBE and EBD are supplementary; angles CBE and EBA are complementary.

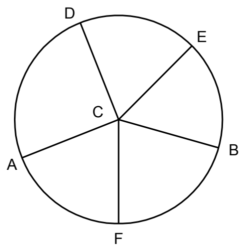
13. A BOUNDARY of a thing is its limit, or where it stops.

For example, a sphere is bounded by one surface, a square by four lines, a straight line by two points.



14. A FIGURE is something contained by its boundary or boundaries – something which cannot be entered or departed from without cutting across its boundary or boundaries.

A straight line, such as FG, though it is bounded by two endpoints, is not a figure, since it is possible to pass through the line without passing through the endpoints which bound it. But a triangle, such as HKL, is a figure, since it is impossible to pass into the triangle or out of it without passing through one of the straight lines that bound it.

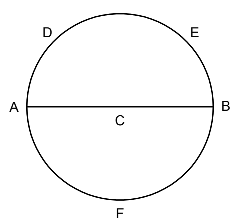


15. A CIRCLE is a plane figure contained by one curved line whose every point is the same distance from a single point inside it.

Figure ADEBFA is a circle, since it is bounded by the one line ADEBFA, and all the points on that line, such as A,D,E,B,F, are the same distance from C; that is, CA = CD = CE = CB = CF.

16. The single point inside a circle which is the same distance from every point along the curved line bounding the circle is called the circle's CENTER. And the curved line bounding the circle is called the circle's CIRCUMFERENCE.

C is the center of the circle ADEBF, and ADEBFA is its circumference.



17. Any straight line drawn from the center of a circle and stopping at the circumference is called a RADIUS of the circle. Thus, by definitions 15 & 16, it is evident that all the radii of a circle are equal.

The plural of "radius" is "radii."

Any straight line drawn through the center of a circle and terminated at each end by the circumference is called a DIAMETER of the circle. It is evident that any diameter of a circle *bisects* the circle, that is, cuts the circle into two equal parts.

CD is an example of a radius of circle C, and ACB is an example of a diameter of circle C.

18. A SEMICIRCLE is the figure contained by a circle's diameter and the circumference cut off by it.

The figure contained by ADEB and the straight line AB is an example of a semicircle.

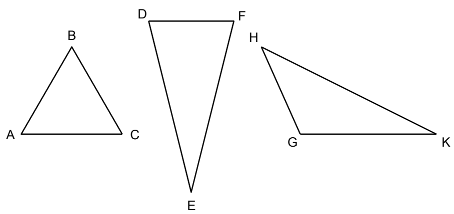
19. A RECTILINEAL FIGURE is a plane figure contained by straight lines only.

A TRIANGLE is a plane figure contained by three straight lines.

A QUADRILATERAL is a plane figure contained by four straight lines.

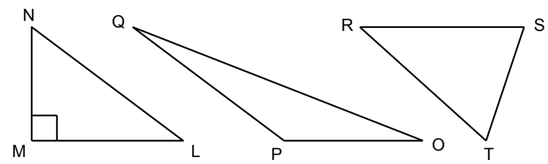
A POLYGON is any plane figure contained by more than four straight lines.

20. Among triangles, an EQUILATERAL TRIANGLE is one with all three sides equal. An ISOSCELES TRIANGLE is one with only two sides equal. A SCALENE TRIANGLE is one without any equal sides.



ABC is an equilateral triangle, DEF is an isosceles triangle, GHK is a scalene triangle.

21. Among triangles, a RIGHT TRIANGLE is one containing a right angle. An OBTUSE TRIANGLE is one containing an obtuse angle. An ACUTE TRIANGLE is one with all three of its angles acute.

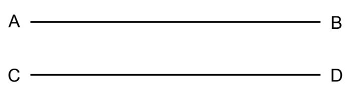


LMN is a right triangle, OPQ is an obtuse triangle, RST is an acute triangle.

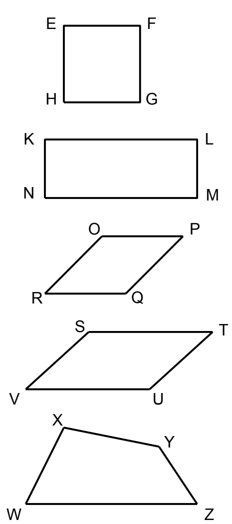
In a right triangle, the side opposite the right angle is called the HYPOTENUSE, and the two sides containing the right angle are called LEGS.

NL is the hypotenuse of triangle LMN; MN and ML are its legs.

22. If two straight lines lie in one plane together, but never meet each other in either direction however far they are extended, they are said to be PARALLEL to each other.



AB and CD are parallels.



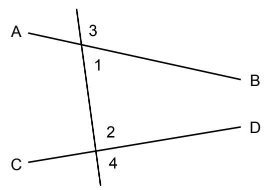
23. Among quadrilaterals, a SQUARE is a quadrilateral with all four sides equal and all four angles right; a RECTANGLE is a quadrilateral with all four angles right but not all four sides equal; a RHOMBUS is a quadrilateral with all four sides equal but no right angles.

EFGH is a square, KLMN is a rectangle, OPQR is a rhombus.

A PARALLELOGRAM is any quadrilateral contained by two pairs of parallel straight lines. Quadrilaterals that are none of the above will be called TRAPEZIA.

STUV is a parallelogram, even though it has no right angles and not all of its sides are equal. WXYZ is a trapezium.

24. Two straight lines are INCLINED TOWARD EACH OTHER if, when cut by a third straight line, the sum of the two interior angles on one side is less than the sum of the two exterior angles on that same side.



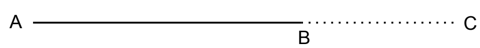
For example, if 1 + 2 *is less than* 3 + 4, *then AB and CD are inclined to each other toward the right*.

GEOMETRICAL POSTULATES

1. A straight line can be drawn from any point to any point.

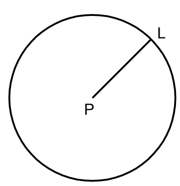
For example, from A to B.

2. Any straight line can be extended continuously in a straight line in either direction and as far as you please.



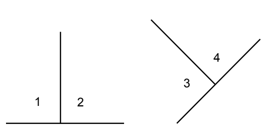
For example, AB can be extended to C, in such a way that ABC is one straight line.

3. A circle can be drawn around any point as its center and with a radius of any given length.



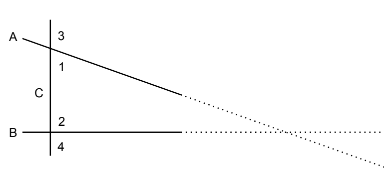
For example, around point P we may draw a circle whose radius shall be any given length, such as PL.

4. All right angles are equal.



Not only are adjacent right angles equal to each other, such as 1 and 2, but even those that are not adjacent, such as 1 and 3.

5. Straight lines inclined towards each other eventually meet, when extended far enough.



For example, let A and B be two straight lines that cut across another straight line C, making angles 1 & 2 *to the right of C. If* 1 + 2 *is less than* 3 + 4, *then A and B must eventually meet on the right side of C at some point.*

GENERAL PRINCIPLES

1. Things equal to the same thing are also equal to each other.

For example, 36 inches is equal to a yard, but 3 feet is also equal to a yard, so 36 inches has to be equal to 3 feet.

2. When equals are added to equals, the wholes are equal.

For example, if Fred and Jack are both exactly five feet tall, and this summer each will grow the exact same amount, then their heights will still be equal.

3. When equals are subtracted from equals, the remainders are equal.

For example, if two brand new pencils have exactly the same length, and then you sharpen them down by exactly the same amount, they will still have equal lengths.

4. Things that can be made to coincide with each other are equal.

That is, if two things are such that neither one goes outside of the other even a little bit, then neither is greater than the other, and so they must be equal. For example, if the bottom surface of a box fits exactly on the surface of a table, nowhere hanging over the edge or letting any part of the table's surface show, then the bottom of the box and the surface of the table have equal areas.

5. Every whole is greater than any one of its parts.

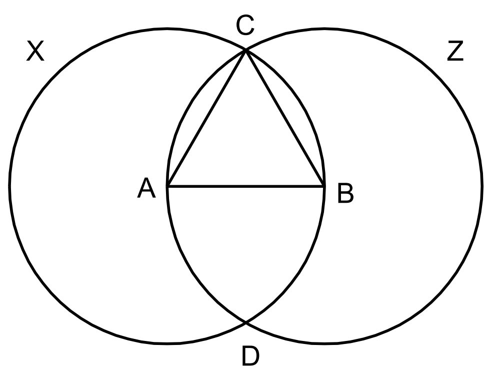
The surface area of a whole lake is greater than the surface area of any part of it; fifty dollars is greater than any part of fifty dollars, etc.

THEOREMS

THEOREM 1: *How to make an equilateral triangle.*

If someone gives us a straight line AB, how can we make an equilateral triangle on top of it so that AB is the base of the triangle?

As follows.



[1] Draw circle X around point A with radius AB (by Postulate 3).
[2] Draw circle Z around point B with radius BA (by Postulate 3).
[3] These two circles obviously must intersect each other, namely at points C and D.
[4] Join points A and C with a straight line (Postulate 1).
[5] Join points B and C with a straight line (Postulate 1).

So we have made a triangle, ABC.
It is in fact equilateral.
Why? Because:
[6] AC = AB, since these two lines are radii of circle A (see Definition 17).
[7] BC = AB, since these are both radii of circle B (Def. 17).
[8] AC = BC, since each is equal to AB (Steps 6 & 7; see Common Notion 1).
[9] So all three sides of the triangle are equal to one another, making it an equilateral triangle (Def. 20).

Q.E.F.

THEOREM 1 Remarks

1. Notice what we have done in this first theorem: we have made a perfectly equilateral triangle *without measuring anything*. There is something surprising about that. And who would have thought that circles are helpful for making triangles?

2. The equilateral triangle is the simplest rectilineal figure, and so it makes sense that we should begin geometry by making it. It has the fewest number of sides, and they are all the same.

3. Despite the simplicity of the equilateral triangle, it is a figure rich with surprising properties, some of which we will discover in this book. Also, it is very useful for other constructions, as we will see once we get to Theorem 7. We have already accomplished something significant in making the equilateral triangle, right here in the very first theorem!

THEOREM 1 Questions

1. Does an equilateral triangle appear to be right, obtuse, or acute?

2. Looking at the diagram, find a way to make a rhombus. Prove all four of its sides are equal. (We have not yet done enough geometry to prove that its angles are not right angles.)

3. Imagine that your compass became so rusty you can no longer adjust it, so that you can make circles only of one size. If the unadjustable radius of your compass is *not* equal to AB in the diagram, but was something less than AB, can you still use it to make an equilateral triangle on AB? (Remember that you still have a straight-edge which lets you extend straight lines as far as you like.)

4. Using a method of construction similar to that for the equilateral triangle, can you find a way to make an isosceles triangle? What about a scalene triangle?

5. What happens if you make 3 more equilateral triangles, one on each side of ΔABC ?

THE SIDE-ANGLE-SIDE THEOREM

THEOREM 2: *If in one triangle a side, the next angle, and the next side, are respectively equal to a side, the next angle, and the next side in another triangle, then all the corresponding sides and angles of the two triangles are equal, and they have equal areas.*

Imagine two triangles ABC and DEF such that

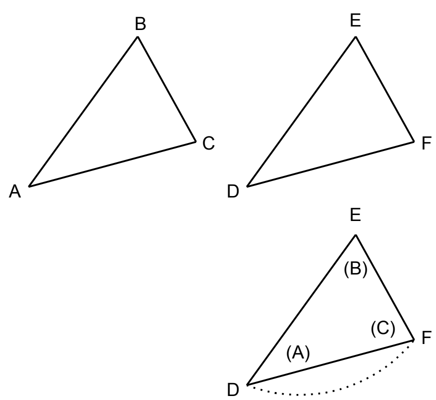
AB = DE

∠ABC = ∠DEF (angle ABC equals angle DEF)

BC = EF

Then the remaining corresponding sides and angles of these two triangles must also be equal, and the two triangles must contain equal areas. To see this,

[1] Imagine moving △ABC so that AB lies on DE; since they are equal, they will coincide, so that A is on D and B is on E.



[2] Since ∠ABC = ∠DEF, BC will fall along EF, and C will fall on F (since BC = EF).

[3] So A, B, C are sitting on D, E, F. Then AC must coincide with DF; for if it fell outside it, as the dotted line, then two straight lines, namely AC & DF, would cut each other at two points and enclose a space (which is impossible).

[4] So AB, BC, AC coincide with DE, EF, DF. Thus the two triangles have been made to coincide exactly, and so they are completely identical (Common Notion 4).

[5] Since the two triangles are identical, it follows that

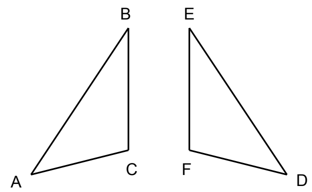
AC = DF & ∠BAC = ∠EDF & ∠BCA = ∠EFD

that is, the remaining sides and angles are equal, and also the triangles have equal areas.

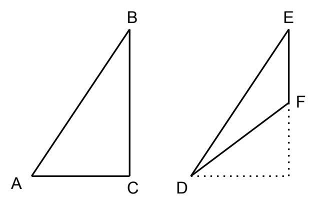
Q.E.D.

THEOREM 2 Remarks

1. This theorem is often called the "Side-Angle-Side" Theorem. If a side, the next angle, and the next side in one triangle are equal to a side, the next angle, and the next side in another triangle, then the two triangles are identical to each other.



2. This "Side-Angle-Side" Theorem is used throughout geometry, and throughout this book. Since the areas of the two triangles are equal, and also the corresponding sides and angles, this theorem is useful for proving the equality of (1) lines, (2) angles, and (3) areas.



3. An important principle is used in this theorem, namely that *Two straight lines cannot enclose a space*. Another way to say the same thing is that *Two straight lines cannot cut each other more than once*. These statements are obvious because of the perfect uniformity of straight lines; after cutting once they cannot bend back to cut each other a second time.

4. The two triangles in this theorem are not only equal in area, but all their corresponding sides and angles are equal. Such triangles are said to be congruent to each other; in fact, any pair of figures which are both the same shape and the same size are called "congruent". The symbol for congruency is ≅. The squiggly line over the equal sign means "similar", so that the two symbols together mean "equal and similar to", or "the same size and the same shape". So when you read "ΔABC ≅ ΔDEF", this means that all the corresponding sides and angles of triangle ABC and triangle DEF are equal, and the two triangles have the same area.

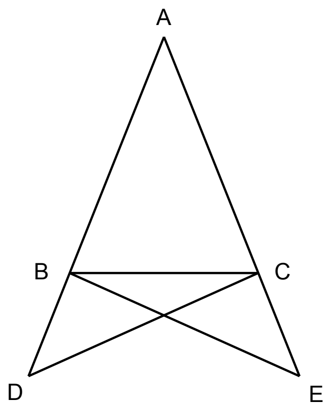
THEOREM 2 Questions

1. When we are told which initial sides and angles are equal, exactly how do we decide which of the remaining angles and sides in the two triangles are "corresponding" sides and angles?

2. What if the two triangles in question are mirror images of each other? Then it is not possible to slide one over on top of the other. So how would we prove the theorem?

3. What if ∠ABC = ∠DEF, AB = DE, but BC is *greater than* EF? What can we say about the two triangles then?

THEOREM 3: *In an isosceles triangle, the base angles are equal to each other, and the angles under the base are equal to each other.*



Imagine an isosceles triangle ABC in which AB = AC. I say that the base angles, namely ABC and ACB, those opposite the equal sides, must be equal to each other, too. Moreover, if you extend AB to D and AC to E, the angles *under* the base must also be equal to each other, namely DBC and ECB. To see it, just …

[1] Cut off CE = BD by drawing a circle around C with a radius equal to BD.

[2] Join BE. Join CD.

[3]	AB = AC	(triangle ABC is given as isosceles)
	BD = CE	(Step 1)
so	AD = AE	(the sums of equals are equal; C.N. 2)
[4]	AC = AB	(triangle ABC is given as isosceles)
[5]	$\angle$BAC is common to triangles ADC & AEB	
[6]	ΔADC $\cong$ ΔAEB	(Steps 3, 4, & 5: Side-Angle-Side)
[7]	$\angle$ADC = $\angle$AEB	(being corresponding angles of the equal triangles)
and	DC = EB	(being corresponding sides of the equal triangles)
and	BD = CE	(Step 1)
so	ΔBDC $\cong$ ΔCEB	(Side-Angle-Side)
[8]	$\angle$EBA = $\angle$DCA	(being corresponding angles in ΔADC & ΔAEB; Step 6)
but	$\angle$EBC = $\angle$DCB	(being corresponding angles in ΔCEB & ΔBDC; Step 7)
so	$\angle$CBA = $\angle$BCA	(the remainders of equals are equal; C.N. 3)
	i.e. the angles at the base of ΔABC are equal to each other.	

[9] And the angles under the base of ΔABC are also equal, namely $\angle$DBC & $\angle$ECB, being corresponding angles of ΔBDC & Δ CEB (Step 7).

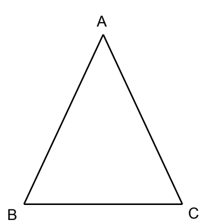
Q.E.D.

THEOREM 3 Remarks

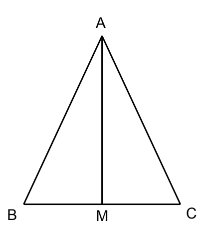
1. Notice the plan of attack, or strategy of this theorem. It is to establish first that the larger overlapping triangles (ADC and AEB) are identical or congruent, and then to use information gained from that to prove the small overlapping triangles under the base are identical or congruent (BDC and CEB). Once that is accomplished, we can derive that the base angles are equal as well as those under the base by using corresponding angles and by subtracting equals from equals. To understand more elaborate proofs it is always helpful to figure out the overall strategy used.

2. *"All isosceles triangles have their base angles equal"* is an example of a universal truth. Such truths apply to an unlimited number of cases. How many isosceles triangles exist, have existed, or will exist? An unlimited number, and they *all* have their base angles equal. We have just proven this, even though we could not possibly inspect every individual isosceles triangle. *When* did Theorem 3 *begin* to be true? Never. It was always true and always will be. Its truth did not depend on us proving it – all the proof did was allow us to come to know it. Theorem 3 was true before any geometer existed. Can the truth of Theorem 3 ever change? No. It is necessarily true and cannot be otherwise. *Where* is Theorem 3 true? Does it apply only in the United States? No. It is true everywhere. Geometry, and all other sciences, aim especially at understanding truths of this kind – universal truths, which are not tied to time and place, and which can never change but must always be so.

THEOREM 3 Questions



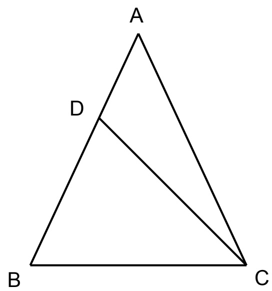
1. You can prove this theorem in another way by flipping the triangle ABC over onto itself: since AB = AC, each will fit on top of the other when we flip the triangle over, and so ∠ACB will coincide exactly with the place where ∠ABC was. Therefore these two angles are equal.



2. In Theorem 7 we will learn how to bisect any angle. Imagine that ∠BAC was already bisected for us by the straight line AM. Given that AB = AC, can you see how to prove once more that ∠ABC = ∠ACB?

3. Use Theorem 3 to prove that all three angles of an equilateral triangle are equal to each other.

THEOREM 4: *If two angles in a triangle are equal to each other, then the sides opposite them are also equal to each other, i.e., the triangle is isosceles.*



Imagine a triangle ABC, in which ∠ABC = ∠ACB. Then AB = AC, too. To see this, assume for a moment that AB and AC are *not* equal – let's see if that is really possible. Suppose, for example, that AB > AC. Then …

[1] From AB, the supposedly greater of the two sides, cut off BD = AC and join CD.

[2] Consider ΔDBC and ΔACB:

since	BD = AC	(we supposedly made it so)
and	∠DBC = ∠ACB	(as given)
and	BC is common to both triangles	
thus	ΔDBC ≅ ΔACB	(Side-Angle-Side), and so their areas are equal.

[3] But that is ridiculous, since ΔDBC is a *part* of ΔACB, and the part is always *less than* the whole (Common Notion 5).

[4] Since something impossible follows from the assumption that AB > AC, therefore the assumption that AB > AC is itself impossible. So AB *is not greater than* AC.

[5] By the same reasoning we can show that AC *is not greater than* AB.

[6] Since AB and AC are such that neither is greater than the other, therefore AB = AC after all.

Q.E.D.

THEOREM 4 Remarks

1. By Theorem 3, an equilateral triangle has all three of its angles equal. Does it follow from Theorem 4 that, since an equilateral triangle has any two of its angles equal to each other, it must therefore be an isosceles triangle? An equilateral triangle is "isosceles" in the sense that it has *at least* two of its sides equal, but not in the more restrictive sense that it has *only* two of its sides equal. We might restate Theorem 4 by saying *"If a triangle has two angles equal, then it has at least two sides equal, namely those opposite the equal angles*."

2. Theorem 4 is the "converse" of Theorem 3. The "converse" of any statement is the statement formed by switching its subject and predicate; for example the converse of the statement

Every bachelor is a man who never married

is the statement that

Every man who never married is a bachelor.

Not all converses of true statements are themselves true. For example,

Every triangle is a figure = TRUE

Every figure is a triangle = FALSE.

That is why we must often *prove* the converse of a true statement. Still, sometimes this is unnecessary. Looking back at Theorem 2, we learned that triangles having certain corresponding parts equal will have *all* corresponding parts equal. The converse is also true, but needs no proof, namely that if two triangles have *all* corresponding parts equal, then they will also have some of their parts equal.

So Theorem 4 proves the converse of Theorem 3, since this is what each proves:

Theorem 3: An isosceles triangle is a triangle with two equal base angles.

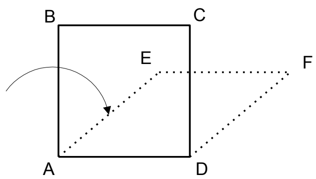
Theorem 4: A triangle with two equal base angles is an isosceles triangle.

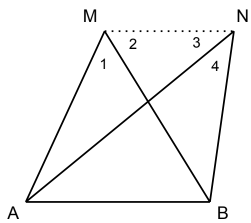
3. The point of proving the converse of a theorem (whenever its converse is true) is to show that the property we showed belongs to some figure belongs *only* to that kind of figure. In Theorem 3 we learned that ALL isosceles triangles have equal base angles – but is having equal base angles unique to isosceles triangles (namely those with at least two equal sides), or can scalene triangles have that property too? Well, in Theorem 4 we learn that ONLY isosceles triangles have equal base angles. As soon as the base angles are equal, the sides opposite them must also be equal.

THEOREM 4 Question

Use Theorem 4 to prove that any triangle with 3 equal angles is an equilateral triangle.

THEOREM 5: *Triangles are rigid.*





Imagine a standing square made out of 4 boards nailed together with one nail at each corner: A, B, C, D. If you gave AB a swift kick, then each nail would act like a hinge-pin, and the square would slant over into a new shape such as AEFD, where all the boards remain the same length, but they are now at new angles to each other. So squares are "flexible".

Can we do the same thing with a triangle? Imagine ΔABM, and suppose another triangle ABN could sit on the same base AB, having different angles from ΔABM, and yet

AM = AN

and BM = BN.

Is that possible? No! To see the impossibility, try to suppose it is possible ...

[1] Join MN.

[2] $\angle 1 + \angle 2 = \angle 3$ (since AM = AN, $\angle AMN = \angle ANM$ by Thm. 3)

[3] $\angle 2 = \angle 3 + \angle 4$ (since BM = BN, $\angle BMN = \angle BNM$ by Thm. 3)

[4] Now, by Step 3 we can substitute $(\angle 3 + \angle 4)$ in place of $\angle 2$ anywhere we please, since they are equal. So in Step 2 let's replace $\angle 2$ with $(\angle 3 + \angle 4)$, which gives us

$$\angle 1 + (\angle 3 + \angle 4) = \angle 3$$

[5] So $\angle 3 = \angle 3 + \angle 1 + \angle 4$ (Step 4 rearranged), which is to say that an angle is equal to itself plus some other angles, which is impossible.

[6] Therefore the original supposition from which this absurdity follows is also impossible, namely that the triangle ABM might be flexible, and we could keep all its sides the same length but change its angles. So *triangles are rigid*.

Q.E.D.

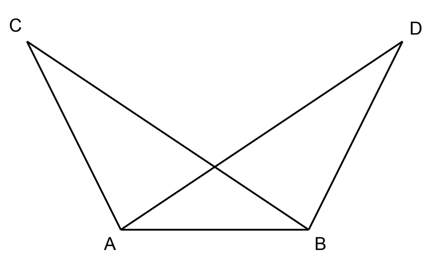
THEOREM 5 Remarks

1. The kind of proof we just used for Theorem 5, like the proof for Theorem 4 before it, is called a "reduction to the absurd" (*reductio ad absurdum* or *reductio ad impossibile* in Latin). It is also called "indirect proof."

The strategy of such a proof is to assume the *opposite* of what we wish to prove, show that something impossible would necessarily follow from such an assumption, and conclude that therefore the original assumption which gave rise to the impossibility is itself impossible. And so the thing we wish to prove is necessarily true – since its opposite turned out impossible.

Theorem 4 was a perfect example of such a line of reasoning. We wanted to show that any triangle with two equal angles is isosceles. Assuming the opposite, namely that we could have a triangle with equal angles whose opposite sides were *not* equal, it would follow that we could cut out a part of that triangle which is equal to the whole triangle. But that's impossible. So we also had to condemn the initial assumption. That is, the sides opposite equal angles in a triangle *must* be equal, lest an absurdity follow.

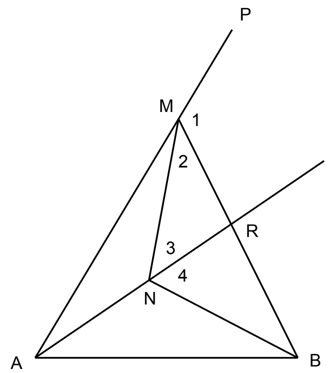
2. Another way to state Theorem 5 is this: given three straight lines making up a triangle, it is impossible to take them apart and put them together again and get a *different* triangle. Try it.



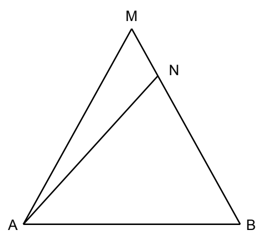
3. It is possible to make two triangles on top of AB which have identical sides but do not coincide, such as △ABC and △ABD. Yet they remain congruent, having identical sides and angles, and it is not AD that AC is equal to, but BD. The triangles are mirror images of each other. Therefore we do not get △ADB by "tilting" △ACB, but by flipping it over.

4. Theorem 5 has some applications in building things. If you look at most bridges and frames for roofs, you find triangular braces everywhere – that's because triangles add rigidity. If we use only rectangles, things can fold over, tilt, collapse!

THEOREM 5 Questions



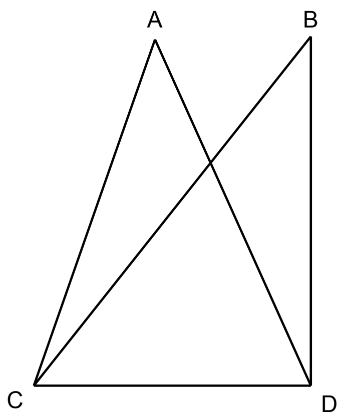
1. What if we supposed triangle ANB fell *inside* the original triangle AMB? Could we still prove the theorem then? Draw yourself a diagram: join MN as before, extend AM to any point P and extend AN until it meets MB at a point R. You will want to use the part of Theorem 3 about angles *under* the base of an isosceles triangle, since you are given that AM = AN (making AMN isosceles) and that BN = BM (making BMN isosceles, too). Begin by writing out the angles which must therefore be equal, and see if you can find something impossible. If so, you will have proved that neither is it possible to *squish* triangle AMB down to a smaller triangle ANB while keeping all the sides the same length.



2. How would the proof go if we assume N fell on MA or on MB?

3. Five sticks are nailed together to make a pentagon, with only one nail at each corner. How many cross braces are needed to make the figure rigid? In how many different ways can this be done?

THEOREM 6: *If the 3 sides of one triangle are equal to the 3 corresponding sides of another triangle, then the two triangles will also have their corresponding angles equal (namely those between equal sides), and they will have equal areas.*



Imagine ΔA, whose three sides are equal to the corresponding sides of ΔB. Since all their sides are equal, let them sit on the same base CD, and place them so that the equal sides share an endpoint, namely C and D, so that

CA = CB
and DA = DB.

[1] Now, if points A and B do not coincide with each other, then triangles will not be rigid; for then ΔBCD would have its three sides all equal to the sides of ΔACD, and yet it would slant a little to the right.

[2] Since triangles *are* rigid (Thm. 5), A and B must coincide.

[3] Since points A & B must coincide, therefore CA coincides with CB, and DA coincides with DB.

[4] Since CA coincides with CB and DA with DB, therefore

∠DCA coincides with ∠DCB
and ∠CDA coincides with ∠CDB
and ∠CAD coincides with ∠CBD

[5] Since things which coincide are equal, therefore these two triangles have all their corresponding angles equal, and are also equal in area.

Q.E.D.

THEOREM 6 Remarks

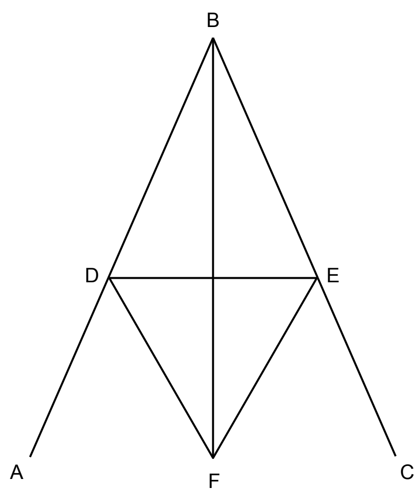
1. This Side-Side-Side Theorem is in the same family as Theorem 2, the Side-Angle-Side Theorem. It is our second Theorem about the conditions required for two triangles to be congruent to each other.

2. As with Theorem 2, it does not matter if the two triangles are mirror images of each other – we have only to flip one of them over, and then the proof can proceed as it does.

THEOREM 6 Question

Using this Theorem, prove that the two equilateral triangles made on the same straight line (one above it and the other below it) are equal to each other.

THEOREM 7: *How to bisect any rectilineal angle.*



Suppose someone gives you an angle ABC.
How can you cut it exactly in half? Easy:

[1] Pick any point D on AB, and make a circle around B with radius BD, cutting off BE = BD.
[2] Join DE.
[3] Make an equilateral triangle DEF on DE (Thm. 1).
[4] Join BF.

This line BF bisects angle DBE, that is, ∠FBD = ∠FBE. Why? Because …

[5] BE = BD (we made it so)

[6] FE = FD (they are sides of an equilateral triangle)

[7] BF is common to both ΔFBD and ΔFBE

[8] So the 3 sides of ΔFBD are equal to the 3 sides of ΔFBE, and so their corresponding angles are equal (Side-Side-Side, Thm. 6).

[9] ∠FBD = ∠FBE (being corresponding angles in ΔFBD & ΔFBE).

Q.E.F.

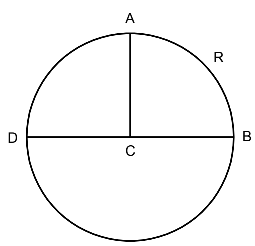
THEOREM 7 Remarks

1. It follows from this Theorem that there is no *smallest* rectilineal angle, since no matter how small it is, we can always use this theorem to bisect it and get two smaller angles.

2. The way to *trisect* any random angle, that is, cut it into 3 equal angles, is not so easy; in fact it is impossible if we limit ourselves to using circles and straight lines in a plane. *Some* angles can be trisected without difficulty, but others, such as the angle of an equilateral triangle, require more sophisticated tools than circles and straight lines.

3. Since we are cutting angles in half, we might as well mention here the mechanical tool for doing this, the *protractor*, and the unit it uses, the *degree*.

Suppose you have a circle whose circumference has been divided into 360 equal parts for you. Each part is one 360^{th} of the way around a circle, and it is called a <u>degree</u>. Degrees, accordingly, can be used to measure either the length of an arc around the circle, or the angle drawn from the center standing on that arc.



If we go along the circumference of a circle from A through R to B, and we have gone one quarter of the way around, then we have gone through 90 of the 360 equal parts of the circumference, or 90°. We can say that the arc ARB is an arc of 90°, or we can say that the angle ACB is an angle of 90°.

To go halfway around the circle from B to D is to go 180° (since that is half of 360°), and that is to open up an angle into a straight line DCB. Since ∠ACB is 90°, ∠ACD must also be 90° (since together they make 180°). Since they are adjacent equal angles and DCB is a straight line, each of them is right. So *a right angle is 90 degrees*. Any angle more than that is obtuse, and any angle less than that is acute.

A <u>protractor</u> is a simple hand tool used for measuring angles – it is basically a circle (or semicircle) with degrees marked off along its circumference and numbered. In geometry, a protractor can be handy for making accurate diagrams, even though we don't need it to prove anything.

THEOREM 7 Questions

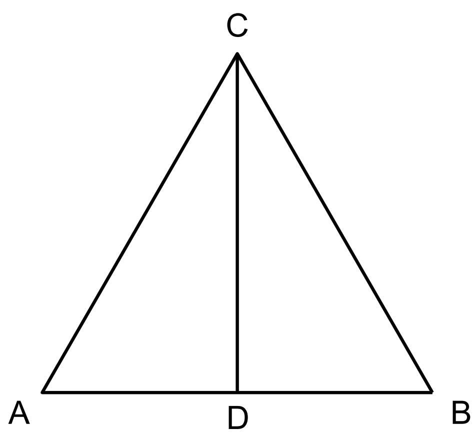
1. Will the construction and proof for Theorem 7 still work if we put the equilateral triangle on *top* of DE?

2. Draw any triangle and, using the method in Theorem 7, bisect its three angles as carefully as you can. What do you notice about the three bisectors?

3. Do you need to draw the sides of the equilateral triangle in order to draw the bisector in Theorem 7? No. That is only for the sake of the proof. Then what is the fewest number of steps needed in order to bisect an angle?

4. If 90° is a right angle, and 180° is two right angles, then give the complementary and supplementary angle for each of the following angles: 30°, 45°, 37.5°, 27.368°.

THEOREM 8: *How to bisect any straight line.*



Suppose you have a straight line AB you need to bisect. How do you do it?

Like this:

[1] Make an equilateral triangle ABC on AB (Thm. 1).

[2] Bisect ∠ACB with a line CD (Thm 7).

Now I say that the point D, where angle-bisector CD meets AB, bisects AB. That is, AD = DB. Why? Because …

[3] AC = CB (being sides of equilateral triangle ACB)

[4] ∠ACD = ∠BCD (since CD bisects ∠ACB)

[5] CD is common to both ΔACD and ΔBCD.

[6] So by the Side-Angle-Side Theorem (Thm 2), the other corresponding sides and angles of ΔACD and ΔBCD are also equal.

[7] AD = DB (being corresponding sides in ΔACD and ΔBCD)

Q.E.F.

THEOREM 8 Remarks

1. From Theorem 8 it follows that there is no smallest straight line, since no matter how small it is we can use this Theorem to bisect it and get two smaller lines.

2. A rough way to bisect a length in practice is simply by measuring with a ruler or tape measure and dividing the length in half numerically. You can also take a piece of paper of the given length and fold it in half, giving the half length.

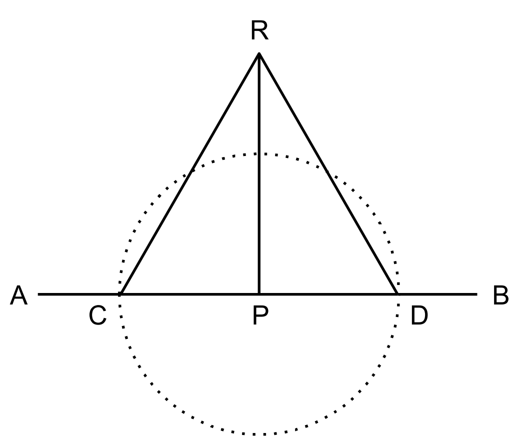
3. Unlike the problem of cutting an angle into 3 equal parts, the problem of cutting a straight line into 3 equal parts is not very difficult. In fact, later in this book, we will find a way to cut any straight line into *any* number of equal parts. Before we get there, see whether you can come up with a way to do it yourself.

THEOREM 8 Questions

1. Looking back to the diagram for Theorem 1, can you see the fewest steps needed in order to bisect a straight line?

2. Draw any triangle you like and bisect the three sides of it. Join each vertex of the triangle to the midpoint of the opposite side. What do you notice about the three straight lines you have drawn?

THEOREM 9: *How to draw a line at right angles to any straight line from any point on it.*



Suppose you have a straight line AB, and I pick a random point P on it. How can you draw a line from P at right angles to AB? As follows.

[1] Pick any point C on AP, and draw a circle around P with radius PC, thus cutting off PD = PC.

[2] Make an equilateral triangle CDR on CD.

[3] Join PR.

We did it: PR is at right angles to AB. How do we know? Because …

[4] PD = PC (Step 1)

[5] CR = DR (being sides of equilateral triangle CDR)

[6] PR is common to ΔRPC and ΔRPD.

[7] So, by the Side-Side-Side Theorem (Thm. 6), all of the corresponding angles of ΔRPC and ΔRPD must be equal.

[8] ∠RPC = ∠RPD (being corresponding angles of ΔRPC & ΔRPD)

[9] So ∠RPC and ∠RPD are adjacent angles, formed by one line RP standing on another, AB, and they are equal to each other. Therefore they are right angles (Def. 9).

[10] Thus a straight line, PR, has been set up at right angles to the given line AB and from the point P on it.

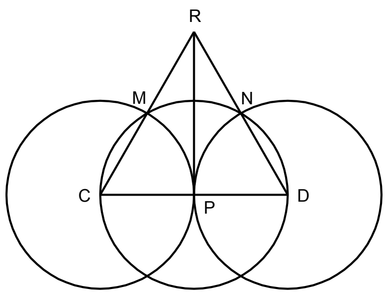
Q.E.F.

THEOREM 9 Remarks

1. Since we can now draw right angles, we can also draw right triangles. In fact, we have drawn two right triangles in the construction for this Theorem, namely ΔRPC and ΔRPD.

2. In order to avoid having to go through all the steps in this construction every time a right angle is needed, carpenters and engineers use a tool called a carpenter's "square", which is a tool shaped like a big right angle or letter "L". You can also use a protractor to mark off angles of 90°. Of course, all such tools have to be constructed by employing geometrical constructions such as the one we have given here.

THEOREM 9 Questions



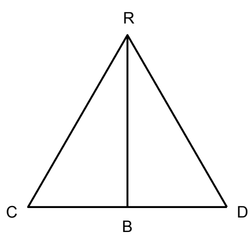
1. Suppose you have a straight line CD that has been bisected at P, so CP = PD. Imagine that three circles of the same radius (namely CP or PD) have been drawn around point C, point P, and point D.

The left circle cuts the middle one at point M, the right circle cuts the middle one at point N.

Extend CM and ND until they meet at a point, R. Join RP.

If you have drawn all of this carefully, and you measure CM and MR, what do you notice? If you measure angle RPD, how many degrees does it seem to be? See if you can settle the question with a proof.

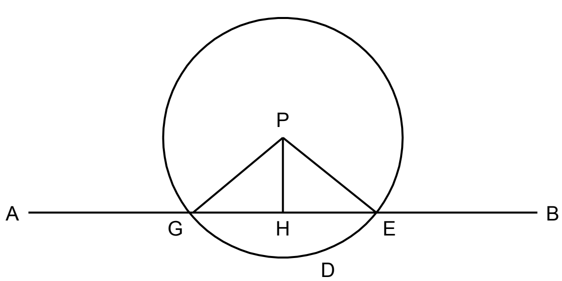
2. What is the fewest number of steps needed to draw a line from a point P at right angles to a line through P?



3. Prove that any point along the perpendicular bisector of a straight line is equidistant from both ends of the line. That is, if RB is perpendicular to CD and bisects it at B, and R is any point along the perpendicular RB, prove that RC = RD.

THEOREM 10: *How to drop a perpendicular line to any straight line from any point above it.*

Now suppose you have a straight line AB, and I pick a random point P above it. How can you drop a line from P which is perpendicular to AB? Easy.



[1] Choose any point D *below* AB.
[2] Draw a circle around P with radius PD, thus cutting AB at G and E.

[3] Join GP; join EP.
[4] Bisect GE at H (Thm. 8).
[5] Join PH.

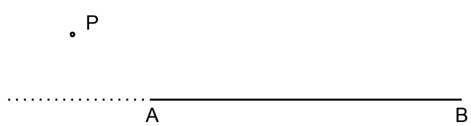
PH is in fact perpendicular to AB. Why? Because …

[6] PG = PE (being radii of circle P)
[7] HG = HE (since we bisected GE at H)
[8] PH is common to ΔPHG and ΔPHE.
[9] So, by the Side-Side-Side Theorem (Thm. 6), all the corresponding angles of ΔPHG and ΔPHE are equal.
[10] ∠PHG = ∠PHE (being corresponding angles of ΔPHG and ΔPHE)
[11] But these two equal angles, ∠PHG and ∠PHE, are adjacent angles formed by one line, PH, standing on another, AB, and so they are right angles (Def. 9).
[12] Hence PH has been drawn perpendicular to AB (Def. 10).

Q.E.F.

THEOREM 10 Remarks

1. Theorem 10 is somewhat the opposite of Theorem 9, since in Theorem 9 the point from which we had to draw a perpendicular was *on* the given line – here in Theorem 10 the point is *above* the given line.



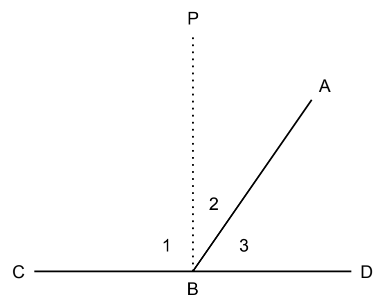
2. What if P is not directly above the straight line AB? Then we simply have to extend AB until it passes underneath P.

3. The practical way to draw a straight line perpendicular to AB from a point P is to use a tool such as a carpenter's square, the tool shaped like the letter L or like a big right angle. Put one leg of the square on AB and slide it along until the other leg touches point P, and then trace the edge of the square from point P down to AB.

THEOREM 10 Question

Do we need to draw PG and PE in order to draw PH? No; they were drawn for the sake of the *proof* that PH is perpendicular. So what is the fewest number of steps actually needed to draw a perpendicular from a point to a straight line?

THEOREM 11: *When one straight line stands on another one, the adjacent angles add up to two right angles.*



Obviously if PB stands upon CD at right angles, then ∠PBC + ∠PBD equals two right angles. Now what if AB stands upon CD, but *not* at right angles to it? Will it still be true that ∠ABC + ∠ABD = 2 rights? Yes.

To see it,

[1] Draw BP at right angles to CD (Thm. 9).

[2] Thus	∠PBC + ∠PBD = Two rights	
[3] But	∠PBC + ∠PBD = ∠1 + ∠2 + ∠3	
[4] So	∠1 + ∠2 + ∠3 = Two rights	(Common Notion 1)
[5] But	∠1 + ∠2 + ∠3 = ∠ABC + ∠ABD	
[6] So	∠ABC + ∠ABD = Two rights	(Common Notion 1)

Q.E.D.

THEOREM 11 Remarks

1. Theorems 9 & 10 began the study of right angles; now we are investigating "two right angles."

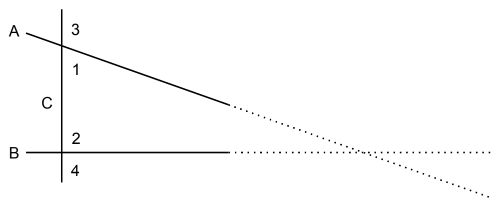
2. This Theorem 11 is rather obvious even without a proof. If CBD is a straight line, then BD must go through half of one full rotation to get to BC, which is 180°. It does not make any difference how we divide up that 180° with another line such as AB; the two angles into which the 180° has been divided must still add up to 180°.

3. We should note in connection with this Theorem that *angles supplementary to the same angle are equal to each other*. For example,
if ∠X + ∠Y = two rights,
and ∠X + ∠Z = two rights,
then ∠Y = ∠Z,

being both supplementary to ∠X. This is obvious, since each is equal to two right angles minus ∠X (recall that all right angles are equal, and that equals with equals subtracted from them leave equal remainders).

4. Recall Postulate 5, which says that inclined straight lines eventually meet. By definition, lines such as A and B are "inclined to each other" if $1 + 2 < 3 + 4$. With the help of Thm. 11, we can now define "inclined straight lines" another way.

We know that	1 + 3 = two rights
and that	<u>2 + 4 = two rights</u>
thus	1 + 2 + 3 + 4 = four rights
so if	1 + 2 is less than two rights,
it follows that	3 + 4 is more than two rights.



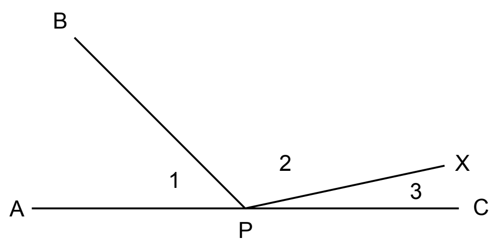
In other words, if 1 + 2 is less than two rights, then these two inside angles are less than the outside angles 3 + 4, and therefore the two lines A and B will incline to each other toward the right, and eventually meet there. So we can now restate our fifth postulate like this: *if two straight lines make less than two right angles on one side of a third straight line, then they will eventually meet on that side*.

THEOREM 11 Questions

1. What is ∠2 called in relation to ∠3?

2. What is ∠3 called in relation to ∠ABC?

THEOREM 12: *If two adjacent angles add up to two right angles, then the two lines other than the line common to both angles are in a straight line with each other.*



Let $\angle APB$ and $\angle BPC$ be two adjacent rectilineal angles beginning at the point P, and suppose that

$$\angle APB + \angle BPC = \text{two rights.}$$

Then AP and PC must be in a straight line with each other.

Why is that?

Simple. Suppose that PC is *not* the extension of AP in a straight line, then

[1] Extend AP in a straight line to some point X (Post. 2)

[2] Thus $\angle 1 + \angle 2 =$ two rights (Thm. 11)

[3] But $\angle 1 + \angle 2 + \angle 3 =$ two rights,
That is, $\angle APB + \angle BPC =$ two rights, since that is how they were given to us.

[4] So $\angle 1 + \angle 2 = \angle 1 + \angle 2 + \angle 3$,
putting together Steps 2 & 3, since each side is supposedly equal to two right angles.

[5] But that is crazy. For $\angle 1 + \angle 2$ is only a *part* of $\angle 1 + \angle 2 + \angle 3$, and the whole never equals a part of itself (Common Notion 5).
And since this impossibility follows from our initial supposition that PC is *not* the extension of AP in a straight line, therefore that initial supposition is also impossible.

[6] Therefore PC *is* the extension of AP in a straight line, i.e. AP and PC are in a straight line with each other.

Q.E.D.

THEOREM 12 Remarks

1. Theorem 12 is the converse of Theorem 11. Theorem 11 showed that angles adding up to a straight line must add up to two right angles; Theorem 12 shows that angles adding up to two right angles must add up to a straight line.

2. Notice that we use Postulate 4 for the first time in this theorem. In Step 4 we say that

$\angle 1 + \angle 2 = \angle 1 + \angle 2 + \angle 3$

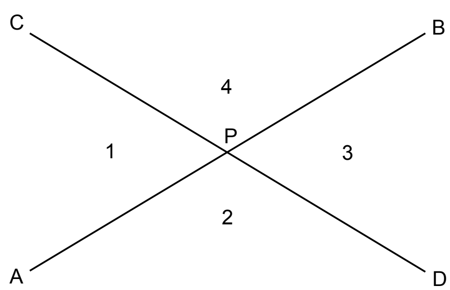
on the grounds that each side is equal to two right angles. We are presuming that "two right angles" is always the same amount, i.e. that *all right angles are equal*.

3. Notice that if from the equation in Step 4 we subtract $\angle 1 + \angle 2$ from both sides, we get $\angle 3 =$ *nothing!* If that were true, then PX would coincide with PC, and AP would be in a straight line with PC, which is what we set out to prove.

THEOREM 12 Question

Does it make any difference to the proof if someone says that PX actually falls *below* PC?

THEOREM 13: *When two straight lines cut each other, they make the vertical angles equal to each other.*



Imagine two straight lines AB and CD cutting across each other at P.
The "vertical angles" in the diagram are equal to each other, that is

1 = 3
2 = 4

Can it be proven? Of course.

[1] $\angle 1 + \angle 2 =$ two rights (Thm. 11).

[2] $\angle 3 + \angle 2 =$ two rights (Thm. 11).

[3] $\angle 1 + \angle 2 = \angle 3 + \angle 2$ (Steps 1 + 2, Common Notion 1)

[4] Subtracting $\angle 2$ from each side of Step 3, we have
$\angle 1 = \angle 3$ (Common Notion 3)

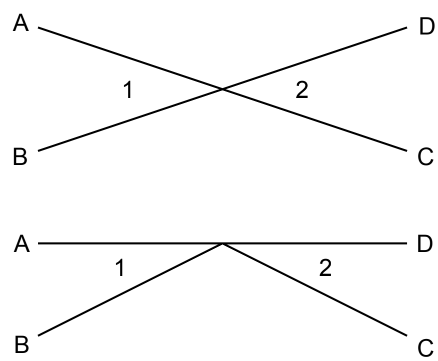
[5] $\angle 2 = \angle 4$ is proven in the same way.

Q.E.D.

THEOREM 13 Remarks

1. Theorem 13 demonstrates a kind of symmetry: the angle on one side of two cutting straight lines is a mirror image of the angle on the opposite side of the vertex.

The reason for this symmetry is clear enough: it is a result of the uniformity of straight lines. Two straight lines cannot make *one* size opening on one side and *another* size opening on the other side of the vertex. They tend *toward* each other on one side in the same way that they tend *away* from each other on the other.



2. What about the converse? If we state the *total* converse of Theorem 13, it is not true, namely *If two rectilineal angles are equal, then they are formed by two cutting straight lines*. Obviously, two rectilineal angles can be equal and yet be far apart from each other. But the *partial* converse of Theorem 13 is true, namely *If two rectilineal angles are equal and share a common vertex, and one leg of one angle is in a straight line with the alternate leg of the other angle, then the remaining legs are also in a straight line with each other*.

To be more concrete, look at the adjoining figure: If 1 = 2, and line A is in line with line C, then line B must also be in line with line D.

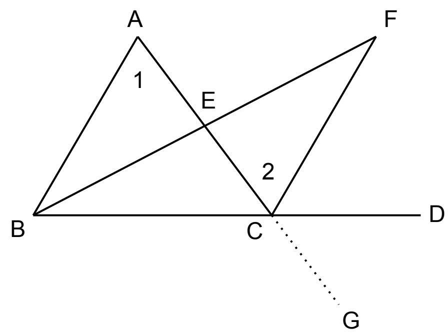
On the other hand, if A is in line with line D, the adjacent leg in the other angle, it is *not* true that B has to be in line with line C.

THEOREM 13 Questions

1. Prove the partial converse of Theorem 13 stated above in Remark 2.

2. Looking back at the diagram for Theorem 13, how many degrees does $\angle 1 + \angle 2 + \angle 3 + \angle 4$ equal?

THEOREM 14: *If any side of a triangle is extended, the exterior angle is greater than either of the interior and opposite angles*.



Take any triangle – ABC will do. Extend any side, say BC, to D. The angle ACD is called an "exterior angle," and this theorem says

$\angle ACD > \angle BAC$

and $\angle ACD > \angle ABC$.

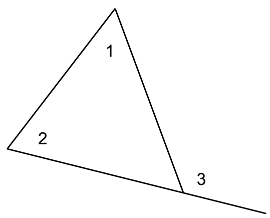
If you want proof, then …

[1] Bisect AC at E. (Thm. 8)
[2] Join BE and produce it far enough to cut off EF = BE.
[3] Join CF. Now,

[4] EF = BE (we made it so)
[5] EA = EC (we bisected AC at E)
[6] $\angle AEB = \angle CEF$ (Thm. 13; they are vertical angles)
[7] So by the Side-Angle-Side Theorem (Thm. 2), the other corresponding angles of ΔAEB and ΔCEF are equal.
[8] $\angle 1 = \angle 2$ (being corresponding angles of ΔAEB & ΔCEF)
[9] But $\angle ACD > \angle 2$ (since the whole is greater than the part)
[10] So $\angle ACD > \angle 1$ (putting together Steps 8 & 9)
[11] So $\angle ACD > \angle BAC$ ($\angle 1$ *is* $\angle BAC$)
And that is one part of what we wanted to prove.
[12] $\angle ACD > \angle ABC$ is proved by bisecting BC, and by extending AC to any point G.
Then by the same argument as before,
$\angle BCG > \angle ABC$
But $\angle ACD = \angle BCG$ (Thm. 13; they are vertical angles)
Thus $\angle ACD > \angle ABC$.
[13] So the exterior angle ACD is greater than either of the opposite and interior angles, namely $\angle ABC$ and $\angle BAC$.

Q.E.D.

THEOREM 14 Remarks



1. Here we learn that
$\angle 3 > \angle 1$
and $\angle 3 > \angle 2$.

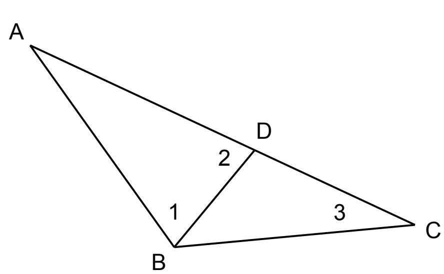
2. In Theorem 27, we will learn that
$\angle 3 = \angle 1 + \angle 2$
which explains why $\angle 3$ is greater than either one of them taken alone.

THEOREM 14 Questions

1. In Step 12 of Thm. 14, I said "by the same argument as before, $\angle BCG > \angle ABC$." Go through the actual steps. Make a diagram for yourself, bisecting BC this time.

2. Join AF in the original diagram or in your copy of it. What shape does AFCB appear to be?

THEOREM 15: *In any triangle, a greater side will have opposite to it a greater angle.*



Look at triangle ABC. If it is given that
$AC > AB$,
it must also be true that the angle opposite AC is greater than the angle opposite AB, that is:
$\angle ABC > \angle BCA$.
Why? Let's see …

[1] Since $AC > AB$, we can cut off $AD = AB$. Join BD.

[2] $\angle 1 = \angle 2$ (Thm. 3, since $AB = AD$)

[3] $\angle 2 > \angle 3$ (Thm. 14; since $\angle 2$ is exterior to ΔBDC)

[4] $\angle 1 > \angle 3$ (Putting Steps 2 & 3 together)

[5] $\angle ABC > \angle 1$ (Since the whole is greater than the part)

[6] $\angle ABC > \angle 3$ (Putting Steps 4 & 5 together)

[7] That is, $\angle ABC > \angle BCA$ ($\angle BCA$ is the same as $\angle 3$)

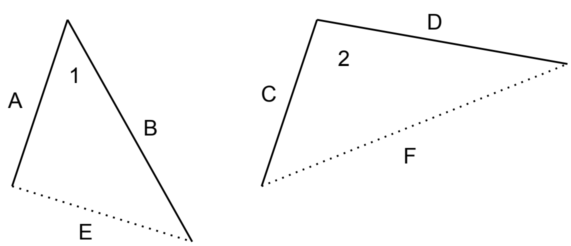
Q.E.D.

THEOREM 15 Remarks

1. It follows from Theorem 15 that in any triangle:
The greatest angle is the one opposite the greatest side.
The least angle is the one opposite the least side.
We already know from Theorem 3 that in any triangle:
Angles opposite to equal sides are themselves equal.

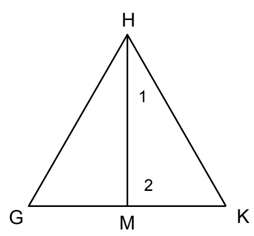
2. These facts might lead us to think that angles in a triangle are somehow proportional to their opposite sides, for example that if one side is double another side, then the opposite angle is double the opposite angle. Unfortunately, that is completely FALSE! To see this better, take a look at Question 2 below.

THEOREM 15 Questions



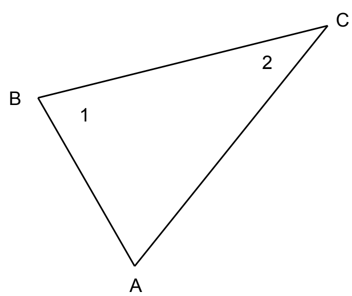
1. Using Theorem 15, prove that if *two* triangles have two sides equal to two sides, but one base greater than the other base, then the included angle is also greater than the included angle. That is,

If $A = C$
and $B = D$
but $E < F$
Show that $1 < 2$



2. Draw an equilateral triangle GHK. Bisect GK at M and join HM. Now just focus on ΔHMK. Clearly HK is double MK, but are their opposite angles respectively double? That is, is ∠2 double ∠1? Prove it is not so.

THEOREM 16: *In any triangle, a greater angle will have opposite to it a greater side*.



Look at triangle ABC. If it is given that

∠1 > ∠2,

then it must also be true that the side opposite ∠1 is greater than the side opposite ∠2, which is to say that

AC > AB

Who says? Pure logic says . . .

[1] There are only three possibilities:
Either AC = AB
Or AC < AB
Or AC > AB

[2] *Assume AC = AB*, and let's see what happens.

[3] Then ∠1 = ∠2 (Thm. 3)
[4] But ∠1 > ∠2 (They are given to us that way)
[5] So AC cannot be equal to AB.

[6] *Assume AC < AB*, and let's see what happens.

[7] Then ∠1 < ∠2
since Thm. 15 says that in any triangle, a greater side stands opposite a greater angle.
[8] But ∠1 > ∠2 (They are given to us that way)
[9] So AC cannot be less than AB.

[10] Since AC is neither equal to AB (Step 5), nor less than AB (step 9), therefore AC is greater than AB.

Q.E.D.

THEOREM 16 Remarks

1. From Theorem 16 we now know that, in any triangle:

The greatest side is the one opposite the greatest angle.
The least side is the one opposite the least angle.

We already know, from Theorem 4, that

Sides opposite equal angles are themselves equal.

2. Theorem 16 is the converse of Theorem 15. The strategy here is the process of elimination. If there are only three possibilities, and we eliminate two of them, then the third one must be true.

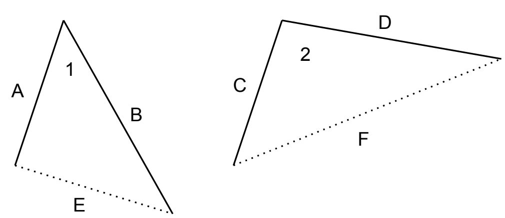
3. Notice there is no construction whatsoever in the proof for Theorem 16; it is a matter of pure logic.

4. Even though angles are not exactly proportional to sides in a triangle, we now know that the order of inequality among angles corresponds to the order of inequality among their opposite sides.

If you have a protractor, but not a ruler, you can still tell which is the greatest side in the triangle just by measuring the angles.

If you have a ruler, but not a protractor, you can still tell which is the greatest angle in the triangle just by measuring the sides.

THEOREM 16 Questions

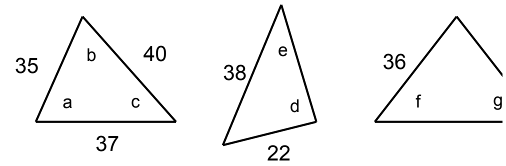


1. Using Theorem 16, prove that if *two* triangles have two sides equal to two sides, but the included angle in the one triangle is less than the included angle in the other, then the base is also less than the base. That is,

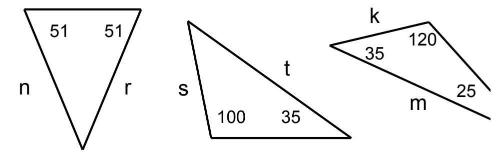
Given that	$A = C$
and that	$B = D$
but that	$1 < 2$
show that	$E < F$

Use the process of elimination (either $E = F$ or $E > F$ or $E < F$), Theorem 6, and the conclusion of Question 1 after Theorem 15.

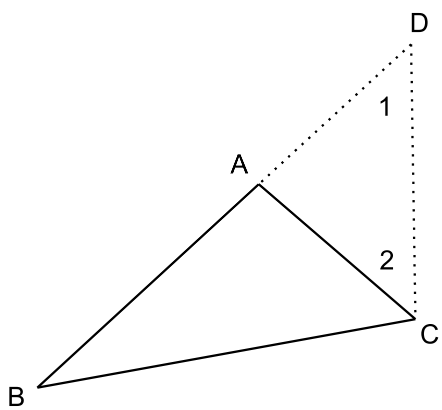
2. Given the lengths of some of the sides in each of these triangles, say what we know about the labeled angles.



3. Given the number of degrees in some of the angles of these triangles, say what we know about the labeled sides.



THEOREM 17: *In any triangle, any side is less than the sum of the other two sides, but greater than their difference.*



Take any triangle ABC. Any two sides of it added together must be greater than the third side, that is:

BA + AC > BC
and AC + BC > BA
and BC + BA > AC

Proof:

[1] Extend BA to D so that AD = AC.
[2] Join CD.

[3] ∠BCD > ∠2 (since ∠2 is part of ∠BCD)
[4] ∠BCD > ∠1 (∠2 = ∠1, since AD = AC)
[5] So in ΔDBC, the side opposite ∠BCD is greater than the side opposite ∠1 (Thm. 16).
[6] BD > BC (being the sides opposite ∠BCD & ∠1 respectively)
[7] BA + AC > BC (BD = BA + AD, and AD = AC)
[8] Likewise we can prove that
AC + BC > BA
and BC + BA > AC
[9] Thus in △ABC any side is less than the sum of the other two. It follows also that any side must be greater than the difference between the other two. For example,
BA > BC – AC. Why? Because
BA + AC > BC (any two sides are greater than the third)
so BA > BC – AC (subtracting AC from both sides).

Q.E.D.

THEOREM 17 Remarks

1. We could also look at the theorem this way: if you had to race someone from A to B, would you run first from A to C, then from C to B? No way. AB is the shortest distance from A to B.

2. A more general version of this theorem is this: *A straight line is the shortest distance between two points*. Every jointed or curved path is longer.

Obviously, that does not mean that the straight path is always the easiest to travel. The shortest distance between two points on either side of a forest is a straight line, but the quickest way through might be a curved path that takes us *around* the trees! Hence the expression "as the crow flies", which means I am telling you the *straight line* distance, although the path you actually take to walk the distance might be curved and therefore much longer. Crows don't have to move around obstacles.

3. Although the straight line is the *shortest* distance between two points, there is obviously no *longest* distance between two points. You can take as convoluted a path from point A to point B as you like, and there will always be one even more convoluted and longer.

This often happens in mathematics, that there is one kind of limit, e.g. a shortest or smallest, but there is no limit in the opposite direction, e.g. a longest or greatest.

4. This theorem shows that there is a condition that must be fulfilled before any 3 straight lines will be able to make a triangle. *Any two of those lines added together has to be greater than the third*. Otherwise, you can't make a triangle with those 3 straight lines. For example, try to make a triangle whose sides are equal to 1, 2, and 3 unit lengths (say 1 inch, 2 inches, and 3 inches). Good luck!

This means that as soon as I give you two sides to make a triangle with, I have placed a restraint on what you can use for the third side: whatever you use *has* to be less than the sum of the two sides I already gave you, but greater than their difference. For example, if I give you a side of 4 feet and a side of 7 feet, you cannot use 24 feet for the other side, since that is bigger than 11 feet (the sum of the two sides I gave you). On the other hand, you cannot use 2 feet, since that is smaller than 3 feet (the difference between the sides I gave you). Any side you choose must therefore be *greater* than 3 feet and yet *less* than 11 feet.

5. This condition is not true with regard to the *angles* of a triangle, i.e. it is *not* true that any 2 angles in a triangle, taken together, have to add up to more than the remaining angle. It is possible, for example, to have a triangle whose angles are 1°, 1°, and 178°.

THEOREM 17 Questions

1. Say which of the following triads of straight lines can be used to form a triangle, and which cannot (say the lengths are all in inches):

(a)	3	4	5
(b)	1	2	3
(c)	1	2	1
(d)	4	5	6
(e)	1	10	11
(f)	1	1	1
(g)	3	2	3
(h)	7	9	8

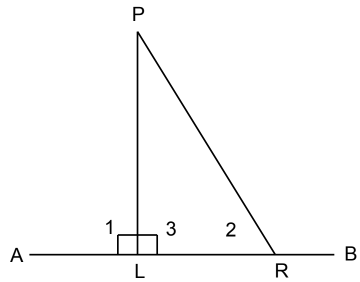
2. If I give you a line 5" long, and another that is 6" long, the Theorem states that any third line able to form a triangle with them must be *between* 1" and 11"; it must be more than 1" long, and less than 11" long. What is the range of possible lengths for the third side of a triangle with one side of 8 miles and another side of 11 miles?

3. Cut out six thin strips of paper with the following lengths in inches: ½, 1, 2, 6, 8, 10, 19. Measure exactly. Color the 10-inch strip red. Next make any triangle you like using any three of these strips as sides.

I will now make a prediction: the triangle you made uses the red strip, and the angle opposite to it is a right angle.

Why was I able to predict that you used the red strip? (The reason I know the angle opposite it is a right angle you will learn in Theorem 37.)

THEOREM 18: *The shortest distance from a point to a straight line is the perpendicular from the point to the line.*

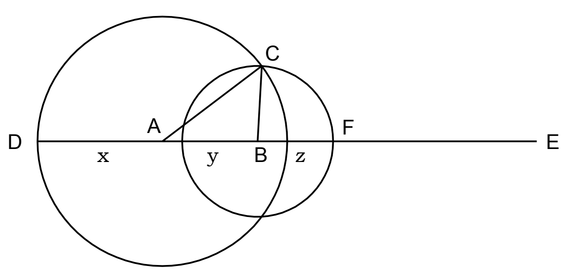


If P is a point and AB a straight line below it, what is the shortest distance between P and AB? Clearly the shortest distance between P and some point on AB is the straight line between them, since the shortest distance between any two points is a straight line (Thm. 17). For example, PB is the shortest distance between P and B. And if R is a random point on AB, PR is the shortest distance between P and R. But which among these sorts of straight lines is shortest? The perpendicular line PL. PL is shorter than any other straight line drawn from P to any random point R on AB. Here's why:

[1] $\angle 1 > \angle 2$ ($\angle 1$ is exterior to ΔPLR; Thm. 14)
[2] $\angle 3 > \angle 2$ ($\angle 3 = \angle 1$, since both are right angles)
[3] Therefore in ΔPLR, the side opposite $\angle 3$ is greater than the side opposite $\angle 2$ (Thm. 16), that is
[4] PR > PL.

Q.E.D.

THEOREM 19: *How to make a triangle out of any three straight lines (provided any two of them together are greater than the third).*



Take any three straight lines X, Y, Z which are such that any two added together make a line greater than the third one. Can you make a triangle out of them? You bet.

[1] Call line X "DA", and extend it out to E as far as you need.

[2] Cut off AB = Y, and BF = Z.

[3] Draw a circle around A with radius AD.

[4] Draw a circle around B with radius BF.

[5] Since AD and BF add up to *more than* AB (given), the circles with radii AD and BF will *overreach* each other along AB. So at least a part of AB is inside *both* circles at once. Thus the circles around A and B overlap.

[6] Since BA and AD add up to *more than* BF (given), thus BD is *greater than* the radius of circle B, and so point D lies *outside* circle B.

[7] Since AB and BF add up to *more than* AD (given), thus AF is *greater than* the radius of circle A, and so point F lies *outside* circle A.

[8] Thus circle A and circle B must *cut each other*. For they overlap each other (by Step 5), and each passes outside of the other, since D on circle A lies outside circle B (Step 6), and F on circle B lies outside circle A (Step 7).

[9] Call the point where circles A and B cut each other point C.

[10] Join AC and join CB, forming triangle ABC.

[11] AC = AD = X
AB = Y
BC = BF = Z

[12] So triangle ABC has been made, with its sides equal to X, Y, and Z.

Q.E.F.

THEOREM 19 Remarks

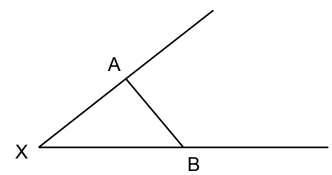
1. Theorem 19 is the converse of Theorem 17.

Thm. 17 says: *If 3 lines can form a triangle, then any 2 of them is greater than the remaining one,*
Thm. 19 says: *If 3 lines are such that any 2 of them is greater than the remaining one, then they can form a triangle*.

THEOREM 19 Question

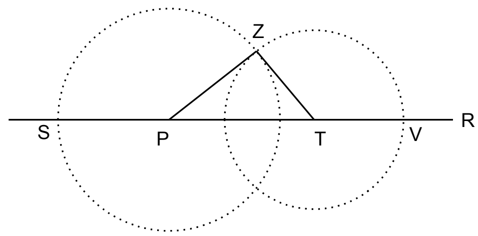
Make a triangle with sides equal to 3", 4", and 5". What do you notice about it?

THEOREM 20: *How to make an angle equal to any angle.*



Suppose an angle X is drawn on the wall, and I point out a straight line PR somewhere else on the wall. Can you draw an angle beginning at P, equal to X, and having PR as one of its sides?

Sure you can. Just . . .



[1] Pick points A & B at random on each of the legs of ∠X, and join them.

[2] Extend RP, and
 cut off PS = XA
 cut off PT = XB
 cut off TV = AB

[3] Make ΔPZT = ΔXAB (Thm. 19)
 And so by the Side-Side-Side Theorem, the corresponding angles of these two triangles must be equal.

[4] ∠ZPT = ∠AXB (being corresponding angles of ΔPZT & ΔXAB)

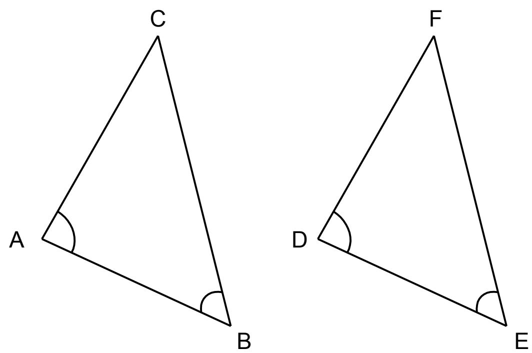
[5] But ∠ZPT begins at P, and PT lies along PR, and ∠AXB is the given angle X.

Q.E.F.

THEOREM 20 Remark

Theorem 20 obviously enables us to do more than duplicate an angle; it enables us to duplicate an entire triangle.

THEOREM 21: *If in a triangle two angles and the side joining them are respectively equal to two angles and the side joining them in another triangle, then all the corresponding sides and angles of the two triangles are equal, and the triangles have the same area.*



Imagine two triangles ABC & DEF, such that

∠BAC = ∠EDF
AB = DE
∠ABC = ∠DEF

Then the remaining sides and angles of these two triangles must be equal, too. Why, you ask? Because:

[1] Pick up ΔABC in your imagination and place AB on top of DE.
Since AB = DE, A will sit on D and B will sit on E.

[2] Since ∠BAC = ∠EDF, therefore AC and DF will lie in line together.

[3] Since ∠ABC = ∠DEF, therefore BC and EF will lie in line together.

[4] Since AC lies on DF (Step 2),
and since BC lies on EF (Step 3),
the meeting point of AC & BC sits on the meeting point of DF & EF.

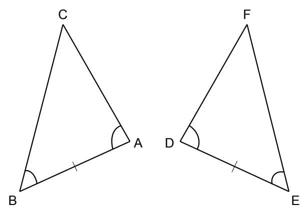
[5] That is, C sits on F.

[6] So if the base AB is placed on the base DE, then
A coincides with D
B coincides with E
and C coincides with F

[7] So the 3 sides of ΔABC coincide with the 3 sides of ΔDEF, and so all of their corresponding sides and angles are equal and they also have equal areas (Side-Side-Side Theorem).

Q.E.D.

THEOREM 21 Remarks



1. This is our third triangle-congruence theorem. We now have:

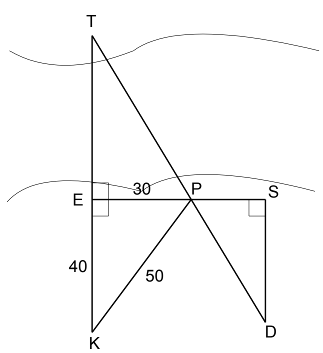
SAS (Side-Angle-Side)
SSS (Side-Side-Side)
ASA (Angle-Side-Angle)

2. As with the other congruence theorems, it is possible for the given triangles to be mirror images of each other. In that case, we must flip over one of the triangles to show that they can coincide.

THEOREM 21 Questions

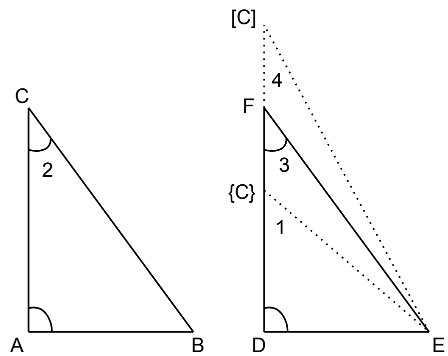
1. Prove Theorem 21 by reduction to the absurd. Start by assuming, if possible, that EF > BC. That will make it possible to cut off a *part* of EF that is equal to BC. Find an absurdity, and you will have found a proof.

2. Some Boy Scouts stood at the edge of a canyon at point E, across from a tree on the other side of the canyon standing at T. They needed to measure the distance across the canyon, namely the distance ET. But how could they measure a distance across a canyon? Here is what they did. They marked E with a stake, and from it sighted along a straight stick, pointing it at T, putting the stick in line with ET. Next, they walked out to some point P further along the edge of the canyon, so that ∠TEP was a right angle, and then they marked P with a tall pole. Then they continued in a straight line to S until PS was the same distance as EP, and then marked S with another stake. Then they walked *away* from the canyon along the line SD so that SD was at right angles to PS. They stopped at the point D where the pole at P was in the same line of sight with the tree at T across the canyon. Then they just measured SD, and said that was the same length as the distance ET across the canyon. Prove they were right.



Incidentally, how could they ensure EP is at right angles to ET? They could stretch a string E in the same line of sight with T, say 40 feet long, then form a triangle with EP being 30 feet long, and KP 50 feet long. Why angle KEP would be 90° we shall see later.

THEOREM 22: *If two triangles have two angles equal to two angles, and a side equal to a side (namely a side opposite an equal angle), then they will have all their corresponding sides and angles equal, and they will also have equal areas*.



Suppose you have two triangles ABC and DEF such that

$\angle BCA = \angle EFD$
$\angle BAC = \angle EDF$
$AB = DE$

Then the remaining sides and angles of the two triangles must be equal, too. To prove it:

[1] Pick up ΔABC and place AB on top of DE.
Since AB = DE, A will sit on D and B will sit on E.

[2] Since $\angle BAC = \angle EDF$,
AC must lie along DF.
But is AC the same *length* as DF?

[3] Suppose, if possible, that AC is *shorter* than DF, and C lands at {C} along DF. *Let's see if that works . . .*

[4] $\angle 1 = \angle 2$ (since ΔD{C}E is the same as ΔACB)
[5] $\angle 3 = \angle 2$ (since the original triangles are given that way)
[6] $\angle 3 = \angle 1$ (putting Steps 4 & 5 together)

[7] So exterior ∠1 is equal to the opposite and interior ∠3 in triangle E{C}F – which is absurd, since the exterior angle of a triangle must always be *greater than* an interior and opposite angle (Thm. 14).

[8] Since the absurdity in Step 7 follows from our assumption in Step 3 that AC < DF, therefore that assumption is also absurd, and so AC is *not less than* DF.

[9] For the same reason, neither can AC be *greater than* DF, such that C would fall as [C] along DF extended. For then exterior ∠3 would equal interior and opposite ∠4.

[10] Since AC is not less than DF (Step 8), and AC is not greater than DF (Step 9), thus AC is equal to DF.

[11] Since AB = DE, ∠BAC = ∠EDF, and AC = DF (Step 10), therefore all corresponding sides & angles of ΔABC & ΔDEF are equal and they also have equal areas (Thm. 2).

Q.E.D.

THEOREM 22 Remarks

1. This is our fourth triangle-congruence theorem. Now we have

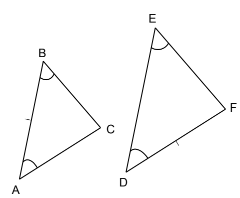
SAS SSS ASA AAS

Essentially, we need 3 pieces of information about a pair of triangles to determine whether or not they are equal; we need to know either that

(a) two sides and an included angle are equal to two sides and an included angle, or that

(b) three sides are equal to three sides, or that

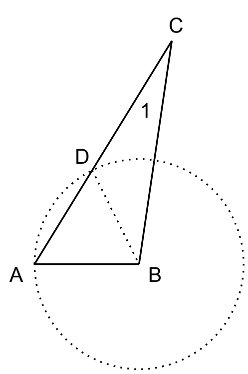
(c) two angles and one side are equal to two angles and one corresponding side.



2. What happens if the two sides given as equal are not corresponding sides? For example, what if AB = DF? Then, as we shall see later, although the triangles might not be congruent, they will still be similar.

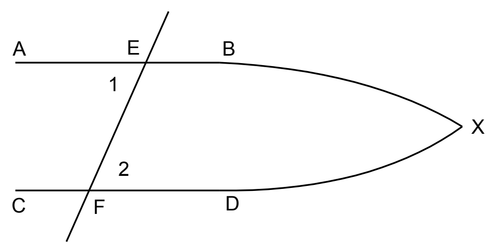
3. The difference between Theorem 21 and Theorem 22 is simply this: in Theorem 21, we are given two triangles having two angles equal to two angles, and the side *adjoining* the equal angles equal to the side *adjoining* the equal angles. In Theorem 22, however, the sides given as equal are *not* the ones adjoining the pairs of equal angles, but another pair of corresponding sides.

THEOREM 22 Question



1. What if two sides and a *non*-included angle are given as equal to two sides and a corresponding *non*-included angle? Should there be an SSA theorem? As a hint, consider △ABC, and suppose a circle around center B with radius AB cuts AC at D. Compare △ABC with △DCB. Will it be possible to make any more triangles with ∠1 and side BC, having another side equal to AB?

THEOREM 23: *If two straight lines are cut by a third making the alternate angles equal, then the two straight lines are parallel.*



Imagine you have three straight lines: AB, CD, EF. Suppose EF cuts AB and CD in such a way that

$$\angle 1 = \angle 2$$

Then AB and CD are parallels.
Let's prove it.

[1] Suppose, if possible, that *AB and CD are not parallel*, but rather EB and FD meet toward the right (when extended far enough) at some point X.

[2] Then they form a triangle EFX.

[3] $\angle 1$ is the exterior angle of triangle EFX,
$\angle 2$ is one of the interior and opposite angles of triangle EFX.
And yet . . .

[4] $\angle 1 = \angle 2$ (they are given to us that way)

[5] And so the exterior angle of triangle EFX is equal to one of its interior and opposite angles, which is absurd, since by Theorem 14 the exterior angle is always greater.

[6] Since the absurdity in Step 5 follows from our initial assumption that *AB and CD are not parallel*, but meet and form a triangle, therefore that assumption is absurd, and we must say instead that AB and CD are parallel, and never meet so as to form a triangle with EF.

Q.E.D.

THEOREM 23 Remarks

1. Here we begin the study of parallels.

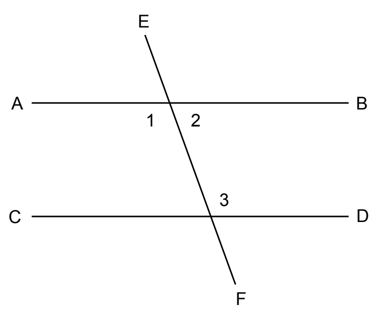
2. It is amazing that we can prove that two straight lines *never* meet, even though we can't actually extend them forever and check. We have managed a way around that!

3. The straight line EF is called a transversal in relation to the straight lines AB and CD, since it cuts across them both.

THEOREM 23 Question

Prove Theorem 23 in another way by the symmetry of the diagram; is there reason to think that what happens on *one* side of EF must also happen on the *other* side? And if so, can you see how that will help prove the theorem? Recall that two straight lines cannot enclose a space.

THEOREM 24: *If two straight lines are cut by a third making interior angles on one side add up to two rights, then the two lines are parallel.*



Suppose AB & CD are cut by EF so that
$\angle 2 + \angle 3$ = two right angles.
Then AB is parallel to CD.

Why?

[1] $\angle 2 + \angle 3$ = two right angles (given)

[2] $\angle 2 + \angle 1$ = two right angles (Thm. 11)

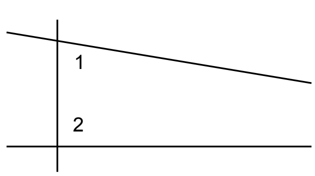
[3] $\angle 2 + \angle 3 = \angle 2 + \angle 1$ (Putting together Steps 1 & 2)

[4] $\angle 3 = \angle 1$ (Subtracting $\angle 2$ from both sides of Step 3)

[5] AB is parallel to CD (Since $\angle 3 = \angle 1$; see Thm. 22)

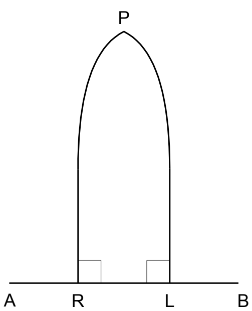
Q.E.D.

THEOREM 24 Remarks



1. Compare Theorem 24 to our fifth postulate. Postulate 5 says that if $1 + 2 < 180°$, then the two lines must meet out toward the right somewhere. Theorem 24 says that if $1 + 2 = 180°$, then the two lines do *not* meet anywhere.

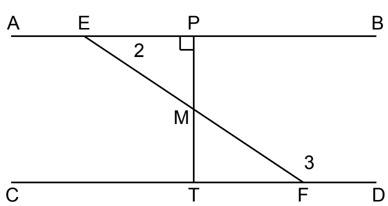
2. Notice that the postulate is about lines that *do* meet, and this theorem is about lines that do *not* meet.



3. It follows from this theorem that there cannot be *two* perpendiculars from one point to one straight line, but only one. Assume that you *could* have two perpendiculars, PR & PL, from the same point P to the straight line AB. Then we have formed a triangle PRL having two right angles in it, namely ∠PRL & ∠PLR. By Theorem 24, RP and LP have to be parallel! And so *either* RP and LP do not meet (and therefore there is no point P common to them), or one of those two angles is less than a right angle.

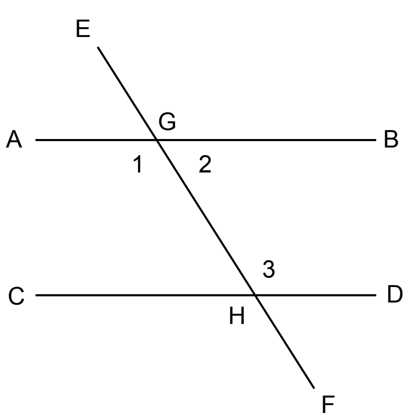
THEOREM 24 Questions

1. What if 1 + 2 > 180°? What happens then?



2. Is it possible to prove Theorem 24 independently of Theorem 23? Try bisecting EF in the diagram at its midpoint M, then drop MP perpendicular to AB, and extend PM to T on CD. As with Theorem 23, think about the symmetry of the figure. If you can prove ∠MTF is a right angle (compare △EMP and △FMT), then you can see that whatever happens on one side of PMT must also happen on the other side of it.

THEOREM 25: *If a straight line cuts two parallels, it makes interior angles on one side add up to two rights, and it makes equal alternate angles.*



Let AB and CD be two parallel straight lines cut by EGHF, making alternate angles 1 & 3, and interior angles 2 & 3 on one side of EGHF.

Then 2 + 3 = two right angles
and 1 = 3

Here's how we know:

[1] *Suppose, if possible, that 1 > 3.*

[2] Then $1 + 2 > 3 + 2$ (adding equals to unequals gives unequals)

[3] But $1 + 2 =$ two rights (Thm. 11)

[4] So two rights $> 3 + 2$ (putting together Steps 2 & 3)

[5] So AB and CD eventually meet on the side of angles 3 & 2, if AB and CD are extended far enough. (Postulate 5, as restated in Thm.11, Remark 4)

[6] But that is absurd, since AB and CD are given to us as parallel straight lines, i.e. lines that never meet no matter how far they are extended.

[7] Therefore our initial supposition, namely that *1 > 3*, is also absurd.

[8] By the same kind of argument, we can show that *1 < 3* is also absurd.

[9] Since 1 is neither greater than 3 (Step 7) nor less than 3 (Step 8), therefore $1 = 3$.

[10] But $2 + 1 =$ two right angles (Thm. 11)

[11] Thus $2 + 3 =$ two right angles (Since $1 = 3$, by Step 9)

Q.E.D.

THEOREM 25 Remarks

1. This theorem is the converse of Theorems 23 and 24.

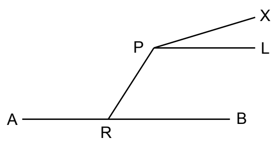
2. In this theorem, we make the first use of our 5th Postulate, namely in Step 5.

3. Notice the order of Theorems 23 & 24 & 25. In Theorems 23 & 24, we are given lines that are disposed to a transversal at certain angles, and *then* we prove that they must be parallel. In Theorem 25 we are given lines that are parallel, and then we prove that they are disposed to a transversal at certain angles. We begin by giving ourselves what we know we can have, namely two lines disposed at certain angles to a transversal; we can construct that. But we do not know how to construct parallel straight lines (or even that they exist) until *after* Theorems 23 & 24. That is why we wait until Theorem 25 to give ourselves a pair of parallels, and ask what has to be true about them.

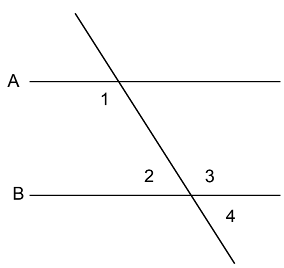
4. We have just proved that if AB & CD are parallel, then ∠2 + ∠3 must add up to 180°. That means that if ∠2 + ∠3 do *not* add up to 180° (but add up to something more or less than that), then AB & CD are *not* parallel.

If they add up to less than 180°, then they meet out toward the right side, which is what the fifth postulate says. If they add up to more than 180°, then they will meet on the *other* side of GH, toward the left, since the angles on *that* side will then add up to less than 180°.

THEOREM 25 Questions

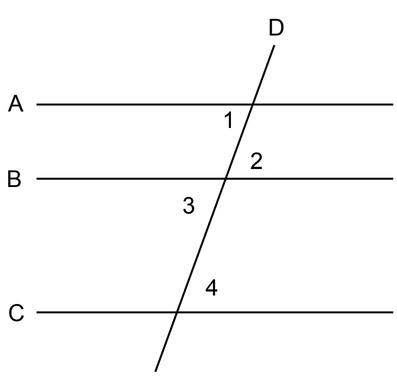


1. Using Theorem 25, prove that only *one* line through a point P can be parallel to any given line AB. Start by assuming the possibility of having *two* parallels to AB drawn through the same point P (call the other one PX), and join P to any random point R along AB, making PR a transversal.



2. If A is parallel to B, and ∠4 = 35°, then how many degrees is ∠1?

THEOREM 26: *Straight lines parallel to the same straight line are parallel to each other.*



Should you require proof for this, let A, B, C be three straight lines such that

A is parallel to B

and B is parallel to C

Then it must also be true that

A is parallel to C.

To prove it:

[1] Cut all three straight lines with another straight line D; in doing so you form angles 1, 2, 3, 4.

[2]	$\angle 1 = \angle 2$	(since A is parallel to B; Thm. 25)
[3]	$\angle 2 = \angle 3$	(vertical angles are equal; Thm. 13)
[4]	$\angle 1 = \angle 3$	(putting Steps 2 & 3 together)
[5]	$\angle 3 = \angle 4$	(since B is parallel to C; Thm. 25)
[6]	$\angle 1 = \angle 4$	(putting Steps 4 & 5 together)
[7]	A is parallel to C	(since $\angle 1 = \angle 4$; Thm. 23)

Q.E.D.

THEOREM 26 Remarks

1. Parallelism, according to Theorem 26, is similar to the relationship of equality. Just as *lines equal to the same line are equal to each other*, so too *lines parallel to the same line are parallel to each other*.

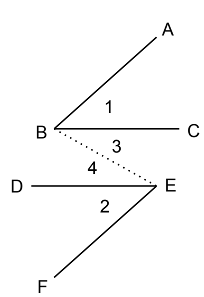
2. Notice this kind of statement does *not* apply to the relationship of perpendicularity: it is not true that *perpendiculars to the same line are perpendicular to each other*.

Some family relationships are like equality and parallelism, and others are more like perpendicularity. For example:

Brothers to the same person are brothers to each other.

On the other hand, we cannot say that

Parents to the same person are parents to each other.



3. Another theorem similar to Theorem 26 is this: *If the sides of one angle are parallel to the sides of another angle, then the two angles are either equal or supplementary*. Consider the case of $\angle ABC$ and $\angle DEF$, where $AB \parallel EF$ and $BC \parallel DE$. Join BE, and

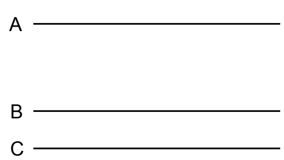
suppose that each angle lies on opposite sides of BE. It is easy to see that $\angle 1 = \angle 2$, i.e. that $\angle ABC = \angle DEF$. For:

	$1 + 3 = 4 + 2$	(since $AB \parallel EF$; Theorem 25)
and	$3 = 4$	(since $BC \parallel DE$; Theorem 25)
thus	$1 = 2$	(subtracting the equal angles 3 & 4 from each side).

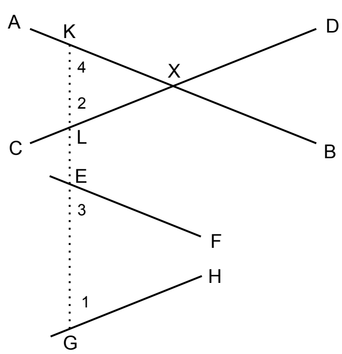
Obviously the converse of this theorem is *not* true, namely "If two angles are equal, then the sides of one are parallel to the sides of the other".

THEOREM 26 Questions

1. What true statement can we make about two straight lines that are perpendicular to the same straight line?

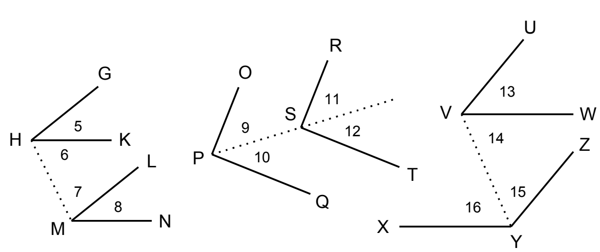


2. There is an easy case of Theorem 25. Suppose A is parallel to B, and C is parallel to B, and B is *in between* A and C. In that case, see if you can prove that A and C are also parallel, without using any prior theorems at all. (If we are instead given that $A \parallel C$ and $B \parallel C$, we need a proof such as the one we gave.)



3. Show that if two straight lines cut, then so do their parallels. Suppose $AB \parallel EF$, and $CD \parallel GH$, and suppose that AB cuts CD at a point X. Prove that GH must also cut EF.

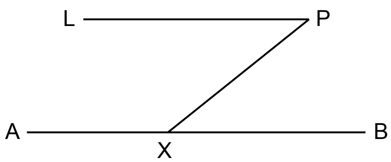
Start by joining GE and extending it. Since GE passes through E, but EF is the only parallel to AB through E, BE must cut AB at some point, say K. Likewise GE must cut CD at some point, say L. You have now formed a triangle KXL. Can you see how to proceed from there?



4. In Remark 3 above, we chose one particular case of two angles whose corresponding sides are parallel, namely the case where each angle falls on an opposite side of the line joining their vertices. See if you can prove the theorem in some of the remaining cases, and state when the angles are equal, and when they are supplementary to each

other. Consider, for example, what happens when (1) Each angle falls on the *same* side of the line joining their vertices, as ∠GHK and ∠LMN, (2) Each angle straddles the line joining their vertices, but in such a way that parallel lines are on the same side of it, as ∠OPQ and ∠RST, or (3) Two of the parallel lines lie on the same side of the line joining the vertices, but the other two parallel lines lie on opposite sides of it, as in ∠UVW and ∠XYZ.

THEOREM 27: *How to draw a line parallel to any straight line, and passing through any point.*



If I give you a straight line AB and a point P, can you draw a straight line through P parallel to AB? Absolutely.

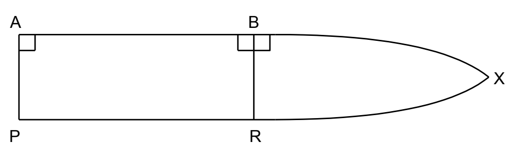
[1] Pick any point X on AB.
[2] Join PX, forming angle PXB.
[3] Draw PL so that ∠LPX = ∠PXB. (Thm. 20)
[4] That makes PL parallel to AB. (Thm. 23)

Q.E.F.

THEOREM 27 Remarks

1. In practice, for example in carpentry, this kind of construction is difficult to use with accuracy. If ∠LPX is even *slightly* unequal to ∠PXB, then LP and AB will become obviously *not* parallel fairly quickly.

THEOREM 27 Questions



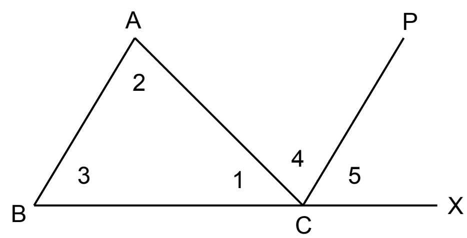
1. A more practical way to draw a line parallel to another line AB through a point P is to draw PA perpendicular to AB using a carpenter's square, go out to the other end of AB and draw BR also perpendicular to AB (and make it equal to AP in length), and then join PR.

Prove that PR is parallel to AB by first assuming the opposite, namely that AB and PR *do* meet, say out to the right at some point X. Use Angle-Angle-Side to show that the two resulting triangles AXP and BXR must be equal – and say why that is impossible.

2. Now prove that quadrilateral ABRP is a rectangle.

THE TRIANGULAR ANGLE-SUM THEOREM

THEOREM 28: *In any triangle the three angles added together equal two right angles, and the exterior angle equals the two interior and opposite angles added together*.



Take any triangle you want, such as ABC, having angles 1, 2, 3. Extend any side such as BC to any point X forming exterior angle ACX.

I say that

(a) ACX = 2 + 3

(b) 1 + 2 + 3 = two right angles

[1] Draw CP parallel to BA (Thm. 27), thus dividing exterior angle ACX into the two angles 4 and 5.

[2] 2 = 4 (since BA is parallel to CP; Thm. 25)

[3] 3 = 5 (since BA is parallel to CP; Thm. 25)

[4] 2 + 3 = 4 + 5 (putting together Steps 2 & 3)
Thus ACX = 2 + 3.

[5] 1 + 2 + 3 = 1 + 4 + 5 (equals added to equals make equals)

[6] 1 + 4 + 5 = two right angles (BCX is a straight line; Thm. 11)

[7] 1 + 2 + 3 = two right angles (putting together Steps 5 & 6)

Q.E.D.

THEOREM 28 Remarks

1. This theorem is actually quite surprising, if you think about it. Triangles can be *wildly* different shapes from each other, and yet the three angles in any of them will always add up to the same sum!

2. Recall Theorem 14, which says that ∠ACX has to be *greater than* ∠2 and also *greater than* ∠3. Now we see, more precisely, that it is greater than either because it is equal to the sum of both of them.

3. A <u>corollary</u> of this theorem, meaning "a side-result we get for free" is that *Any two angles of a triangle add up to LESS than 180°*, because only all three together add up to 180°. This corollary is the exact converse of the 5th Postulate.

Corollary: In a triangle, 2 angles add up to less than 180°

5th Postulate: 2 angles (on one side of a transversal) that add up to less than 180° are angles in a triangle, i.e. the two outside lines forming the angles must eventually meet and form a triangle.

4. Another corollary to the theorem is this: *In any right triangle, the right angle is the greatest angle*. That has to be, since if there were another angle equal to or greater than the right angle, the sum of the three angles in the triangle would be *greater than* 180°. Also, since the side opposite the greatest angle is the greatest, it follows that in any right triangle the hypotenuse is the greatest side.

Yet another corollary is this: *In any obtuse triangle, the obtuse angle is the greatest*. The same reason applies, and obviously the side opposite the obtuse angle must be the greatest.

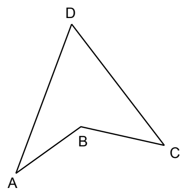
5. It also follows from this theorem that *In any right triangle, the angles other than the right angle are complementary to one another*. Since the right angle plus the other two equals two right angles, therefore the other two together equal one right angle, i.e. they are complements of each other.

THEOREM 28 Questions

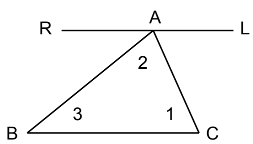
1. Draw any triangle on a piece of paper and label its angles 1, 2, 3. Next cut out the triangle with a pair of scissors. Now tear off each angle of the triangle and line up the three angles so that they all come to a single point. Do the three of them add up to a

straight line? See if you can come up with two reasons why this procedure does not constitute a proof that *the angles in every triangle add up to exactly to two right angles*.

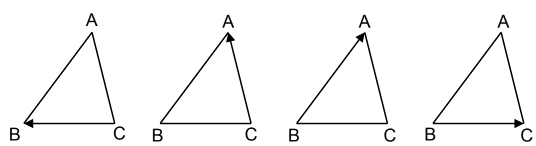
2. Using Theorem 28, you should be able to find the angle-sum of *any* rectilineal figure, by cutting it into triangles and adding up the angles. Start with a quadrilateral: what is the angle-sum of any quadrilateral? What is the angle-sum of any pentagon? Of any hexagon? Can you find a general rule, so that given the number of sides in a convex polygon, you can state the sum of its angles?



3. What if one or more of the sides of a polygon is "dented in"? For example, take quadrilateral ABCD, where $\angle ABC$ is "dented in". Does that affect the sum of the angles in the figure?



4. Prove Theorem 28 by drawing a straight line RAL through point A parallel to the base BC.



5. Place any straight edge (such as a pencil) along BC, then rotate it through $\angle BCA$; from CA, next rotate the straight edge through $\angle CAB$; from AB, finally rotate the straight edge through $\angle ABC$. You have rotated the straight edge through the 3 angles of the triangle. Can you see that the straight edge has turned through exactly one half a full rotation, namely 180°?

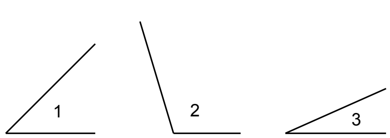
6. Using Theorem 17, prove that every angle in an equilateral triangle is acute. Prove, further, that every angle in an equilateral triangle is 60°.

Can a scalene triangle be right? Acute? Obtuse?

Can an isosceles triangle be right? Acute? Obtuse?

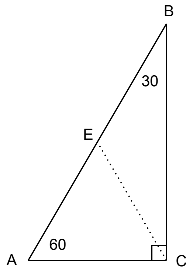
7. Armed with Theorem 28, go back to Theorem 1 Question 2 and prove that the figure is a rhombus.

8. In Theorem 19, we showed how to make a triangle given any three straight lines (as long as any two of them summed up to more than the third line). Can we make a triangle with any three angles? Yes, as long as the three of them add up to two right angles.

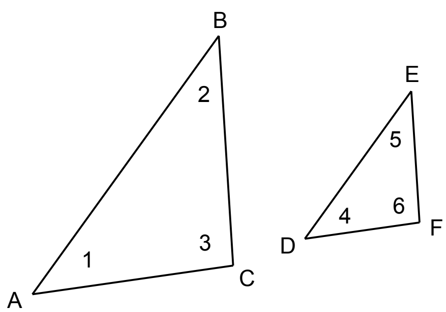


Suppose $\angle 1$, $\angle 2$, $\angle 3$ add up to 180°. Given a straight line AB, can you make a triangle, using AB as one of the sides, whose three angles are $\angle 1$, $\angle 2$, and $\angle 3$?

9. Since we now know that the angles of a triangle add up to two right angles, or 180°, it is possible to make a triangle whose angles are 30°, 60°, and 90°, since these add up to 180°. Suppose triangle ABC is just such a triangle. Prove that in such a triangle, AB is double the length of AC, and furthermore that this triangle is actually the left half of an equilateral triangle. Start by making an equilateral triangle on AC; since the angles of an equilateral triangle each equal 60°, therefore ΔAEC will sit right inside angle BAC. And since AE = AC, and AC < AB (since AB is the hypotenuse and thus the longest side of triangle ABC), AE < AB, too, and thus E will land somewhere between A and B. See if you can prove that it lands exactly in the middle of AB, and you will be nearly done. (Hint: prove EC = EB first, using Thm.3.)



THEOREM 29: *If two triangles have two angles equal to two angles, then the remaining angle is equal to the remaining angle.*



Picture two triangles ABC & DEF which are such that

1 = 4
2 = 5

Then it must also be true that

3 = 6

Why? Well, because

[1] 1 + 2 + 3 = two rights (Thm. 28)

[2] 1 = 4
2 = 5 (the angles are given that way)

[3] 4 + 5 + 3 = two rights (putting together Steps 1 & 2)

[4] 4 + 5 + 6 = two rights (Thm. 28)

[5] 4 + 5 + 3 = 4 + 5 + 6 (putting together Steps 3 & 4)

[6] 3 = 6 (subtracting 4 + 5 from both sides)

Q.E.D.

THEOREM 29 Remarks

1. Just by applying Thm. 28, we see that given any two angles of a triangle, the third can be determined. In fact, even if we do not know the two angles of a triangle separately, but we know their sum, then we also know what the remaining angle must be.

For example, if $\angle 1 = 30°$, and $\angle 2 = 50°$, then since they add up to 80°, the remaining angle, $\angle 3$, must be 100° in order for all three of the angles to add up to 180° (as Thm. 28 says they must).

And if $\angle 1 + \angle 2 = 121°$, then even if we don't know their values separately, we still know that $\angle 3$ must be 59°, in order for all the angles to add up to 180°.

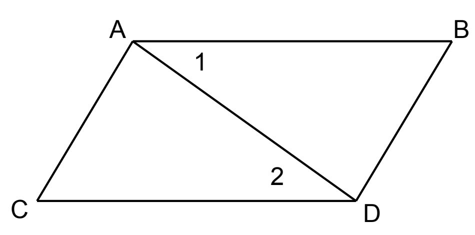
2. Notice that we cannot say the same thing about the *sides* of a triangle, namely that given any 2 of them, we can figure out the length of the third side. Suppose, for example, that Side One = 5", and Side Two = 6". What is the third side? There's no telling. Could it be 8"? Yes, since 5, 6, 8 are such that any two are greater than the third. Could it be 10"? Yes, since 5, 6, 10, are also such that any two are greater than the third. Could it be 3"? Yes, since 5, 6, 3 are also such that any two of them are greater than the third.

THEOREM 29 Questions

1. Are triangles ABC and DEF the same size? Are they the same shape?

2. If ΔABC has an angle of 35°, and another angle of 42°, then what is the value of the remaining angle?

THEOREM 30: *If one pair of opposite sides in a quadrilateral are both parallel and equal, then the quadrilateral is a parallelogram.*



Consider quadrilateral ABCD, in which AB & CD are both parallel and equal. Then ABCD is a parallelogram, meaning that AC & BD are also parallel, and what's more AC = BD.

On what grounds can we make such a claim?

[1] Join AD, forming ΔACD and ΔABD.

[2] $\angle 1 = \angle 2$ (since AB is parallel to CD; Thm. 25)

[3] AB = CD (these lines are given that way)

[4] AD is common to ΔACD & ΔABD.

[5] So, by the Side-Angle-Side Theorem, all the corresponding sides and angles of ΔACD & ΔABD are equal.

[6] $\angle CAD = \angle BDA$ (being corresponding angles of ΔACD & ΔABD)

[7] AC is parallel to BD (since $\angle CAD = \angle BDA$; Thm. 23)

[8] AC = BD (being corresponding sides of ΔACD & ΔABD)

Q.E.D.

THEOREM 30 Remark

We could also state Theorem 30 in another way: *If two straight lines are parallel and equal, then the straight lines joining their corresponding endpoints are also parallel and equal.*

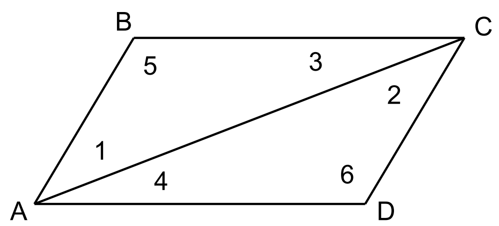
THEOREM 30 Questions

1. Construct a parallelogram on a given straight line CD, and having a given point A above CD as one of its corners.

2. Prove that every square is a parallelogram.

3. Prove that every rectangle is a parallelogram.

THEOREM 31: *In any parallelogram, the opposite sides and angles are equal, and the diagonal bisects the parallelogram.*



If ABCD is a parallelogram, then by definition AB is parallel to CD, and BC is parallel to AD. But it will also be the case that

$\angle ABC = \angle ADC$
$\angle BAD = \angle BCD$
AB = CD
BC = DA

And AC, the diagonal, will bisect the area of the parallelogram.

[1] 1 = 2 (since AB is parallel to CD; Thm. 25)

[2] 4 = 3 (since BC is parallel to AD; Thm. 25)

[3] AC is common to ΔABC & ΔADC.

[4] So, by the Angle-Side-Angle Theorem, all the corresponding sides and angles of ΔABC & ΔADC will be equal, and the triangles themselves are equal in area.

[5] So the diagonal AC cuts the parallelogram ABCD into two triangles of equal area, i.e. AC bisects the area of the parallelogram, and

[6] AB = CD (being corresponding sides of ΔABC & ΔADC)

[7] BC = DA (being corresponding sides of ΔABC & ΔADC)

[8] 5 = 6 (being corresponding angles of ΔABC & ΔADC), that is, $\angle ABC = \angle ADC$.

[9] 1 + 4 = 2 + 3 (putting together Steps 1 & 2), that is, $\angle BAD = \angle BCD$.

Q.E.D.

THEOREM 31 Remark

Theorem 31 is the partial converse of Thm. 30, which says that in a quadrilateral, if one pair of sides is both parallel and equal, then the other pair is parallel and equal. 31 says that in a quadrilateral, if both pairs of sides are parallel, then both pairs of sides are equal.

THEOREM 31 Questions

1. Prove that if ABCD is a quadrilateral, and AB = BC = CD = DA, then also AB $\parallel$ CD and AD $\parallel$ BC. (In other words, prove that every rhombus is a parallelogram.)

2. Prove that if ABCD is a quadrilateral, and $\angle$ABC = $\angle$ADC and $\angle$BAD = $\angle$BCD, then also AB $\parallel$ DC and AD $\parallel$ BC.

This, together with the solution to Q.1 above, proves the full converse of Thm. 31.

3. Prove that the diagonals of any parallelogram bisect each other.

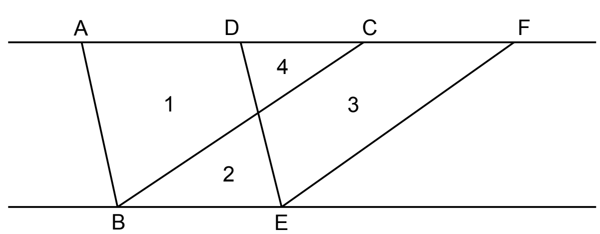
4. Prove the converse: that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

5. Prove that the diagonals of a rectangle are equal.

6. Prove the converse, that if the diagonals of a quadrilateral are equal, then the quadrilateral is a rectangle.

7. Prove that the diagonals of any equilateral quadrilateral are perpendicular to each other; then prove the converse, that if the diagonals of a quadrilateral are perpendicular to each other, the quadrilateral is equilateral (i.e. has all its sides equal to each other, and so is either a square or a rhombus).

THEOREM 32: *Parallelograms on the same base and in the same parallels have equal areas.*



Imagine two parallels AF and BE that contain two parallelograms ABED and CBEF which stand on base BE. Then ABED and CBEF have equal areas. Surprised? Well, here's proof:

[1]	AD = BE	(being opposite sides of a parallelogram)
	<u>CF = BE</u>	(being opposite sides of a parallelogram)
so	AD = CF	
[2]	AD + CD = CF + CD	(adding CD to both sides of AD = CF)
i.e.	AC = DF	

[4] AC = DF (Step 3)
AB = DE (being opposite sides of a parallelogram)
BC = EF (being opposite sides of a parallelogram)
so by the Side-Side-Side Theorem, ΔABC & ΔDEF are congruent triangles and so have the same area.

[5] 1 + 4 = 4 + 3 (simply renaming the areas of the equal triangles)

[6] 1 = 3 (subtracting area 4 from each side of Step 5)

[7] 1 + 2 = 2 + 3 (adding area 2 to each side of Step 6)

[8] That is, ABED = CBEF

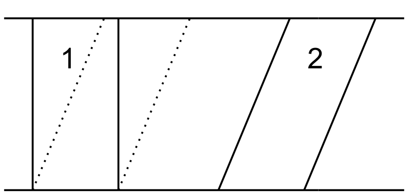
Q.E.D.

THEOREM 32 Remarks

1. Notice how parallelograms ABED & CBEF have the same area, but do not have the same shape. One cannot simply be placed right on top of the other one so as to coincide with it. This shows that things can be equal, or have the same quantity, without being exactly identical in every way, e.g. without having the same shape.

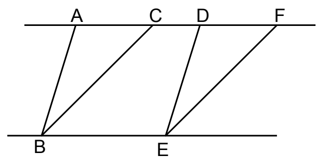
The word congruent is commonly used to name the relationship between figures that have not only the same size, but also the same shape, as we remarked after Theorem 2 (Remark 4). Congruent triangles, for example, would be those which are entirely identical, according to the theorems of Side-Angle-Side or Side-Side-Side or Angle-Side-Angle. You might say that "congruence" is a particular kind of equality, the most perfect kind of equality.

2. Theorem 32 is a bit surprising. Parallelogram CBEF could be tilted toward the right without limit, making it a wildly different shape from parallelogram ABED, and yet the two would always have exactly the same area.

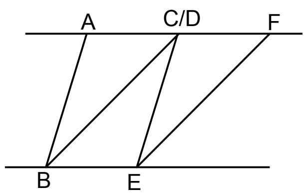


3. Notice it easily follows that parallelograms on *equal* bases and in the same parallels have equal areas. If parallelograms 1 and 2 are in the same parallels and have equal bases, just slide 2 until its base coincides with that of 1, and we are back to Thm. 32.

THEOREM 32 Questions

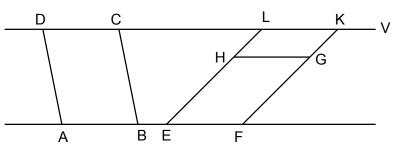


1. Prove Thm. 32 in the case where C lands between A and D.

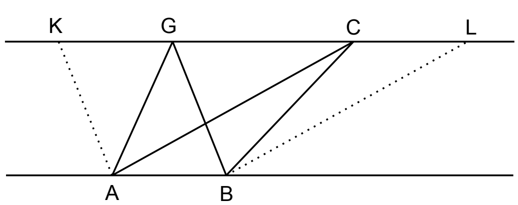


2. Prove Thm. 32 in the case where point C lands right on point D.

3. Prove the converse, namely that if parallelograms ABCD & EFGH have the same area and their equal bases AB & EF are in a straight line, then their tops DC & HG must also be in a straight line. Start by assuming DC & HG are not in a straight line, but, say, DC is higher up, and extend DC to V, and EH & FG to L & K forming a new parallelogram ELKF.



THEOREM 33: *Triangles on the same base and in the same parallels have equal areas; and a triangle on the same base and in the same parallels as a parallelogram has half the area of the parallelogram.*



Let there be two triangles, ABC and ABG, standing on base AB and inside the same set of parallels AB and GC.

Then ΔABC and ΔABG have the same area (even if they don't have the same shape). We prove it like this:

[1] Draw AK parallel to BG, making parallelogram ABGK.

[2] Draw BL parallel to AC, making parallelogram ABLC.

[3] ABGK = ABLC (Thm. 32)

[4] $\Delta ABG = \frac{1}{2}ABGK$ (Thm. 31)
$\Delta ABC = \frac{1}{2}ABLC$ (Thm. 31)

[5] $\Delta ABG = \Delta ABC$,
since by Steps 4 & 5 these two triangles are halves of two equal things, namely the parallelograms ABGK & ABLC.

[6] Also, it is evident that ΔABC is equal to half of parallelogram ABGK, since it is half its equal, namely parallelogram ABLC.

Q.E.D.

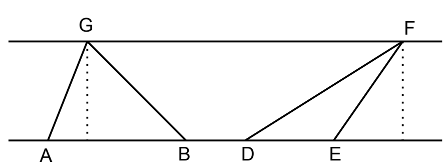
THEOREM 33 Remarks

1. Obviously, it follows that a triangle is half of any rectangle that stands in the same parallels and has an equal base.

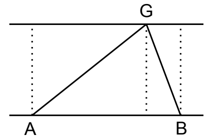
2. Notice that in Step 5 we use the principle that the halves of equal things are equal.

3. The converse for this Theorem can be proved in the same way that the converse for Theorem 32 is proved (see Q.3 after Thm. 32). And, just like with Thm. 32, the two triangles need not have the same base, but *equal* bases is enough.

4. What we call the "base" of a triangle is not any special side of it – any of the three sides can be called its "base". In this case we call the "base" the side which lies along one of the parallels.



5. Notice that since ΔABG and ΔDEF are inside the same set of parallels, the perpendicular dropped from G to the bottom parallel will be equal to the perpendicular dropped from F to the bottom parallel. For that reason, these triangles can be said to have the same height or altitude. The "height" or "altitude" of a triangle is the perpendicular drawn from its vertex to the base of the triangle. And just as a triangle can have three "bases", so too it has three "heights" (or "altitudes"), namely one in relation to each base.



6. Obviously, the perpendiculars drawn from A and B to the upper parallel will also be equal to the "height" of triangle ABG.

THEOREM 33 Questions

1. If a rectangle has Side One = 3”, and Side Two = 4”, then what is the total area of the rectangle in square inches? What operation does one perform on 3 & 4 to get the right answer? Why is that the operation performed? Draw a diagram that shows your procedure was right.

2. What if one side of a rectangle is 2 and the other is 2½? Will that change the procedure? Draw a diagram of the rectangle, and divide it into squares showing that the answer is still correct using the same procedure.

Accordingly, the formula for calculating the area of a rectangle is

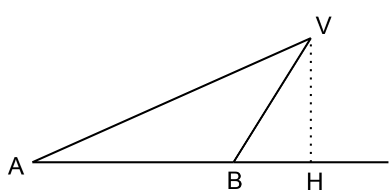
$A = bh$

Where A means the area of the rectangle, and b means the length of its base, and h means the height of the rectangle, and putting b and h right next to each other means to multiply them. The result is the number of unit square areas in the rectangle.

3. Show that the formula for finding the area of a triangle is

$A = \frac{1}{2}bh$

Where A means the area of the triangle, and b means the length of its base, and h means the height of the triangle.



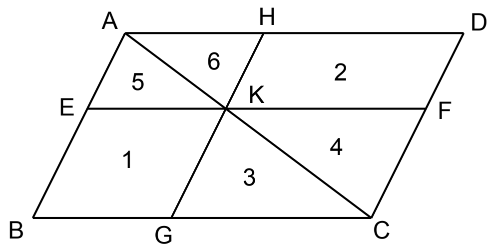
4. Look at triangle ABV. VH has been dropped perpendicularly to AB extended.

VH = 8 feet

AB = 13 feet

How many square feet of area is in triangle ABV? Just apply the second part of Thm. 33.

THEOREM 34: *In any parallelogram, the "complements" of any two parallelograms about the diagonal are equal.*



Take a parallelogram ABCD and join the diagonal AC. Pick any point K on AC, and draw

HKG parallel to AB

EKF parallel to AD

thus forming parallelograms EH and GF sharing the original diagonal as their own diagonals, and also forming two "complementary" parallelograms EG & HF (or 1 & 2). These complementary parallelograms must be equal in area, because . . .

[1] 5 = 6 (AK bisects parallelogram EH)

[2] 3 = 4 (KC bisects parallelogram GF)

[3] 5 + 3 = 6 + 4 (putting together Steps 1 & 2)

[4] 5 + 3 + 1 = 6 + 4 + 2,
since diagonal AC bisects the area of parallelogram ABCD.

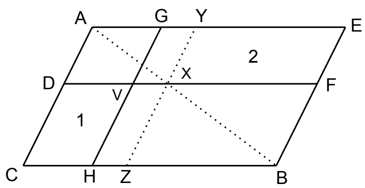
[5] 1 = 2,
subtracting the equal areas 5 + 3 and 6 + 4 from each side of the equal areas in Step 4.

Q.E.D.

THEOREM 34 Remarks

1. It is also true that if we join and extend GE, and then join and extend FH, and then extend CA, these three straight lines will all meet each other in the same point! In fact, that will still be true even if EF and HG do not pass through the same point along the diagonal AC (just as long as they are each parallel to one of the sides of the original parallelogram). We don't have the tools to prove it, yet, though.

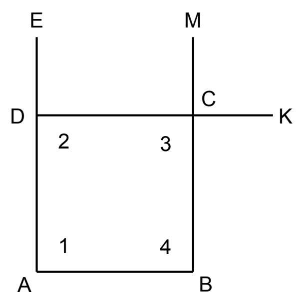
2. Notice that the little parallelograms EH and GF seem to be the same shape as the original parallelogram BD.



Now prove the converse of Thm. 34. Let 1 & 2 (DVCH and GVFE) be parallelograms of equal area drawn inside the opposite angles of the larger parallelogram ACBE by drawing DVF$\|$CB, and GVH$\|$AC. Join AB. You should be able to show that V lies on AB.

Start by assuming that V does *not* lie on diagonal AB, but instead DVF passes through AB at another point X. Then draw YXZ$\|$AC and see what happens when you apply Thm. 34.

THEOREM 35: *How to make a square.*



If I give you a straight line such as AB, can you build a square on it so that AB is one of the sides? No problem.

[1] Draw AE perpendicular to AB, and make it as long as you need (Thm. 9).

[2] Cut off AD = AB (by drawing a circle around A, with radius AB).

[3]	Draw DK parallel to AB	(Thm. 27)
	Draw BM parallel to AD	(Thm. 27)
	Call the intersection of DK and BM the point C.	
[4]	ABCD is a parallelogram	(since AB$\|$CD, AD$\|$BC)
[5]	1 + 2 = two rights	(Thm. 25)
but	1 = one right	(we made AD $\perp$ AB)
so	2 = one right	
[6]	But the angles opposite 1 & 2 are equal to them, since ABCD is a parallelogram (Thm. 31),	
so	3 = one right	(being opposite $\angle$1)
and	4 = one right	(being opposite $\angle$2)
[7]	And the sides opposite AB & AD are equal to them, since ABCD is a parallelogram (Thm. 31),	
so	DC = AB	
and	CB = AD	

[8] But AB = AD (we made them so)

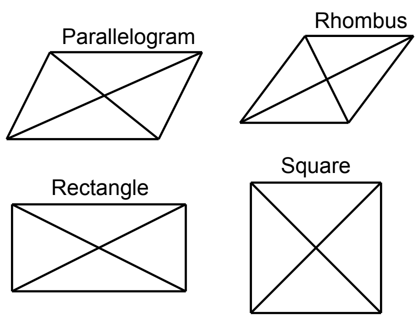
[9] So AB = AD = DC = CB (putting together Steps 7 & 8)

[10] So ABCD is equilateral (Step 9), and has all right angles (Steps 5 & 6), and so it is a square (Def. 23)

Q.E.F.

THEOREM 35 Remarks

1. We have now made two regular rectilineal figures, the equilateral triangle and the square. A regular rectilineal figure is one all of whose sides are equal and all of whose angles are equal. We have made the 3-sided regular figure, and the 4-sided; we will learn how to do the 5-sided, 6-sided, 8-sided, 10-sided, 12-sided, and 15-sided later.



2. As you know, rectangles, rhombuses, and squares are special kinds of parallelograms. Moreover, a square can be considered as a special kind of rectangle, namely one whose sides are all equal. A square can also be considered as a special kind of rhombus, namely one which has been "squared up".

Because a square is a parallelogram, its diagonals bisect each other; because a square is a rectangle, its diagonals are equal to each other; and because a square is equilateral (like a rhombus), its diagonals are perpendicular to each other.

3. Make a square on two consecutive sides of your original square, and then add a fourth square in between the two added squares. You now have one big square made out of the four little squares.

Look at one of the little squares and compare it to the big square.

How do the lengths of their sides compare? How do their areas compare? Does the area of a square increase in the same proportion as the length of its side increases?

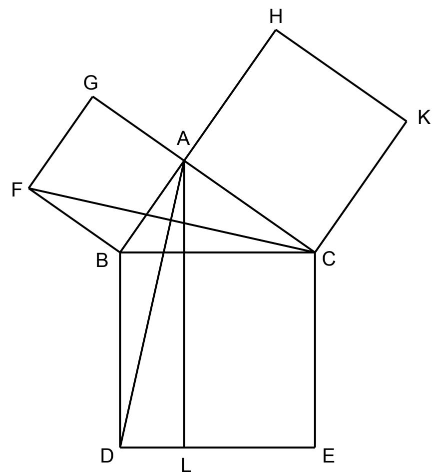
THEOREM 35 Questions

1. Using a compass and a straight edge, what is the fewest number of steps in which you can make a square on a given straight line as one of its sides? Count each circle or straight line you make as one step. See if you can find the top two corners of the square in 5 steps.

2. As we will see later, among all quadrilaterals, the square has the greatest area for a given perimeter, and conversely the least perimeter for a given area. We can say the same thing about the equilateral triangle among all triangles. Proof will come later.

THE PYTHAGOREAN THEOREM

THEOREM 36: *In any right triangle, the square on the hypotenuse is equal to the sum of the two squares on the remaining sides.*



ΔABC has a right angle at A. If we make a square on each side, as ABFG, ACKH, BCED, then the area of the square on hypotenuse BC, namely BCED, is equal to the area of the other two squares combined. Let's prove it.

[1] Drop AL perpendicular to DE, thus forming rectangles BL & LC.
Join AD, join CF.

[2] Since ∠GAB & ∠BAC are both right angles, therefore GAC is one straight line (Thm. 12).

[3]	FB = AB	(ABFG is a square)
	BC = BD	(BCED is a square)
	∠FBC = ∠ABD	(each is ∠ABC plus a right angle)
so	ΔFBC = ΔABD	(Side-Angle-Side)

[4] 2ΔFBC = 2ΔABD (the doubles of equal areas are equal)
[5] 2ΔFBC = ☐ABFG,
since this triangle and parallelogram share a common base FB and stand within the same parallels FB & GAC (Thm. 33).
[6] 2ΔABD = rectangle BL,
since this triangle and parallelogram share a common base BD and stand within the same parallels BD & AL (Thm. 33).
[7] ☐ABFG = rect. BL (putting together Steps 4, 5, 6)
[8] ☐ACKH = rect. CL (by a corresponding argument)
[9] ☐ABFG + ☐ACKH = rect. BL + rect. CL
[10] rect. BL + rect. CL = ☐BCED (both parts of a whole equal the whole)
[11] ☐ABFG + ☐ACKH = ☐BCED (putting together Steps 9 & 10)

Q.E.D.

THEOREM 36 Remarks

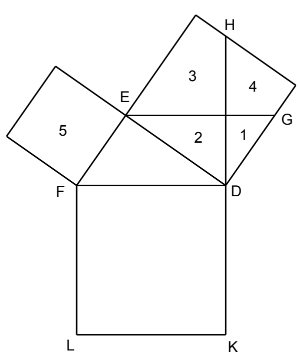
1. This is the most famous theorem in all of geometry. The first general proof of it is attributed to Pythagoras, a Greek philosopher who lived five centuries before Christ. Legend has it he was so delighted by the Theorem that he went out and sacrificed an ox in thanksgiving to the gods for so beautiful a truth.

2. In the diagram for this theorem, FC, AL, and BK all pass through one point! If you draw a careful diagram, you can verify this for yourself – but we won't burden ourselves with a proof for it right now.

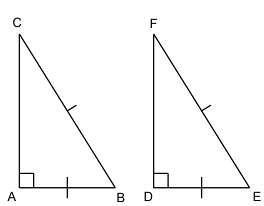
3. Numerically, the way to calculate the area of a square, as with any rectangle, is to multiply the number of unit lengths in one side by the number of those same unit lengths in an adjacent side. But since any two adjacent sides of a square are equal, it follows that we need only multiply the numerical length of one side times itself to obtain the number of unit squares in the whole square. The algebraic symbol for the product of any number n times itself is n^2. So if a is the hypotenuse of a right triangle, and b and c are its other two sides, we can express the Pythagorean Theorem this way:

$$a^2 = b^2 + c^2$$

THEOREM 36 Questions

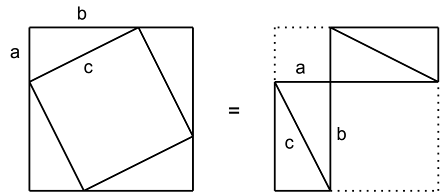


1. Illustrate the Pythagorean Theorem by making a puzzle. If DEF is a right triangle, and DF the hypotenuse, and FDKL the square on it, then extend KD to H on the larger of the two other squares, and draw EG parallel to DF. If you draw this carefully, and then cut out areas 1, 2, 3, 4, 5, you will see that they can be placed together to make a square identical to DFLK.

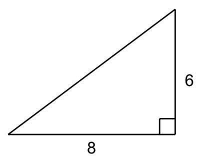


2. We can now prove another "Triangle Congruence Theorem", although it is limited to right triangles. *If we have two right triangles ABC and DEF, and two sides of one of them equal two corresponding sides in the other, then the triangles are identical.* This is true even if the corresponding sides are *not* around the right angle.

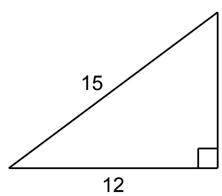
Let AB = DE, and BC = EF. Using the Pythagorean Theorem, show that these two triangles are identical.



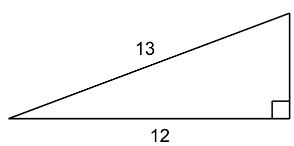
3. See the two figures at left? Each is made by taking four identical right triangles with sides a, b, c (c being the hypotenuse). Can you verify the Pythagorean Theorem by looking at these two figures? In the left figure, we see that the square on c is equal to the big square minus four triangles. In the right figure, we see that the squares on a and b together are equal to the big square minus four triangles.



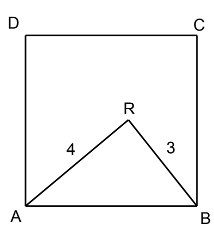
4. Given the triangle at left, with legs of 6 and 8 containing a right angle, how long is the hypotenuse?



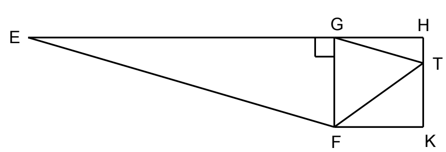
5. Given the triangle at left, with one leg of 12 and the hypotenuse of 15, how long is the other leg?



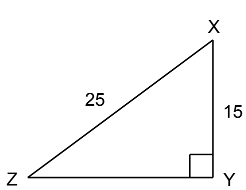
6. Given that the hypotenuse of a right triangle is 13 units long, and one leg is 12, how long is the other leg?



7. ARB is a right triangle, AB its hypotenuse. AR = 4, RB = 3, and ABCD is a square. What is the area of the figure ARBCD?

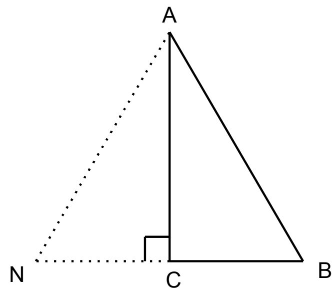


8. EFG is a right triangle, EF its hypotenuse. EF = 25, EG = 24, and GFKH is a square. What is the area of triangle GTF?



9. XY = 15, XZ = 25. What is the area of triangle XYZ?

THEOREM 37: *If the square on one side of a triangle equals the sum of the squares on the remaining sides, then the triangle is a right triangle.*



Suppose you have a triangle ABC in which ☐AB = ☐AC + ☐BC. Then the angle opposite AB is a right angle, i.e. ∠ACB is a right angle. Here's proof:

[1] Draw CN at right angles to AC, and cut off NC = BC.

[2] Join NA.

[3]	☐AN = ☐AC + ☐NC		(Pythagorean Theorem)
[4]	☐AN = ☐AC + ☐BC		(By Step 1 NC = BC, so ☐NC = ☐BC)
[5]	☐AB = ☐AC + ☐BC		(ΔABC is given this way)
[6]	☐AB = ☐AN		(Putting together Steps 4 & 5)
[7]	so	AB = AN	
[8]	but	NC = BC	(Step 1)
[9]	and	CA is common to ΔANC & ΔABC	
[10]	so	ΔANC ≅ ΔABC	(Side-Side-Side)
[11]	thus	∠ACN = ∠ACB	(Corresponding angles of ΔANC & ΔABC)
[12]	but	∠ACN is a right angle	(We made it so)
[13]	so	∠ACB is a right angle	(Putting together Steps 11 & 12)

Q.E.D.

THEOREM 37 Remarks

1. Theorem 37 is simply the converse of the Pythagorean Theorem: so not only do all right triangles have the Pythagorean property, but *only* right triangles have that property.

2. It follows that the square on the side of an acute or obtuse triangle does *not* equal the sum of the squares on the remaining two sides.

3. A triangle whose sides are 3 units long, 4 units long, and 5 units long is a right triangle, and the longest side, 5, is its hypotenuse. This fact is often used in carpentry. If you need to make a very large right angle, say at the corner of a house's foundation, then a little carpenter's square won't do. You might be off (or the carpenter's square might be off) of 90° by only a miniscule amount near the square, but by the time you extend straight lines far from the square, you might have an appreciable difference between where you end up and where a precise 90° angle would end up. You can get around this by marking lengths along a string (with knots) that are 3, 4, and 5 units long; if you

stretch them tight and form a triangle with them (there is only 1 triangle you can form with them!), then the angle between the sides of lengths 3 & 4 (feet or yards or whatever) will be a right angle.

THEOREM 37 Questions

1. Using Theorem 37, prove that a triangle having one side 3 feet long, another side 4 feet long, and the remaining side 5 feet long is a right triangle, and that the 5-foot long side is the hypotenuse.

2. Notice that 3, 4, and 5 fit the Pythagorean equation $a^2 = b^2 + c^2$,

since $5^2 = 3^2 + 4^2$

i.e. $25 = 9 + 16$.

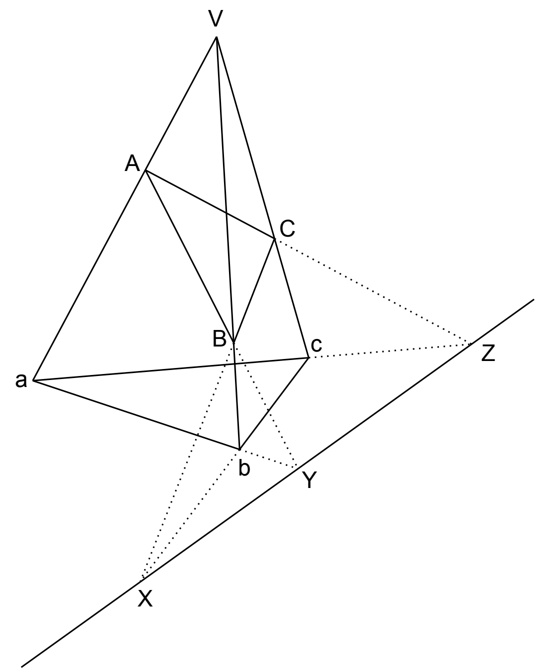
3, 4, and 5 are not the only whole numbers that do that. Any three whole numbers which do that are called "Pythagorean Triples". Here are all the Pythagorean Triples in which no member is greater than 50:

3	4	5	14	48	50
5	12	13	15	20	25
6	8	10	15	36	39
7	24	25	16	30	34
8	15	17	18	24	30
9	12	15	20	21	29
9	40	41	21	28	35
10	24	26	24	32	40
12	16	20	27	36	45
12	35	37	30	40	50

In each case, the squares of the first two numbers, when added together, equals the square of the third number. Obviously, some of these triples are based on others, e.g. 30, 40, 50, is based on 3, 4, 5. Can you see any others that are based on 3, 4, 5?

HOOK THEOREMS

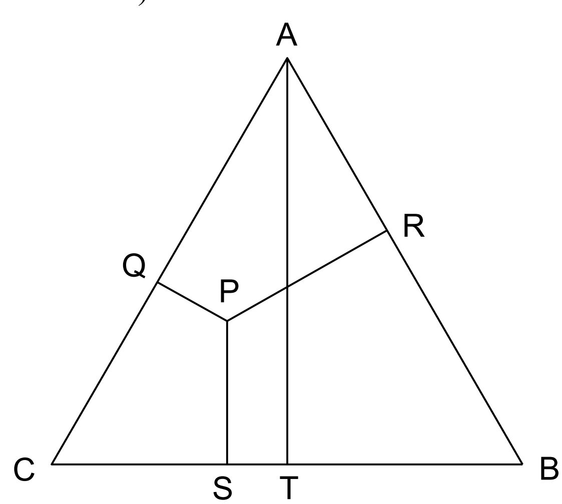
(1) DESARGUES' THEOREM.



As an inducement to further study, I will end each chapter of this book with a theorem or two, stated but left unproved. Each of these will have some bearing on the material of the chapter to which it is appended, and in some cases it will be possible to prove it by using the theorems of the chapter itself. But in most cases, it will simply be a theorem of particular interest, illustrating some of the wonders of geometry which grow out of the elements covered in this book. I will call these "HOOK THEOREMS." And as the first of these "hooks", I give you Desargues' Theorem: If two triangles, ABC, abc, in the same plane, be such that the three straight lines joining their vertices (aA, bB, cC) all intersect in one point V, then the three points of intersection of their corresponding sides (X, Y, Z) will lie in a straight line.

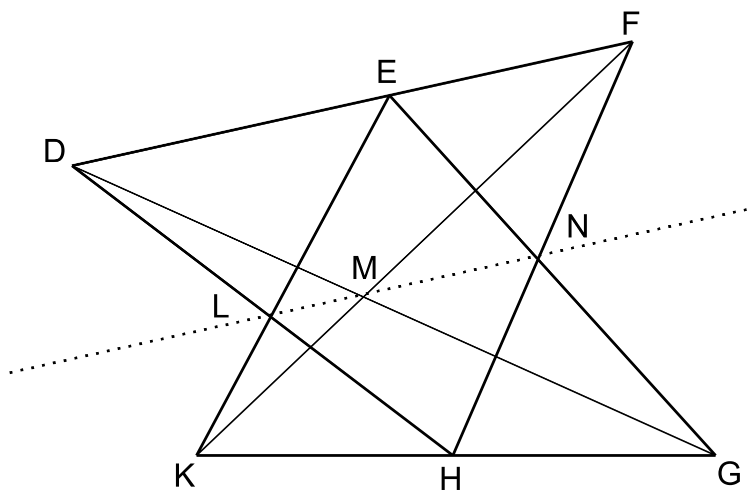
(2) SUM OF PERPENDICULARS FROM A POINT IN AN EQUILATERAL TRIANGLE.

Here is another "hook theorem." Take an equilateral triangle ABC, and pick any point P inside it. Now draw the perpendiculars from P to the three sides: PQ, PR, PS. If you add up those three lengths, their sum is equal to AT, the altitude of the triangle (the perpendicular drawn from A to BC).



(3) PAPPUS' THEOREM.

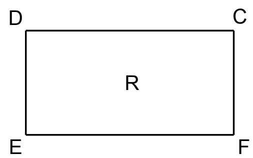
Here is a third hook. If DF, KG be any two straight lines in a plane, and E, H be any points on them, and we form three "X" figures by joining DH & EK (intersecting at L), EG & FH (intersecting at N), DG & FK (intersecting at M), then L, M, N will lie in a straight line.



Chapter Two

Squares
and
Rectangles

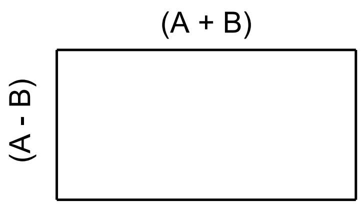
DEFINITIONS



1. A rectangle is said to be CONTAINED BY any two of its adjacent sides, and any two straight lines are said TO CONTAIN a rectangle made with adjacent sides equal to them.

For example, rectangle R *is contained by* DE *and* EF.

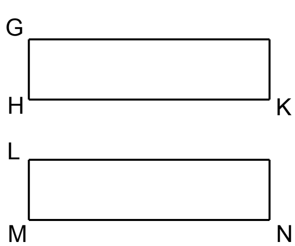
And if A *and* B *are two straight lines equal to* DE *and* EF *respectively*, *then* R *is the rectangle which* A *and* B *contain*.



2. This is not so much a definition, but more an introduction of a way of to symbolize rectangles. Up till now, we have designated a rectangle such as R by its four corners, calling it "Rectangle DEFC". But since every rectangle is uniquely determined by the lengths of any two of its adjacent sides, we can also use these to identify a rectangle.

For example, we can designate DEFC *by writing* DE · EF, *which means "The rectangle contained by* DE *and* EF.*" Likewise,* A·B *means "The rectangle contained by lines* A *and* B.*"*

If we wish to name the side of any rectangle as the sum of or difference between two straight lines, then we simply place parentheses around the sum or difference. For example, (A + B)(A – B) *means "The rectangle contained by* (A + B) *as one side, and by* (A – B) *as the adjacent side."*

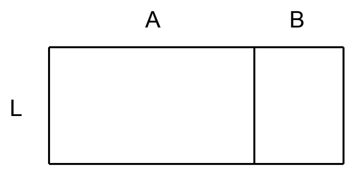


3. IDENTICAL RECTANGLES are those contained by equal corresponding sides. We can also call such rectangles CONGRUENT.

For example, if GH = LM *and* HK = MN, *then* GH · HK *and* LM · MN *are identical rectangles, having the same shape and area.*

THEOREMS

THEOREM 1: *The rectangle contained by any line* L *and the sum of two other lines* (A + B) *is equal to the sum of the rectangles contained by* L *and* A *and by* L *and* B.



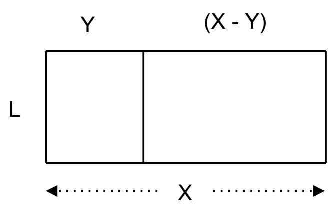
That is to say $L\ (A + B) = L \cdot A + L \cdot B$

This is obvious just from the diagram.

Q.E.D.

THEOREM 1 Remarks

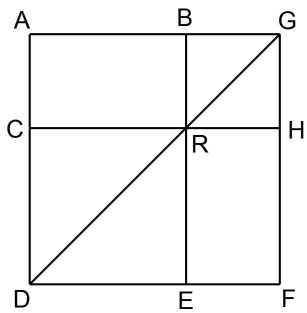
1. This simple theorem is not limited to just two lines. If we had a third line C, then it is likewise true that the rectangle contained by L and (A + B + C) is equal to the sum of the rectangles $L \cdot A + L \cdot B + L \cdot C$.



2. This theorem is obviously true for subtraction as well, i.e. $L\ (X - Y) = L \cdot X - L \cdot Y$

3. This theorem is a cousin to a very similar theorem about numbers, since the number of square units of area in a rectangle is found by multiplying the numerical measure of the sides. The theorem about numbers says *Any number* n *multiplied by the sum of any two numbers* (m + p) *is equal to the sum of the products* n × m *and* n × p. As with the rectangles, there can be more than two numbers in the sum. Thus if we add a third number, *r*, it will be true that $n\,(m + p + r) = n \times m + n \times p + n \times r$.

THEOREM 2: *If through any point on the diagonal of a square two lines are drawn parallel to each side of the square, then the four resulting figures inside are two squares along the original diagonal and two identical complementary rectangles.*



Take any square AGFD, and choose any random point R along its diagonal DG. Now draw BRE parallel to AD, and CRH parallel to DEF. I say that CRED and BGHR are squares, and that ABRC and RHFE are identical rectangles.

[1] Look at △DAG and △DFG. Each is evidently an isosceles right triangle, and because CR is parallel to AG, △DCR is clearly an isosceles right triangle. Again, because ER is parallel to FG, △DER is clearly an isosceles right triangle. Thus CRED is evidently a square. For the same reasons, BGHR is also a square.

[2] Now CAB is a right angle, and AB ∥ CR and AC ∥ BR, so it is clear that ABRC is a rectangle. Likewise RHFE is a rectangle. Moreover, since CR = RE (being sides of square CRED) and BR = RH (being sides of square BGHR), therefore ABRC and RHFE are identical rectangles.

Q.E.D.

THEOREM 2 Questions

1. What figures do we get if DR = RG?

2. What figures do we get if BE and CH are drawn parallel to the sides of the square, but they do not intersect along either diagonal of the square?

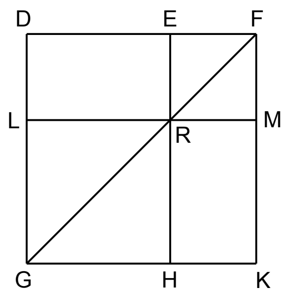
3. What if AGFD were a rectangle? What four figures would we get? By drawing two lines parallel to its sides, could we ever get a square? Could we ever get two squares? Four squares?

4. What if AGFD were a rhombus? What four figures would we get?

5. Prove the converse, i.e. prove that if two little squares situated in opposite corners of a big square touch at their corners, their common corner lies on the diagonal of the original square (and the other two figures are identical rectangles).

6. Prove another form of the converse, i.e. prove that if two identical rectangles in opposite corners of a square touch at their corners, that common corner lies on the diagonal of the original square (and the other two figures are squares).

THEOREM 3: *Given any two lines, the square on their sum is equal to the sum of the squares on each plus twice the rectangle they contain.*



This theorem, while not fascinating in itself, is useful for better upcoming theorems, and has the virtue of being easily proved. Start with a figure like that in Theorem 2, namely a big square DFKG, its diagonal GF, any point R on it, and LRM ∥ DF, and ERH ∥ FK.

[1] A brief examination of the figure reveals that
☐DF = ☐LR + ☐EF + 2 DE · ER

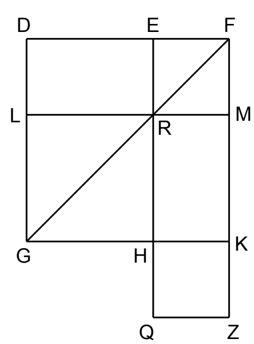
[2] Now since LR = DE,
and since ER = EF,
we can substitute these lengths in the Step 1 equality, and say that
☐DF = ☐DE + ☐EF + 2 DE · EF

[3] Which is to say that given any two lines DE and EF, the square on their sum (☐DF) is equal to the sum of the squares on them (☐DE + ☐EF) plus twice the rectangle they contain (2 DE · EF).

Q.E.D.

THEOREM 3 Remarks

1. This theorem has a cousin theorem about numbers, namely that *The square of the sum of two numbers is equal to the sum of the squares of each number plus twice their product*. That is, if n and m are our numbers, then $(n + m)^2 = n^2 + m^2 + 2nm$.



2. This theorem was about the square on the *sum* of two lines. Something similar is true about the square on the *difference* between two lines. Let DF and FE be the given lines, so that ☐DE is the square on their difference. Now add a square HKZQ onto the figure. Then obviously

☐LRHG + 2 DF· FM = ☐DFKG + ☐HKZQ,

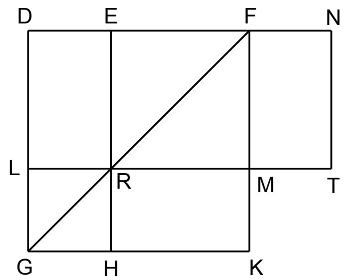
or ☐LRHG = ☐DFKG + ☐HKZQ – 2 DF· FM

(subtracting 2DF·FM). Renaming these quadrilaterals by their sides (or lines equal to their sides), we get

☐DE = ☐DF + ☐EF – 2 DF· FE.

In other words, given any two lines DF and FE, the square on their difference (☐DE) is equal to the sum of the squares on them (☐DF + ☐EF) minus twice the rectangle they contain (2 DF · FE). *Q.E.D.*

THEOREM 4: *The difference between any two squares equals the rectangle contained by the sum and the difference of their sides.*



Like Theorem 3, this theorem is not so much interesting now as useful later. But it's short! Lay out the same figure as in Theorem 3, but this time tack on rectangle FNTM, identical to DERL, or DE · ER.

[1] Inspecting the diagram verifies that

☐DF – ☐LR = ☐EF + 2 DE · ER

but <u>DN · NT = ☐EF + 2 DE · ER</u>

so ☐DF – ☐LR = DN · NT

[2] Since LR = DE

and DN = (DF + DE)

while NT = FM = EF = (DF – DE)

we can substitute these lengths in the equality of Step 1, and say that

☐DF – ☐DE = (DF + DE)(DF – DE)

[3] In other words, the difference between any two squares (☐DF – ☐DE) equals the rectangle contained by the sum of their sides (DF + DE) and the difference between their sides (DF – DE).

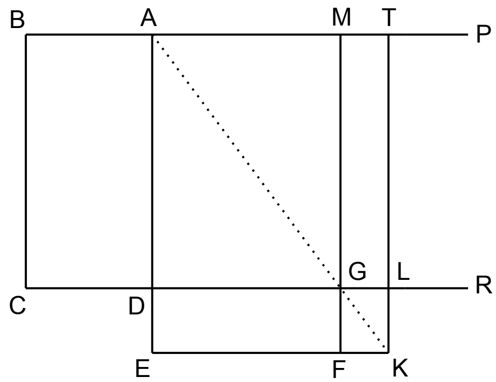
Q.E.D.

THEOREM 4 Remarks

1. The corresponding truth about numbers is that *The difference between the squares of two numbers is equal to the product of the sum of those numbers and the difference between those numbers*. That is, supposing the numbers are n and m, and n is the greater of the two, then $n^2 - m^2 = (n + m)(n - m)$

For example $7^2 - 4^2 = (7 + 4)(7 - 4)$.

THEOREM 5: *How to make a single rectangle equal to any two given rectangles.*



Suppose you have two rectangles with no length of any side common to both, and yet you would like to make one big rectangle equal to the two of them. Easy. Put their corners together as ABCD and DEFG in such a way that CD and DG are in a straight line. Extend BA out to P, and CDG out to R, as far as needed. Next,

[1] Join AG.

[2] Extend AG and EF to where they intersect at K.
[3] Extend FG up to M, and KL up to T, thus completing the large rectangle AEKT, and the two rectangles along its diagonal, AMGD and GLKF.
[4] Now, DEFG = MGLT
since these rectangles are complementary parallelograms (Ch.1, Thm. 34).

[5] And MG = AD (being opposite sides of rectangle AMGD)

[6] Therefore MGLT is equal to DEFG in area, but has one side equal to a side of rectangle ABCD. So if we slide MGLT over until it is right up against ABCD, the two of them will form one rectangle together. And the area of that rectangle will be equal to ABCD + DEFG.

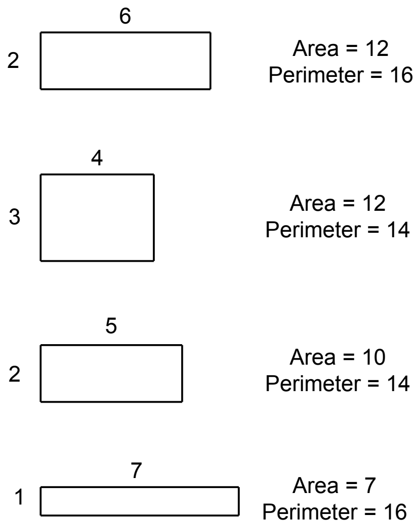
Q.E.F.

THEOREM 5 Remarks

1. Notice that we can use this construction to add together *as many rectangles as we please* into one big rectangle. Suppose, instead of just two rectangles, you had three, namely 1, 2, and 3, and you wanted to make a big rectangle equal to the three of them. Using the theorem, make a rectangle R which is equal to 1 + 2. Using the theorem again, make a rectangle T which is equal to R + 3. Since R = 1 + 2, thus T = 1 + 2 + 3.

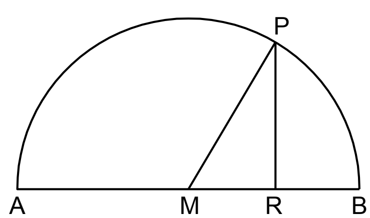
2. Notice that rectangles DEFG and MGLT have the same *area* but they do not have the same *shape*. That leads us to expect a difference in their *perimeters*, or the total length around their sides. In fact, it is quite possible for two different rectangles to contain the *same* area, but with *unequal* perimeters.

For example, consider two rectangles, one of which is 2” × 6”, the other of which is 3” × 4”. Each has an area of 12 square inches, but the 2” × 6” rectangle has a perimeter of 16 inches, whereas the 3” × 4” rectangle has a perimeter of only 14 inches.



A 2” × 5” rectangle also has a perimeter of 14 inches, but it has less area than the 3” × 4” rectangle, i.e. only 10 square inches of area. Stranger still, a 1” × 7” rectangle has a *bigger* perimeter than the 3” × 4” rectangle, namely 16 inches, and yet it has *less* area (namely 7 square inches). So if your yard is 30 yards by 40 yards, and mine is 10 yards by 70 yards, then the fence around your yard is *shorter* than the fence around my yard, and yet your yard is *bigger* than mine.

THEOREM 6: *If a perpendicular is drawn from a circle’s circumference to its diameter, the square on it equals the rectangle contained by the segments of the diameter.*



In any circle with center M and any diameter AMB, choose any point along the circumference and drop PR perpendicular to AB. I say that □PR = AR · RB.

The proof:

[1] Join PM.

[2] Now, □PR + □MR = □MP (Pythagorean Theorem)
so □PR = □MP – □MR (subtracting □MR from each side)

[3] Thus □PR = □MB – □MR (MP = MB, being radii of the circle)

[4] But □MB – □MR = (MB + MR)(MB – MR),
because of Theorem 4 above, the theorem about the difference of two squares.

[5] Thus □PR = (MB + MR)(MB – MR),
putting Steps 3 & 4 together.

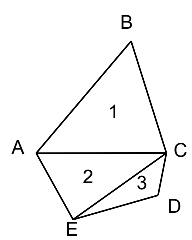
[6] i.e. $\square PR = (AM + MR)(MB - MR)$,
since $AM = MB$, both being radii of the circle.

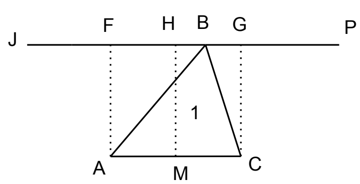
[7] But $(AM + MR)$ is just AR,
and $(MB - MR)$ is just RB.
So $(AM + MR)(MB - MR) = AR \cdot RB$.

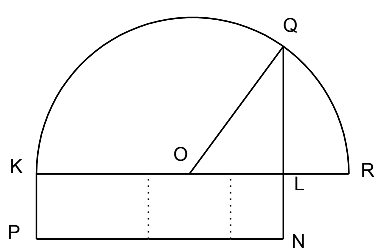
[8] Thus $\square PR = AR \cdot RB$ (putting together Steps 6 & 7)

Q.E.D.

THEOREM 7: *How to make a square that has the same area as any given rectilineal figure*.







Suppose you have a rectilineal figure ABCDE with as many sides as you want (in this case, five). How can you make a square that is equal to it in area? The general procedure is to cut it up into triangles, turn each triangle into a rectangle, turn all the rectangles into one big rectangle, and turn that big rectangle into a square.

[1] You can always divide a rectilineal figure into triangles by joining its vertices together, say into $\Delta 1$ and $\Delta 2$ and $\Delta 3$.

[2] Now make a rectangle equal to each triangle.

To do this, just take each triangle, such as $\Delta 1$, and draw a line JP through its vertex parallel to its base AC. Next, draw AF and CG perpendicular to base AC, forming rectangle ACGF. Since $\Delta 1$ and ACGF are on the same base and in the same parallels, therefore $\Delta 1$ is *half* of rectangle ACGF (Ch.1, Thm. 33).

Bisect AC at M, and draw MH perpendicular to base AC, dividing ACGF into two rectangles AFHM and HMCG. These rectangles are identical, since $AM = MC$ and since HM is a side common to both. Since they have equal areas, and since the two together make up rectangle ACGF, either one of these rectangles is *half* of rectangle ACGF.

But that means each of these rectangles is equal to $\Delta 1$, which is also half of rectangle ACGF. That is how to make a rectangle equal to a triangle. And so, equal to $\Delta 1$, $\Delta 2$, $\Delta 3$ etc., we will have rectangle 1, rectangle 2, rectangle 3 etc.

[3] Now make a rectangle equal to rectangle 1 + rectangle 2, using Theorem 5 above.
Next, make a rectangle equal to *that* rectangle plus rectangle 3, again using Theorem 5. The result is a rectangle equal to $\Delta 1 + \Delta 2 + \Delta 3$. Continuing this process as often as needed, we can make a rectangle equal to any number of triangles added together.

[4] Let KLNP be the finished product, namely a rectangle whose area is equal to
$\Delta 1 + \Delta 2 + \Delta 3$.

[5] Extend KL to R so that LR = LN.
Bisect KR at O and make a circle with center O and radius OR.
Extend NL to Q on the circle's circumference.

[6]	Now	$\square QL = KL \cdot LR$	(Ch.2, Thm.6)
[7]	But	$KL \cdot LR = \Delta 1 + \Delta 2 + \Delta 3$	(Step 4)
[8]	So	$\square QL = \Delta 1 + \Delta 2 + \Delta 3$	(putting together Steps 6 & 7)
[9]	i.e.	$\square QL = ABCDE$	(ABCDE is composed of those 3 triangles)

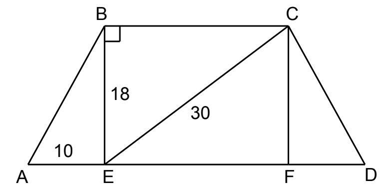
So if we build a square on QL, it will be equal in area to the figure ABCDE.

Q.E.F.

THEOREM 7 Remarks

1. This theorem obviously does not enable us to make a square equal to *any* plane figure, but only to a *rectilineal* figure. The construction requires us to resolve the given figure into triangles, which we can’t do if we are given, for example, a circle.

2. To make a square equal to a given *rectangle*, we can skip right down to Step 5 in the construction and start making our semi-circle. There is no need to divide the rectangle into triangles.



ABCD is an isosceles trapezium, which means that AB = CD and BC ‖ AD. Let BE and CF each be drawn perpendicular to BC. Supposing that

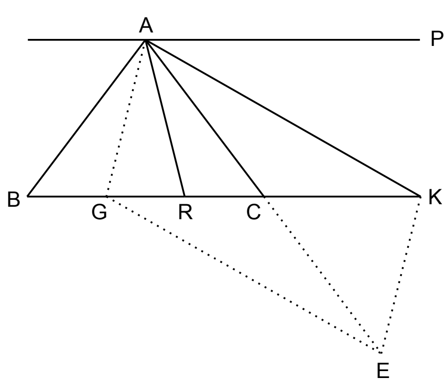
AE = 10 feet long,
BE = 18 feet long,
EC = 30 feet long,

find out how long is the side of a square with an area equal to the area of trapezium ABCD.

Start by noticing that BCEF is in fact a rectangle (which you can prove because BC ‖ AD). Then prove that ΔABE ≅ ΔDCF. Then use the Pythagorean Theorem to determine the length of BC. Having done that, you will be able to calculate the areas of all the triangles in the trapezium. Adding these together, you will have the area of the trapezium itself, and taking the square root of that area, you should be able to find the length of the side of the square equal to ABCD.

THEOREM 8: *An isosceles triangle has less perimeter than any other triangle under the same height and on an equal base.*

This theorem, along with Theorems 9 and 10, is interesting in its own right, but I include all of them here for the sake of the upcoming Theorem 11 ...



Suppose ABC is an isosceles triangle. Draw AP ‖ BC. Let ARK be any other triangle in the same parallels (slid over, if necessary, until its vertex coincides with vertex A of △ABC), and let its base RK be equal to BC – thus the two triangles have equal areas. Nonetheless, I say that the perimeter of ΔABC is less than the perimeter of ΔARK.

[1] Cut off BG = RC. Join AG. Thus it is evident that △ARG is also isosceles.

[2]

Now	GC = BC – BG	(obviously)
but	BC = RK	(given)
and	BG = RC	(we made it so)
so	GC = RK – RC	
i.e.	GC = CK	

[3] Thus if we extend AC to E so that AC = CE as well, it is obvious that AKEG is a parallelogram, since AC = CE and GC = CK.

[4]	Now	EK + AK > AE	(since AKE is a triangle)
	or	GA + AK > 2AC	(GA = EK, opp. sides of a parallelogram)
	or	GA + AK > BA + AC	(BA = AC)
	thus	RA + AK > BA + AC	(GA = RA; Step 1: △ARG is isosceles)

[5] If we now add the lines RK and BC, given as equal, to either side of this inequality, we get

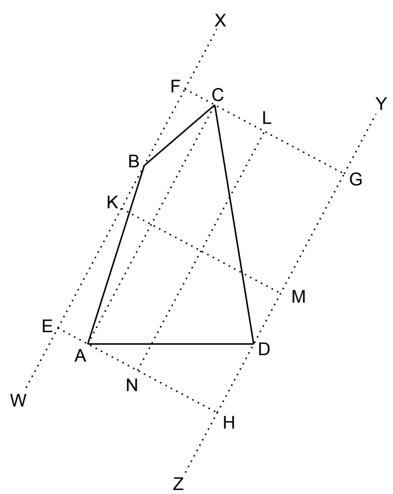
RA + AK + <u>RK</u> > BA + AC + <u>BC</u>

which is to say that the perimeter of △ARK is greater than that of △ABC.

Q.E.D.

THEOREM 9: *Given any quadrilateral which is not equilateral, how to make an equilateral quadrilateral with the same area but less perimeter.*

Suppose ABCD is a quadrilateral, but it is *not* equilateral. The following steps will make an equilateral quadrilateral equal to it, but having less perimeter.



[1] Join any diagonal, say AC.
Draw WBX and YDZ parallel to AC.
Draw FCG and EAG perpendicular to WX.
Thus EFGH is a rectangle.

[2] Bisect its sides at K, L, M, N. Join KM and join NL thus cutting EFGH into four identical rectangles, and forming at the same time the obviously equilateral quadrilateral KLMN (whose sides are not joined in the diagram, in order to avoid clutter).

[3]	Now it is evident that △AKC is isosceles, and so too △AMC.		
	Thus	△AKC = △ABC, but has less perimeter	(Ch.2, Thm.8)
	and	<u>△AMC = △ADC, but has less perimeter</u>	(Ch.2, Thm.8)
	so	quadrilateral AKCM = quadrilateral ABCD, but has less perimeter	

[4]	Again it is clear that △KLM is isosceles, and so too △KNM.		
	Thus	△KLM = △KCM, but has less perimeter	(Ch.2, Thm.8)
	and	<u>△KNM = △KAM, but has less perimeter</u>	(Ch.2, Thm.8)
	so	quadrilateral KLMN = quadrilateral AKCM, but has less perimeter	

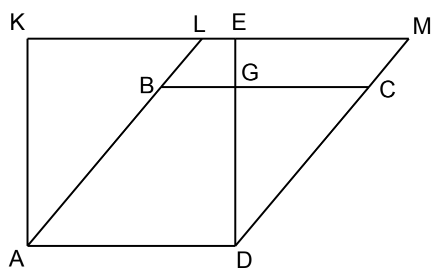
[5] Putting together Steps 3 & 4, it is clear that quadrilateral KLMN equals quadrilateral ABCD, but has less perimeter. And KLMN is equilateral.

Q.E.F.

THEOREM 9 Remark

It is of course possible that ABC should be isosceles, and thus AKC would have the same perimeter as ABC, coinciding with it, and not less perimeter. But that does not affect the proof, since ABCD has to have *some* unequal adjacent sides, being given as not equilateral.

THEOREM 10: *Any square of the same area as a rhombus has less perimeter than the rhombus.*



Let ABCD be a rhombus. Then the square which has the same area as it must have a lesser perimeter. To see why, simply

[1] Draw DG perpendicular to AD. Clearly DG is less than DC, the side of the rhombus, since DC is the hypotenuse in right triangle DGC. So extend DG to E so that DE = DC, the side of the rhombus, and complete square DEKA. Extend AB and DC to L and M on the uppermost parallel.

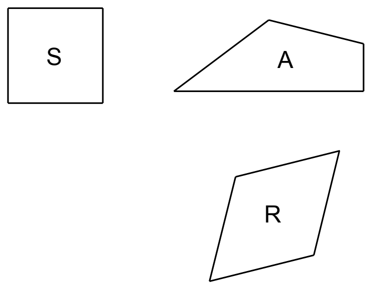
[2] DEKA is a square with the same perimeter as rhombus ABCD, because they both have four sides equal to AD.

But	DEKA = ALMD	(parallelograms on same base, under same height)
and	<u>ALMD > ABCD</u>	(whole is greater than part)
so	DEKA > ABCD	

[3] Now any square of less area than DEKA has less perimeter than it.
But the square equal to ABCD has to be of less area than DEKA (by Step 2).
Therefore the square equal to ABCD has less perimeter than DEKA,
i.e. less perimeter than ABCD.

Q.E.D.

THEOREM 11: *A square has the least perimeter of all quadrilaterals with the same area, and the greatest area of all quadrilaterals with the same perimeter.*

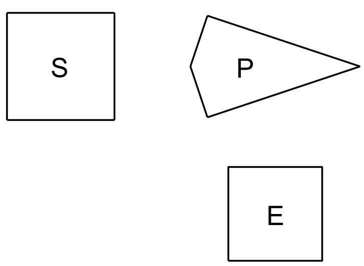


PART ONE: Suppose you have a square S and any other non-square quadrilateral A having the same area. Then the perimeter of S is less than the perimeter of A. Proof:

[1] If A is a rhombus, then it is clear that square S has less perimeter, since the two of them have equal areas (Thm.10).

[2] If A is not a rhombus, and thus is not equilateral, then make an equilateral quadrilateral R with the same area as A but less perimeter (Thm.9). If R happens to be a square, then since it has the same area as square S, it will be identical to it, and thus again it is clear that S, just like its twin R, must have less perimeter than A.

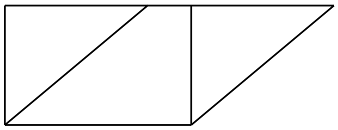
[3] If R is a mere rhombus, then it is clear that square S has less perimeter than it, since they have equal areas (Thm.10). But R was made with less perimeter than A. Therefore S has still less perimeter than A. *Q.E.D.*



PART TWO: Now suppose you have a square S and any other non-square quadrilateral P having the same perimeter. Then the area of S is greater than the area of P.

Make square E equal in area to P (Thm.7). Since square E is equal to P, it has a *lesser* perimeter than P (by Part One above). But square S has a perimeter *equal* to that of P, and thus has a greater perimeter than that of square E. But a square with a greater perimeter has a greater area. So S is greater than E, and therefore is greater than P.

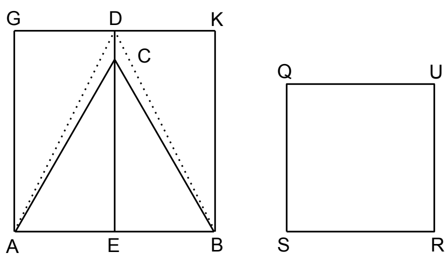
Q.E.D.



1. We have seen that a rhombus can be made with the same area as any non-equilateral quadrilateral, while using less perimeter. It is easy to see that a rectangle can be made with an area equal to any rhombus, while using less perimeter (just take the rectangle on the same base and under the same height and you will see it – remember that a hypotenuse is the longest side of any right triangle). Now in Theorem 11 we see that the square (which is equilateral *and* right angled) contains the maximum quadrilateral area for any perimeter, and also that it has the minimum perimeter for any given quadrilateral area.

It can similarly be proved (in fact, more easily), using Theorem 8, that the equilateral triangle has, among triangles, the maximum area for a given perimeter, and it also has, among triangles, the minimum perimeter for a given area.

There is a lesson here: the more uniform the rectilineal figure is, the greater an area it can encompass with a given perimeter, and the less perimeter it will need to encompass a given area.



2. But what about comparing the equilateral triangle and the square? Which one of them makes "better" use of a given perimeter? Which one holds more area for a given perimeter? Let's see. Say equilateral triangle ABC and square SQUR each have a perimeter of 12 inches.

Then SR = 3
and AB = 4.

Drop CE at right angles to AB.
Since CB is the hypotenuse in ΔCEB, thus CB > CE.
Make ED = CB, so now ED = 4.
Complete square AGKB.
Since ΔABD > ΔABC
thus 2ΔABD > 2ΔABC
so AGKB > 2ΔABC, since AGKB is equal to 2ΔABD, being a parallelogram on the same base with △ABD and under the same height. But AGKB, being a square whose side is 4 inches, has an area of 16 square inches. Therefore ΔABC has an area less than half of that, i.e. ΔABC is less than 8 square inches.

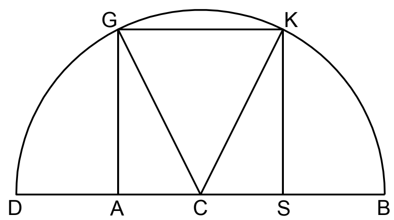
On the other hand, square SQUR has an area *greater* than 8 square inches; since its side is 3 inches, it has an area of 9 square inches.

So when a square and an equilateral triangle have the same perimeter, the square has a greater area.

There is another lesson here: the greater the number of sides in the uniform rectilineal figure, the more area it can hold with a given perimeter.

3. If among rectilineal figures those which are more uniform and have a greater number of sides can hold more area within a given perimeter, what plane figure would hold the *most* area within a given perimeter?

THEOREM 12: *How to extend a straight line so that the square on the original line is equal to the rectangle contained by the whole new line and the extension.*



Suppose you had a straight line AS, and you want to extend AS to some point B so that

$$\square AS = AB \cdot BS$$

Here's how to do it:

[1] Draw any square ASKG.
[2] Bisect AS at C.
[3] Join CG; join CK.

[4] Now, AC = CS (each is half of AS)
and AG = KS (being two sides of square ASKG)
and ∠CAG = ∠CSK (each is a right angle in square ASKG)
so the corresponding sides of ΔCAG and ΔCSK are equal. (Side-Angle-Side)

[5] So CG = CK (being corresponding sides of ΔCAG and ΔCSK)
Thus the circle around C as center and with radius CG will pass through both G and K. Draw it and call the points where it cuts AS extended "D" and "B".

[6] Since SK is perpendicular to DB, the diameter of the circle, therefore
$\square SK = DS \cdot BS$ (Ch.2, Thm. 6)

[7] But $\square SK = \square AS$ (SK = AS, being two sides of square ASKG)
So $\square AS = DS \cdot BS$

[8] Now CD = CB (being radii of the circle)
and <u>CA = CS</u> (each is half of AS)
so DA = SB (being remainders of equals with equals subtracted)
so DA + AS = AS + SB (adding the same thing, AS, to both sides)
i.e. DS = AB

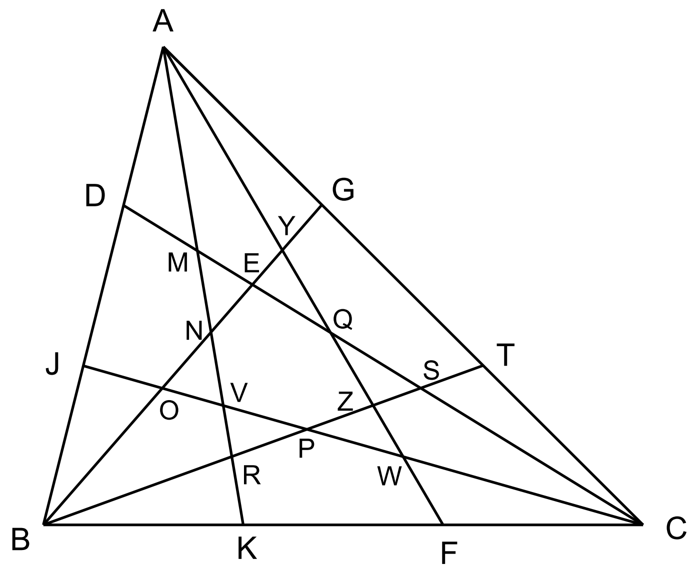
[9] But $\square AS = DS \cdot BS$ (Step 7)
so $\square AS = AB \cdot BS$ (DS = AB; Step 8)

Q.E.F.

Line ASB is cut at S in a famous ratio called "The Golden Section," which we will learn more about in Chapter 6. In the next chapter, we will need this construction to make a regular pentagon.

“HOOK”: MARION’S THEOREM.

In 1993 *Mathematics Teacher* published this interesting theorem: Take any triangle ABC, and cut each side into three equal segments. Then join each vertex to the two section-points on the opposite side. This will form a hexagonal figure in the middle. This hexagonal figure has an area exactly one tenth that of the whole triangle. Similar theorems result from cutting the opposite sides into other odd-numbered portions.

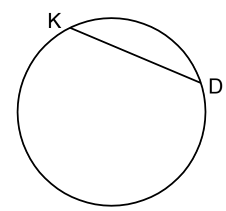


Chapter Three

Circles

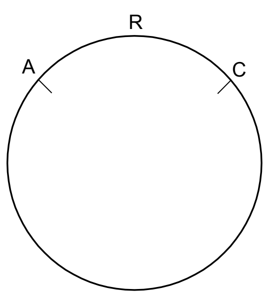
DEFINITIONS

1. EQUAL CIRCLES are those with equal radii.



2. A CHORD of a circle is any straight line joining two points on its circumference.

For example, KD *is a chord*.

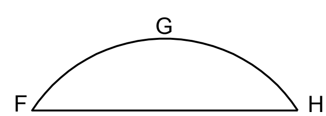


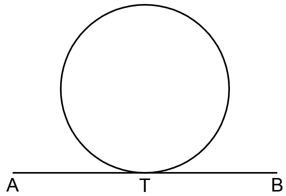
3. An ARC of a circle is any portion of its circumference.

For example, ARC *is an arc*.

4. The side of a circular arc on which lies the straight line joining its endpoints is its CONCAVE side. The other side is its CONVEX side.

For example if FH, *the straight line joining the endpoints of arc* FGH, *lies below* FGH, *then the underside of arc* FGH *is concave, and its top side is convex*.



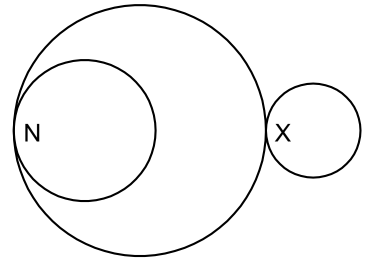


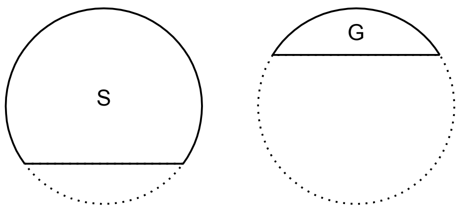
5. A STRAIGHT LINE TANGENT TO A CIRCLE is one which touches the circle at some point but does not go inside it when it is extended in either direction.

For example, ATB *is tangent to the circle at* T.

6. Two CIRCLES EXTERNALLY TANGENT TO EACH OTHER are those which touch each other at some point on their convex sides, but do not cut into each other there. A CIRCLE INTERNALLY TANGENT TO ANOTHER CIRCLE is one whose convex side touches the concave side of another circle at some point without cutting out of it there.

For example, the two circles outside each other are externally tangent at X, *and the circle inside the other circle is internally tangent to it at* N.



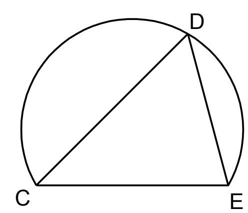


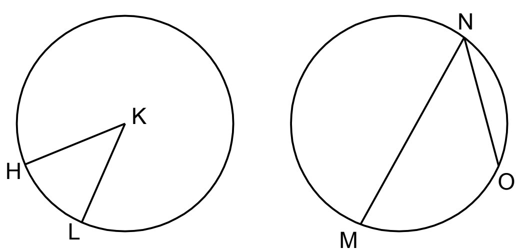
7. A SEGMENT OF A CIRCLE is a figure contained by any arc of the circle's circumference and the straight line (called the BASE of the segment) joining the endpoints of that arc.

For example, S *is a segment of a circle, and so is* G.

8. An ANGLE IN A SEGMENTis the angle joining the endpoints of a segment's base to any point on its circumference.

For example, CDE *is an angle in a segment.*



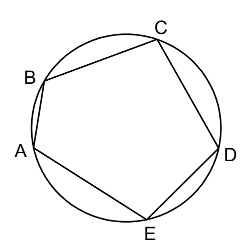


9. A rectilineal angle drawn inside a circle is said to STAND ON the arc that it cuts off, and to be AT THE CENTER if its vertex lies on the center of the circle, but AT THE CIRCUMFERENCE if its vertex lies on the circumference of the circle.

For example, angle HKL *stands on arc* HL, *and it is at the center of the circle. Angle* MNO *stands on arc* MO, *and it is at the circumference.*

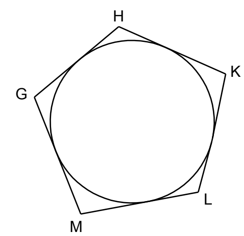
10. A rectilineal figure is said to be INSCRIBED IN A CIRCLE when the vertex of every one of its angles lies on the circle's circumference. When this happens, the circle is said to be CIRCUMSCRIBED AROUND A RECTILINEAL FIGURE.

For example, ABCDE *is inscribed in a circle, and the circle is circumscribed around it*.



11. A rectilineal figure is said to be CIRCUMSCRIBED AROUND A CIRCLE when every side of it is tangent to the circle. When this happens, the circle is also said to be INSCRIBED IN A RECTILINEAL FIGURE.

For example, GHKLM *is circumscribed around a circle, and a circle is inscribed in it*.

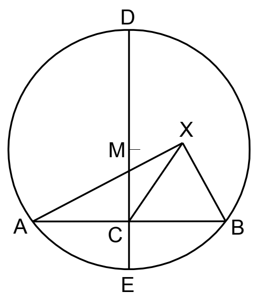


12. A REGULAR POLYGON is one all of whose sides are equal and all of whose angles are equal.

For example, a square, if we call it a polygon (often the name "polygon" is reserved for rectilineal figures having more than four sides), is a regular polygon, since all four of its sides are equal AND all four of its angles are equal. On the other hand, a rhombus is not a regular figure, since not all of its angles are equal, although all its sides are equal. Likewise a rectangle, although its four angles are all equal, is not regular, since not all its sides are equal. A regular polygon must be both equilateral and equiangular.

THEOREMS

THEOREM 1: *How to find the center of a circle.*



Up till now, we have been *making* our own circles, and so we *started* with the center. But what if I make my own circle, and then give it to you without showing you where the center is that I used to make it? Can you find the center? Here is the way to do it:

[1] Pick any two points A and B on the circumference.

[2] Join AB, and bisect it at C.

[3] Draw CD perpendicular to AB, extend it to E, and bisect DE at M.

Then M is the center of the circle. Why? First of all, the center has to lie somewhere along DE, for if it did not, but was somewhere else like the point X, then

[4] Join XA, join XC, join XB.

[5] Now AX = XB (X is supposedly the center of the circle)
AC = CB (we bisected AB at C)
<u>CX is common to ΔACX and ΔBCX</u>
so the corresponding angles of ΔACX and ΔBCX are equal, because of the Side-Side-Side Theorem.

[6] So ∠ACX = ∠BCX (being corresponding angles of ΔACX & ΔBCX)
But these two angles are adjacent, and therefore they are both right angles.
So ∠ACX is a right angle.

[7] Thus ∠ACX = ∠ACD (since both are right angles; we made ∠ACD right)
But that is *impossible*, since ∠ACD is only part of ∠ACX.

[8] Therefore our initial assumption, namely that the center of the circle does *not* lie on the line DE, is itself impossible. So the center of the circle *does* lie on DE somewhere.

[9] Since the center of the circle is equidistant from D and E, it must lie at the midpoint of DE, that is, at M.

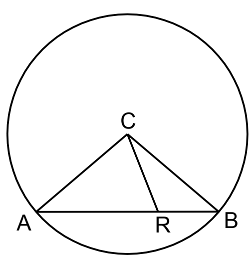
Q.E.F.

THEOREM 1 Remarks

1. From this Theorem it is plain that *Whenever a line bisects any chord of a circle at right angles, the center of the circle must lie on that line*.

2. Plainly, then, if the perpendicular bisectors of two different chords in a circle intersect each other at one point, that point must be the center.

THEOREM 2: *Any chord of a circle falls entirely inside that circle*.



Let A and B be any two points on a circle. Then every point between A and B lies inside the circle. Proof:

[1] Choose any random point R along AB. Join CR.

[2] Thus $\angle CRA > \angle CBR$ ($\angle CRB$ is exterior to $\triangle CRB$)

but $\underline{\angle CAR = \angle CBR}$ ($\triangle ACB$ is isosceles)

thus $\angle CRA > \angle CAR$

and so the sides opposite these angles in $\triangle CAR$ are unequal in the same order, that is

$CA > CR$

[3] So CA, the radius of the circle, is greater than CR. Therefore R lies inside the circle. Thus every point along AB lies inside the circle.

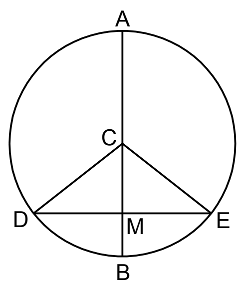
Q.E.D.

THEOREM 2 Remarks

1. This Theorem is rather obvious for *big* chords drawn inside the circle, but what about very small ones? The smaller the chord is, the closer it gets to the circumference of the circle, and the less distinguishable from the circumference it becomes. Without this Theorem, we might think that *very small chords* could actually coincide with a very small part of the circumference, or that a very small portion of a circle is no different from a straight line.

2. A corollary of this Theorem: *no straight line can have more than two points in common with the circumference of a circle.* So a straight line cannot cut the circumference of a circle more than twice.

THEOREM 3: *If a straight line through the center of a circle bisects a chord not drawn through the center, it cuts it at right angles.*



Suppose AB passes through C, the center of a circle, and it also bisects chord DE at M. Then AMB is at right angles to DME. Here's why:

[1] Join CD; join CE.

[2] Now CD = CE (they are radii)
and MD = ME (it is given that M bisects DE)
and <u>MC is common to ΔCMD and ΔCME</u>
so the corresponding angles of ΔCMD and ΔCME are equal (Side-Side-Side).

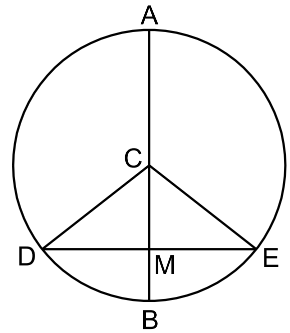
[3] Thus ∠CMD = ∠CME, being corresponding angles of ΔCMD and ΔCME.
But these two angles are adjacent; therefore they are each right angles.
Thus AMB is at right angles to DME.

Q.E.D.

THEOREM 3 Question

Why must we stipulate that the bisected line is not through the center?

THEOREM 4: *If a straight line through the center of a circle cuts a chord at right angles, it bisects it.*



Suppose AB passes through C, the center of a circle, and also cuts DE through M at right angles. Then DM = ME. Here's why:

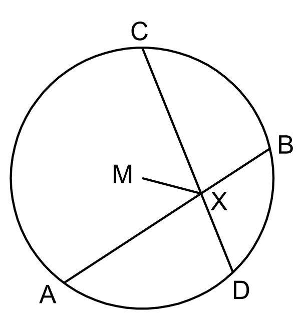
[1] Join CD; join CE.

[2] Now CD = CE (they are radii)
and ∠CDM = ∠CEM (because ΔCDE is isosceles)
and <u>∠CMD = ∠CME</u> (they are given as right angles)
so the corresponding sides of ΔCMD and ΔCME are equal, because of the Angle-Angle-Side Theorem.

[3] Thus DM = ME (being corresponding sides of ΔCMD & ΔCME)

Q.E.D.

THEOREM 5: *If two straight lines in a circle cut each other, but their intersection is not the center of the circle, then they do not bisect each other.*



Imagine that AB and CD are drawn inside a circle, and they cut each other at a point X, but X is *not* the center of the circle. Then X cannot bisect both AB *and* CD. Here's what happens if you suppose it does:

[1] Assume CX = XD
and AX = XB

[2] Find M, the center of the circle. (Thm. 1)

[3] Join MX.

[4] Now ∠CXM is a right angle, since M is the center and X is supposedly the midpoint of CD (Thm. 3).

[5] But ∠BXM is a right angle, too, since M is the center and X is supposedly the midpoint of AB (Thm. 3).

[6] So ∠CXM = ∠BXM,
that is, the whole is equal to the part, which is *impossible*.
So it is also impossible for AB and CD to bisect each other.

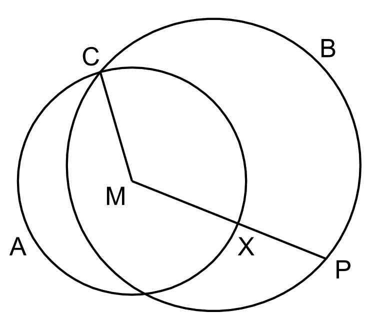
Q.E.D.

THEOREM 5 Questions

1. What if one of the lines is a diameter? Does that affect the proof?

2. Can *one* of the two lines (which cut each other not through the center) be bisected by the other?

THEOREM 6: *If two circles cut each other, they cannot have the same center.*



Conceive of two circles, circle A and circle B, that cut each other at some point C. Then they have different centers. Why?

[1] Let M be the center of circle A. Join MC.

[2] Since circles A and B cut each other, each passes into and back out of the other, so choose any point P on circle B that is *outside* circle A. Join MP.

[3] Since M is *inside* circle A (being its center), and P is *outside* circle A (that is how we chose it), thus MP must pass *out* of circle A at some point, X. Thus MX is only part of MP, and so

$MX < MP$
but $\underline{MX = MC}$ (these being radii of circle A)
so $MC < MP$

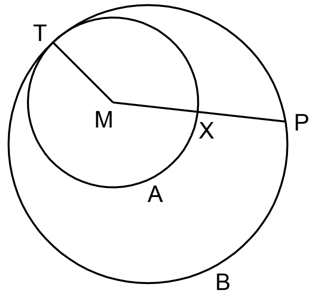
[4] Hence MC and MP, being unequal, are not radii of circle B. And thus M cannot be the center of circle B. But M is the center of circle A. Therefore circles A and B have different centers.

Q.E.D.

THEOREM 6 Remarks

The flip side of this Theorem is this: *If two circles DO have the same center, then they don't cut each other*.

THEOREM 7: *If two circles touch one another, then they cannot share the same center*.



Imagine circle A touching circle B internally at a point T. These two circles cannot possibly have the same center. Why not?

[1] Let M be the center of circle A. Join MT.

[2] Since circle A touches circle B *inside* it, then at least some of circle B must be *outside* of circle A. Choose any point P on circle B that is outside circle A. Join MP.

[3] Since M is *inside* circle A (being its center), and P is *outside* circle A (that is how we chose it), thus MP must pass *out* of circle A at some point, X. Thus MX is only part of MP, and so

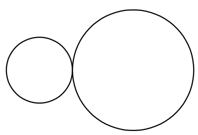
$MX < MP$
but $\underline{MX = MT}$ (these being radii of circle A)
so $MT < MP$

[4] Hence MT and MP, being unequal, are not radii of circle B. And thus M cannot be the center of circle B. But M is the center of circle A. Therefore circles A and B have different centers.

Q.E.D.

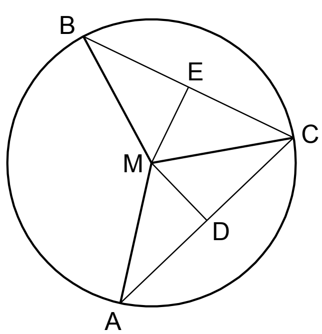
THEOREM 7 Remarks

1. Notice that the proof of the Theorem is only about *internally* tangent circles. What about *externally* tangent circles? Can they have the same center? Obviously not. If they are outside each other, and touch each other at some point but don't cut into each other there, then clearly they have different centers, because each circle's center is *inside* it, and therefore *outside* the other one. Besides, nothing prevents us from making an argument similar to that above for the case of externally tangent circles.



2. Taking Theorems 6 and 7 together, we see now that *If two distinct circles DO have the same center, then their circumferences do not have any points in common at all*. Circles which have the same center are called "concentric" circles – like the circular waves on the surface of a pond into which we have dropped a pebble.

THEOREM 8: *If three lines drawn from one point inside a circle to three points on its circumference are equal to each other, then that point is the center.*



Imagine a circle with a point M inside it, and three points along the circumference, A, B, C. If MA = MB = MC, then M has to be the center of the circle. Now the proof.

[1] Bisect BC at E; join ME.

[2] Bisect AC at D; join MD.

[3] Now AM = MC (we are given that the three original lines from M are equal)
and AD = DC (we bisected AC at D)
and <u>MD is common to ΔAMD and ΔCMD</u>
so the corresponding angles of ΔAMD and ΔCMD are equal (Side-Side-Side)

[4] Thus ∠ADM = ∠CDM, being corresponding sides of ΔAMD and ΔCMD. But these two angles are also adjacent; therefore they are right angles.

[5] So MD is at right angles to AC (Step 4) and it also bisects AC (Step 2), and therefore the center of the circle lies somewhere along MD, as we saw in the way to find a center (Ch.3, Thm.1).

[6] But likewise we can prove that the center has to lie along ME somewhere, since ME will also be at right angles to BC, and it bisects it.

[7] Therefore the center of the circle is a point common to both MD and ME. But M is the only point common to ME and MD (since two different straight lines can never have more than one point in common, because they cannot cut each other twice). Therefore M is the center of the circle.

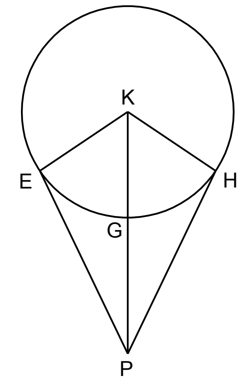
Q.E.D.

THEOREM 8 Remarks

1. Why 3 lines? If 2 lines drawn from one point to the circumference of a circle are equal, isn't that enough to conclude that the point is the center? No. Just look back at the diagram for this Theorem. AD = DC, and yet D is not the center of the circle. 3 is the minimum number of equal lines required to conclude that we have found the center, and 4 is superfluous.

It often happens in mathematics (and elsewhere in life) that *Three is enough*. We need at least 3 straight lines to make a rectilineal figure – two won't do. And things often come in threes. For example, there are three basic relationships between two comparable quantities: greater than, less than, equal to. And accordingly there are three main species of triangle (equilateral, isosceles, and scalene), and three species of angle (right, obtuse, and acute). If you are watchful for it, you will see, over and over again, that important things often come in threes.

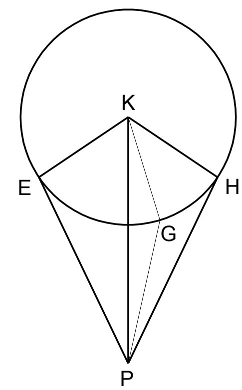
2. Can three equal lines be drawn to a circle from a point *outside* the circle? No. It will be possible to draw *two* equal lines from it to the circle, but it will be impossible to draw *three* equal lines to the circle. Assume for a moment that a point P has been chosen outside a circle, and PE = PG = PH. To see the impossibility of this, draw from center K the lines KE, KG, KH. Now, if PGK happens to be a single straight line, then



	KG = HK	(radii)
and	GP = PH	(they are assumed equal)
so	KG + GP = PH + HK	
i.e.	KP = PH + HK	

which is impossible, since PHK is a triangle, and so side KP has to be *less* than the sum of the remaining sides.

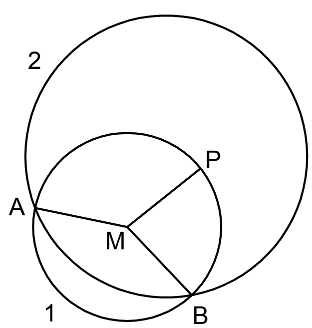
On the other hand, if PGK is *not* a straight line, then join PK. In this scenario, PGK is a triangle instead of a straight line, and



	KG = KH	(radii)
and	GP = HP	(they are assumed equal)
and	KP is common to ΔPGK and ΔPHK	
so	ΔPGK ≅ ΔPHK	(Side-Side-Side)

which is impossible, since then these two triangles would have equal areas, whereas ΔPGK is only *part* of ΔPHK.

THEOREM 9: *The circumferences of two distinct circles cannot have more than two points in common.*



Imagine two different circles, circle 1 and circle 2. The circumferences of these two circles cannot have more than two points in common. Why not?

[1] Suppose they have two points in common, namely A and B. Now choose any third point P on the circumference of circle 1.

[2] Join MA, MB, MP.

[3] Since circles 1 and 2 have common points at A and B, therefore they cannot have the same center (Thms. 6 & 7). But M is the center of circle 1. Therefore M is not the center of circle 2.

[4] Now if three equal straight lines from M fall upon the circumference of a circle, then M will be its center (Thm.8). But M is not the center of circle 2 (Step 3), and hence three equal straight lines cannot be drawn from M to the circumference of circle 2.

[5] Since MA, MB, MP are three equal straight lines from M (being radii of circle 1), hence they do not all fall on the circumference of circle 2 (Step 4). But MA and MB do fall on the circumference of circle 2 (given). Therefore MP does not.

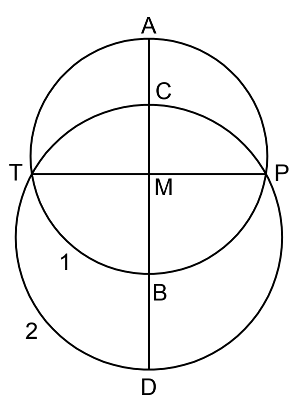
[6] Therefore any random point P on circle 1 other than A and B cannot lie on circle 2. Therefore circles 1 and 2 cannot have more points in common than A and B.

Q.E.D.

THEOREM 9 Question

So the *maximum* number of times two circles can cut each other is twice. Is it possible for two circles to cut each other only once?

THEOREM 10: *If he circumferences of two circles have two points in common, then they cut each other at those two points.*



Suppose the circumferences of circles 1 and 2 have two points in common, namely T and P. I say the circles cut each other, rather than touch each other, at these two points.

[1] Join TP and bisect it at M. Draw a perpendicular to TP at M, cutting circle 1 at A and B, and circle 2 at C and D.

[2] Since TP is a chord in both circles, therefore it lies inside both (Thm.2). Thus the circles overlap each other.

[3] Since TP is a chord in both circles, therefore its perpendicular bisector is a diameter of both circles (Thm.1). Since TM is perpendicular to these diameters, therefore the square on TM equals the rectangles contained by the segments into which M divides the diameters (Ch.2, Thm.6), i.e.

$$\begin{array}{ll} & \square TM = AM \cdot MB \\ \text{and} & \underline{\square TM = CM \cdot MD} \\ \text{thus} & AM \cdot MB = CM \cdot MD. \end{array}$$

[4] Now MB cannot equal MD, since then B and D would be the same point, and thus circles 1 and 2 would have three points in common, which is not possible (Thm.9). Therefore they are unequal. Suppose

MB < MD.

But then, since the rectangles are equal, it is necessary that

AM > CM.

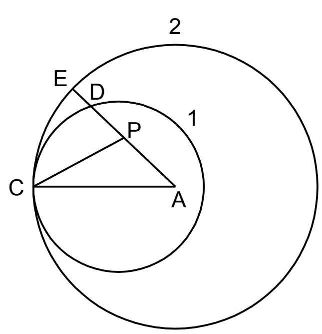
[5] Since MA > MC (Step 4), thus point A on circle 1 lies outside circle 2.
Since MB < MD (Step 4), thus point D on circle 2 lies outside circle 1.
Thus each circle falls partly outside the other.

[6] Since circles 1 and 2 partly overlap (Step 2) and each falls partly outside the other (Step 5), therefore each cuts into and out of the other. Since T and P are the only points shared by their circumferences, therefore these are the points at which they cut one another.

THEOREM 10 Remarks

A corollary to this Theorem, or another way of stating it, is that *If two circles touch one another, whether internally or externally, their circumferences have no point in common other than the one point of contact*.

THEOREM 11: *If one circle is internally tangent to another, the straight line joining their centers (when extended) passes through their point of contact.*



Circle 1 is internally tangent to circle 2 at point C. Find A, the center of circle 2, and join AC. I say that the center of circle 1 lies along AC.

[1] Inside circle 1, pick any random point P not on line AC. Join AP and extend it until it cuts circle 1 at D and circle 2 at E. Join PC.

[2]	Now	AP + PC > AC	(since APC is a triangle)
	but	<u>AC = AE</u>	(being radii of circle 2)
	so	AP + PC > AE	
	or	AP + PC > AP + PE	(AE is AP + PE)

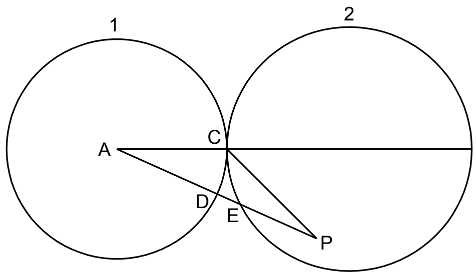
[3] So $PC > PE$ (subtracting AP from each side in Step 2)
but $\underline{PE > PD}$ (whole and part)
so $PC > PD$

[4] Thus PC and PD, being unequal, are not radii of circle 1, and therefore P is not the center of circle 1. Thus no point which is not on AC can be the center of circle 1. Therefore the center of circle 1 lies along AC.

[5] Hence the straight line joining center A and the point of contact, C, passes through the center of circle 1. And therefore the straight line joining the centers of the circles, when extended, passes through the point of contact.

Q.E.D.

THEOREM 12: *If two circles are externally tangent, the straight line joining their centers passes through their point of contact.*



Circle 1 and circle 2 are externally tangent at point C. Find A, the center of circle 1, and B, and join AC, extending it through circle 2 until it meets it again at point B. I say that the center of circle 2 lies along CB.

[1] Inside circle 2, pick any random point P not on line CB. Join AP, which must pass out of circle 1 at a point (D), and into circle 2 at another point (E).

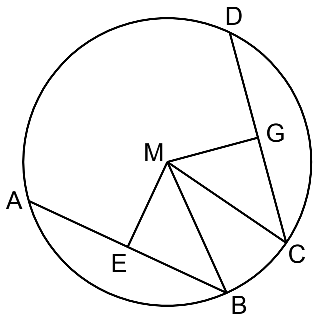
[2] Now $AC + PC > AP$ (APC is a triangle)
but $\underline{AC = AD}$ (these being radii of circle 1)
so $AD + PC > AP$
or $AD + PC > AD + DE + EP$

[3] So $PC > DE + EP$ (Subtracting AD from both sides)
Thus $PC > PE$ (Subracting DE from the lesser side)
And so, being unequal, PC and PE are not radii of circle 2, and therefore P is not the center of circle 2. Thus no point which is not on CB can be the center of circle 2. Therefore the center of circle 2 lies on CB.

[4] Thus the straight line joining center A and the point of contact, C, passes through the center of circle 2 when extended. That is, the line joining the centers passes through the point of contact.

Q.E.D.

THEOREM 13: *In any circle, equal chords are equally distant from the center.*



Imagine a circle in which two equal chords, AB and CD, have been drawn. Then the perpendiculars drawn to them from the center will be equal.

Find the center, M, and drop ME at right angles to AB, and MG at right angles to CD. I say that ME = MG. Here's the proof:

[1] First, AE = EB,
since ME is drawn from the center at right angles to AB (Ch.3, Thm.4).

[2] So EB = ½AB
So too CG = ½CD

[3] But AB = CD (given)
so EB = CG (the halves of equals are equal)

[4] Therefore □EB = □CG (squares built on equal sides are equal)

[5] So □MB – □EB = □MB – □CG,
since this is subtracting equal squares from the same square.

[6] But □MB = □MC,
since the squares on equal sides are equal, and MB = MC, since both are radii of the circle.

[7] So □MB – □EB = □MC – □CG,
putting together Steps 5 & 6.

[8] But □MB – □EB = □ME (Pythagorean Theorem)
and □MC – □CG = □MG (Pythagorean Theorem)

[9] Therefore □ME = □MG (Putting together Steps 7 & 8)

[10] So ME = MG,
since the sides of equal squares have to be equal.

So the perpendicular distances of equal chords from the center of the circle are equal.

Q.E.D.

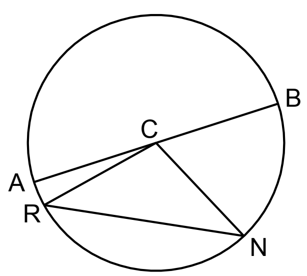
THEOREM 13 Remarks

Why do we call the perpendicular distance between a point and a line *the* distance between them? Because 1. there is only one perpendicular distance, and 2. it is the least distance.

THEOREM 13 Questions

Prove the converse, using the same diagram. Given that ME = MG, i.e. that the perpendicular distances of chords AB and CD from the center are equal, prove that the chords themselves are equal.

THEOREM 14: *The diameter is the longest chord inside a circle, and chords closer to the center are greater than those further away from it.*

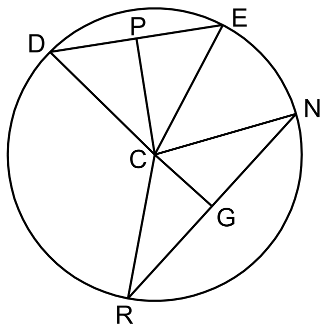


Imagine a circle with diameter AB, whose midpoint C is therefore the center of the circle. Now take any other chord RN inside the circle. I say that AB > RN. Why?

[1] Join CR; join CN.

[2]	Now	CR + CN > RN	(since RCN is a triangle)
[3]	But	CR = AC	
	and	CN = CB	(since all these are radii of the circle)
[4]	So	AC + CB > RN	(putting together Steps 2 & 3)
	i.e.	AB > RN	

Q.E.D.



That's the first part of the Theorem. For the next part, take any other chord DE, whose perpendicular distance from the center (namely CP) is *greater* than RN's perpendicular distance from the center (namely CG). I say that RN > DE. Why?

[1] Join CD; join CE.

[2] Now ☐CG < ☐CP,
since we are given that CG < CP, and a square on a lesser line is a lesser square.

[3] Now whenever we subtract a *lesser* thing from something, it leaves a *greater* remainder than if we subtract something *greater* from it. So if we subtract ☐CG from ☐CD, it leaves a *greater* remainder than if we subtract ☐CP from ☐CD. That is,

☐CD – ☐CG > ☐CD – ☐CP

[4] But ☐CD = ☐CR,
since CD = CR, being radii of the circle, and the squares on equal lines are equal.

[5] So ☐CR – ☐CG > ☐CD – ☐CP (putting together Steps 3 & 4)

[6] But ☐CR – ☐CG = ☐RG (Pythagorean Theorem)
and ☐CD – ☐CP = ☐DP (Pythagorean Theorem)

[7] So ☐RG > ☐DP (putting together Steps 5 & 6)

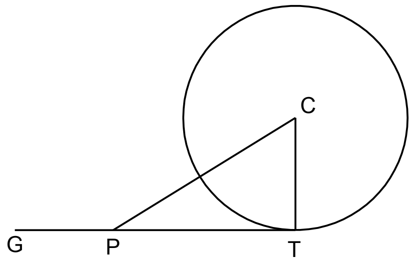
[8] i.e. RG > DP, since the side of a greater square is greater.
Thus 2RG > 2DP, since double a greater line remains greater than double the lesser line.

[9] But 2RG = RN,
since CG is drawn from the center and at right angles to RN (Ch.3, Thm.4).
Also 2DP = DE, for the same reason.

[10] So RN > DE (putting together Steps 8 & 9)

Q.E.D.

THEOREM 15: *A straight line drawn at right angles to the end of a circle's radius is tangent to the circle.*



Conceive of a circle with center C and radius CT, and TG drawn at right angles to CT.

I say that TG is tangent to the circle, i.e. that although it touches it at T it falls entirely outside the circle forever after, not cutting into the circle at all. Here's why.

[1] Choose any random point P along TG other than T. Join CP.

[2] Looking at triangle CPT, since angle CTP is right, therefore ∠CPT is acute.

[3] Hence ∠CTP > ∠CPT
thus CP > CT (opposite a greater angle is a greater side)

[4] But CT is a radius of the circle. Therefore, since CP is greater than a radius of the circle, P must lie outside the circle.

[5] But P is just a random point on TG other than T. Hence every point on TG other than T itself lies outside the circle. Therefore TG is tangent to the circle at T.

Q.E.D.

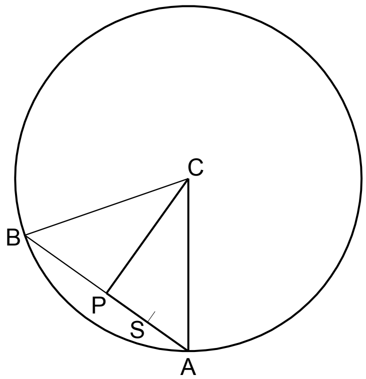
THEOREM 15 Remarks

1. Will TG cut into the circle on the *other* side, namely to the *right* of CT? No. Because ∠CTG is a right angle, therefore when we extend GT to the right, the angle adjacent to it will also be a right angle, and so the exact same proof will apply on the *right* side as well.

2. We could state the Theorem another way: a straight line drawn at right angles to a circle's radius has one and only one point in common with the circumference of the circle, namely the endpoint of the radius to which it is drawn at right angles.

3. Notice that this Theorem gives us an easy construction for a tangent to any point on the circumference of a circle: if the point is T, merely find the center of the circle (Thm.1), say C, and join CT, and then draw a straight line at right angles to CT through point T.

THEOREM 16: *Any straight line drawn at less than a right angle to the end of a circle's radius cuts into the circle.*



Conceive a circle with center C, radius CA, and a straight line AS drawn at any acute angle CAS. I say that AS cuts through the circle.

[1] Drop CP perpendicular to AS (extended, if need be).

[2] Extend AP to B so that AP = PB. Join CB.

Since	AP = PB	
and	CP is common	
and	$\underline{\angle CPA = \angle CPB}$	(both being right)
thus	$\triangle CPA \cong \triangle CPB$	(Side-Angle-Side)
hence	CB = CA	

Thus, since CB is equal to the radius CA and is drawn from the center C, CB is also a radius of the circle, and therefore B is on the circumference of the circle.

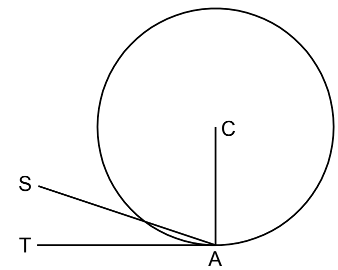
[3] But B lies along the line AS, and A is another point on AS on the circumference of the circle. Therefore line AS (extended, if need be) joins two points on the circumference of the circle, and therefore it is a chord (Thm.2).

Q.E.D.

THEOREM 16 Remarks

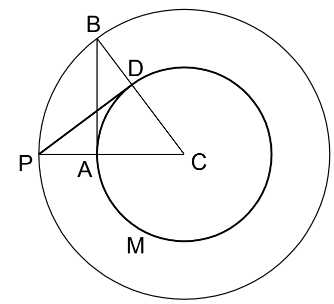
1. From this Theorem it follows that it is impossible to draw *two* tangents to a circle touching it at the same point. For each and every point on the circumference of a circle, there is one *and only one* tangent, namely the straight line drawn through it at right angles to the radius drawn to it.

2. This Theorem is truly surprising. To see why, draw the tangent AT at right angles to CA, and draw a line AS making CAS an acute angle as close to a right angle as you like. By the Theorem, AS must cut into the circle at A and out again at some other point (very close to A). Now there is a space between the circumference of the circle and the tangent AT *all the way down to the point A*. And a straight line has no thickness, and so in a way it doesn't take up any space. And yet we *still* can't find room to stick a straight line like AS in between the tangent and the circle without cutting into the circle!



THEOREM 17: *How to draw a straight line tangent to any circle from any point outside it.*

If I give you a circle M, and a point P outside it, can you draw a line from P tangent to circle M? Sure. All you have to do is …



[1] Find the center, C.

[2] Join CP, cutting circle M at A, and draw another circle around C with radius CP.

[3] Draw AB at right angles to PAC, cutting circle P at B.

[4] Join CB, cutting circle M at D.

[5] Join DP. This line DP is in fact tangent to circle M. Here's why:

[6]		AC = CD	(being radii of circle M)
	and	CB = CP	(being radii of circle P)
	and	<u>∠ACB = ∠PCD</u>	(being in fact the same angle)

so the corresponding angles of ΔACB and ΔPCD are equal (Side-Angle-Side)

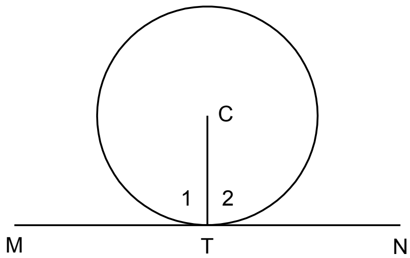
[7] So ∠CDP = ∠CAB (being corresponding angles of ΔACB & ΔPCD)

[8] Thus ∠CDP is a right angle,
since it equals ∠CAB, which we made a right angle.

[9] But CD is a radius of circle M, and so DP, drawn at right angles to CD, is tangent to circle M (Thm.15).

Q.E.F.

THEOREM 18: *If a straight line is tangent to a circle, then the radius joined to the point of tangency is perpendicular to the tangent.*



Picture a circle with center C, to which straight line TN is tangent at T. Then ∠CTN is a right angle. The proof:

[1] Extend NT to M. Since NT is tangent to the circle, therefore NTM does not cut the circle, but lies outside the circle.

[2] If ∠2 were *less* than a right angle, then TN would cut the circle at T, and so NTM would not be tangent to the circle (Thm.16). But NTM *is* tangent to the circle, and therefore ∠2 is *not* less than a right angle.

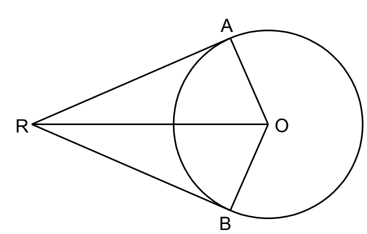
[3] If ∠2 were *more* than a right angle, then ∠1 would be *less* than a right angle (since together they add up to two rights), and so TM would cut the circle at T, and so NTM would not be tangent to the circle (Thm.16). But NTM *is* tangent to the circle, and therefore ∠2 is *not* more than a right angle.

[4] Since ∠2 is neither *less* than a right angle (Step 2), nor *more* than a right angle (Step 3), therefore it *is* a right angle. That is, ∠CTN is a right angle.

Q.E.D.

THEOREM 18 Remarks

1. We can now show that the two tangents to a circle from any one point R must be equal. Let O be the center of a circle, and join RO. Calling the tangents RA and RB, draw the lines OA and OB. Now, ∠RAO = 90°, and likewise ∠RBO = 90° (by Theorem 18). So ΔRAO and ΔRBO are both right triangles, and the Pythagorean Property applies.

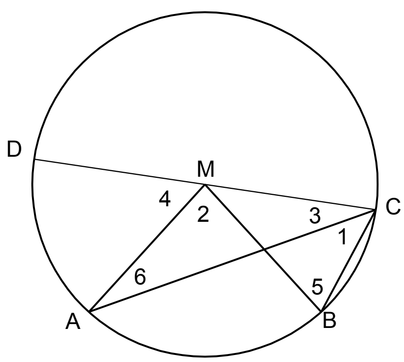


i.e.	☐RO = ☐OA + ☐RA	(ΔRAO is a right triangle)
or	☐RO = ☐OB + ☐RA	(since OA = OB, therefore ☐OA = ☐OB)
but	☐RO = ☐OB + ☐RB	(ΔRBO is a right triangle)
thus	☐RA = ☐RB	(since each is equal to ☐RO – ☐OB)
so	RA = RB.	

2. We can see from Theorem 18 that *If a straight line is drawn at right angles to a circle's tangent at the point of tangency, the line will pass through the center of the circle*. Use the same diagram as in Theorem 18. We are given that MTN is tangent at T, and we want to show that the perpendicular at T passes through the center. Find center C and join TC. Then by Theorem 18 TC is perpendicular to MTN. Therefore the line perpendicular to MTN at the point of tangency passes through the center of the circle.

3. Prove that there are only two tangents to a circle from a given point outside it.

THEOREM 19: *An angle at the center of a circle is double an angle at the circumference if they stand on the same arc.*



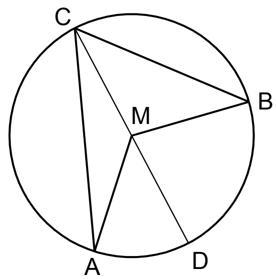
Consider a circle with center M, and an angle AMB standing on arc AB, and another angle ACB also on arc AB but having its vertex C at the circumference instead of at the center. Then, surprisingly, ∠AMB is *double* ∠ACB. To see why, first join CM and extend it to D.

Letting numbers designate angles,

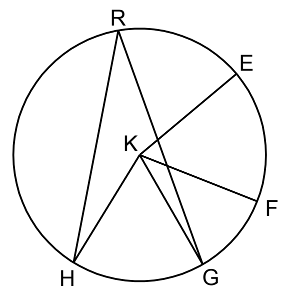
[1]		$4 + 2 = 3 + 1 + 5$	($4 + 2$ is exterior to △BMC)
	but	$\underline{5 = 3 + 1}$	(△BMC is isosceles)
	so	$4 + 2 = 3 + 1 + 3 + 1$	
[2]	and	$4 = 3 + 6$	(4 is exterior to △AMC)
	but	$\underline{6 = 3}$	(△AMC is isosceles)
	so	$4 = 3 + 3$	
[3]	thus	$(3 + 3) + 2 = 3 + 1 + 3 + 1$	(Steps 1 & 2)
	thus	$2 = 1 + 1$	(subtracting 3 from each side twice)

Hence angle 2 is double angle 1, i.e. ∠AMB is double ∠ACB.

Q.E.D.



1. Does it make any difference if CM goes *between* the legs of the angles?



2. The angle at the center and the angle at the circumference do not actually have to be on the *same* arc in order for the theorem to hold. They need only be on *equal* arcs. In a circle with center K let arc EF = arc HG. Choose a random point R on the circumference and draw RH and RG. Then it is still true that

$\angle EKF = 2\angle HRG$

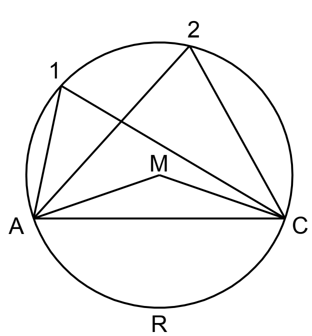
Start by drawing KH and KG. Now, since the arcs EF and HG are equal, therefore if pie-piece EKF is rotated clockwise so that E is on G, then also F will be on H. Therefore KE and KG will coincide while KF and KH also coincide. In other words, ∠EKF will coincide with ∠GKH. Therefore

$\angle EKF = \angle GKH$

but $\underline{2\angle HRG = \angle GKH}$ (Thm.19)

so $2\angle HRG = \angle EKF$

THEOREM 20: *In a circle, angles at the circumference standing on the same arc are equal.*



Imagine a circle with center M, and let ARC be any arc of its circumference. Now let ∠1 and ∠2 be any two angles at the circumference standing on arc ARC. I say ∠1 = ∠2.

Join MA and join MC to find out why.

[1] $\angle 1 = \frac{1}{2}\angle AMC$,

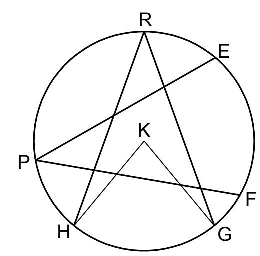
since ∠1 is at the circumference, ∠AMC is at the center, and they stand on the same arc (Thm.19).

[2] $\angle 2 = \frac{1}{2}\angle AMC$,
since $\angle 2$ is at the circumference, $\angle AMC$ is at the center, and they stand on the same arc (Thm.19).

[3] Therefore $\angle 1 = \angle 2$,
since each is half of $\angle AMC$ (Steps 1 & 2).

Q.E.D.

THEOREM 20 Remarks

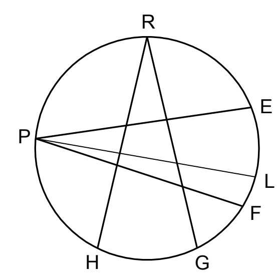


1. As with the last theorem, this Theorem does not depend on the angles at the circumference being on the *same* arc; they need only be on *equal* arcs. So if arc EF = arc HG, and P and R are points chosen on the circumference, then

$\angle EPF = \angle HRG$

Take center K, and draw KH and KG.

Now $\angle HRG = \frac{1}{2}\angle HKG$ (Thm.19)
But $\underline{\angle EPF = \frac{1}{2}\angle HKG}$ (since arc EF = arc GH; Thm.19, Remark 2)
So $\angle EPF = \angle HRG$



2. The converse of this last Remark is also true: *The arcs on which stand equal angles at the circumference are equal*. That is,

if $\angle EPF = \angle HRG$
then arc EF = arc HG.

If you doubt it, assume for a moment that the arcs are *not* equal; suppose arc EF is *greater* than arc HG. Therefore a *part* of arc EF will be equal to arc HG, say

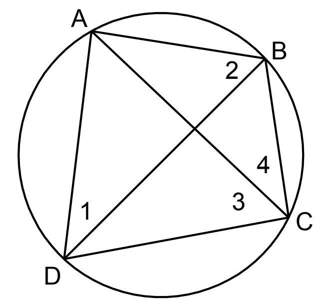
arc EL = arc HG

then $\angle EPL = \angle HRG$ (by Remark 1 just above)
but $\underline{\angle EPF = \angle HRG}$ (given)
so $\angle EPF = \angle EPL$,

which is *impossible*, since the whole never equals a part of itself. So it is impossible that arc EF should be *greater* than arc HG. Likewise HG cannot be greater than EF. Therefore the arcs are equal.

3. Theorem 20 is just as true if the two angles are not in the *same* circle, but instead are in two different but *equal* circles 1 and 2. That is, if $\angle X$ at the circumference of circle 1 stands on arc A, and $\angle Z$ at the circumference of circle 2 stands on arc B, and arc A = arc B, then also $\angle X = \angle Z$.

THEOREM 21: *The opposite angles of a quadrilateral inscribed in a circle add up to two right angles*.



Imagine a quadrilateral ABCD whose four corners lie on the circumference of a circle.
Then $\angle ABC + \angle ADC$ = two right angles
and $\angle DCB + \angle DAB$ = two right angles.
Join AC and join BD to begin the proof.

[1] First $\angle 1 = \angle 4$
(both stand on arc AB; Thm.20)

[2] And $\angle 2 = \angle 3$ (both stand on arc AD; Thm.20)

[3] So $\angle 1 + \angle 2 = \angle 3 + \angle 4$ (adding together Steps 1 & 2)

[4] But $\angle 1 + \angle 2 + \angle DAB$ = two rights (DAB is a triangle)

[5] So $\angle 3 + \angle 4 + \angle DAB$ = two rights (putting together Steps 3 & 4)

[6] i.e. $\angle DCB + \angle DAB$ = two rights ($\angle 3 + \angle 4$ *is* $\angle DCB$)

[7] But since the angles of any quadrilateral add up to four right angles (since it is made up of two triangles, each of whose angles add up to two right angles), and since $\angle DCB + \angle DAB$ equal two rights, therefore the remaining angles of quadrilateral ABCD must add up to the remaining two right angles. That is, $\angle ABC + \angle ADC$ = two rights.

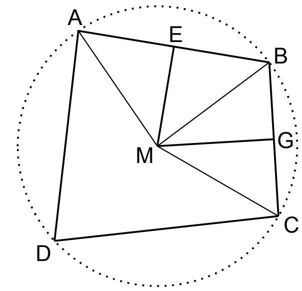
Q.E.D.

THEOREM 21 Remarks

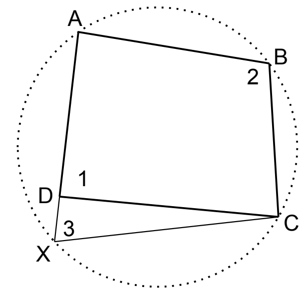
1. It follows from this that a quadrilateral whose opposite angles do *not* add up to two right angles cannot be inscribed in a circle. Take two identical equilateral triangles and put two of their sides together – you now have a rhombus one pair of whose opposite angles are 60° and 60°. Such a quadrilateral cannot be inscribed in a circle – draw it for yourself and confirm it with a diagram.

A quadrilateral that can be inscribed in a circle is sometimes called a "cyclic" quadrilateral. "Cyclic" comes from the Greek word for "circle". Since a pair of angles that add up to two right angles are called *supplementary* angles, we can restate Theorem 21 like this: *In cyclic quadrilaterals, opposite angles are supplementary*.

2. The converse of Theorem 21 is also true, and it goes like this: *If the opposite angles in a quadrilateral are supplementary, then we can circumscribe a circle around it*.



Suppose quadrilateral ABCD is such that its opposite angles are supplementary. Bisect AB at E and BC at G. Draw lines perpendicular to AB and BC at E and G, and where these two perpendiculars meet call M. Join MA, MB, MC.

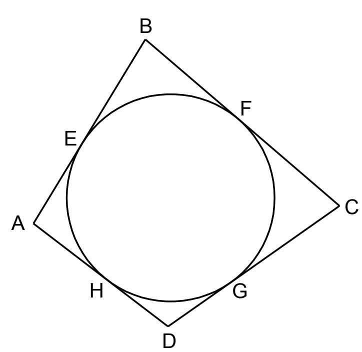


Now $\Delta AEM \cong \Delta BEM$ (Side-Angle-Side)
so MA = MB
and $\Delta CGM \cong \Delta BGM$ (Side-Angle-Side)
so MB = MC
Thus MA = MB = MC, and so the circle with center M and radius MA will pass through A, B, and C.

But will it pass through D, the other corner of quadrilateral ABCD? Yes, it must.

For suppose it did not, but rather D fell *inside* the circle. Then extend AD until it cuts the circle at X. Join CX.

Now $\angle 3 + \angle 2$ = two rights (quadrilateral ABCX is inscribed in a circle)
but $\angle 1 + \angle 2$ = two rights (quadrilateral ABCD is given that way)
so $\angle 1 = \angle 3$, that is, an angle exterior to triangle CDX is equal to one of its interior and opposite angles, which is *impossible* (Ch.1, Thm.14). So it is impossible for D to fall *inside* the circle. Likewise it is impossible for it to fall *outside* the circle. Therefore D falls on the circumference of the circle.



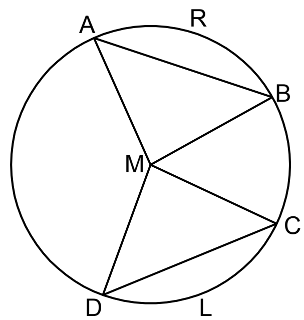
3. Theorem 21 has a cousin theorem about quadrilaterals that are *circumscribed around* circles. Consider quadrilateral ABCD, circumscribed around a circle and so having its four sides tangent to the circle at E, F, G, H. Although its opposite angles do not have to add up to two right angles, its opposite *sides* must have the same sum, that is AD + BC = AB + DC.

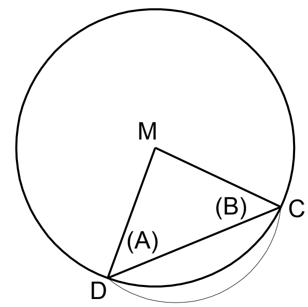
The proof is easy. Since AH is tangent to the circle at H, and AE is tangent to it at E, therefore AH and AE are two tangents to the same circle from the same point, and thus

AH = AE (Thm.18, Remark 1)
and HD = DG (Same reason)
and BF = BE (Same reason)
and $\underline{CF = CG}$ (Same reason)
so AH + HD + BF + CF = AE + DG + BE + CG
or (AH + HD) + (BF + CF) = (AE + BE) + (DG + CG)
i.e. AD + BC = AB + DC.

THEOREM 22: *In any circle, the arcs cut off by equal chords are equal.*

Imagine a circle with any center M and having two equal chords AB and CD in it cutting off arcs ARB and CLD. Then these two arcs must be equal. To see the proof, draw these lines: MA, MB, MC, MD.





[1]	First,	AB = CD	(the chords are given as equal)
	and	MA = MC	(being radii of the circle)
	and	MB = MD	(being radii of the circle)
	so	ΔAMB ≅ ΔCMD	(Side-Side-Side)

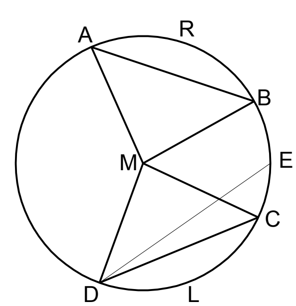
[2] Since ΔAMB and ΔCMD are congruent, therefore if we were to rotate the pie-piece MARB clockwise so that A is on C, then also B would be on D, and so the arc ARB would then coincide with the arc CLD (for if it fell outside as the dotted line, the radii from M would not all be equal!).

[3] Since the arcs ARB and CLD can be made to coincide with each other, therefore they are equal.

Q.E.D.

THEOREM 22 Remarks

1. The converse of this Theorem is also true. That is, *In any circle, the chords of equal arcs are equal*. For instance,
if arc ARB = arc CLD
then AB = CD.
If possible, assume that
AB > CD
then make DE = AB, which is easily done by making a circle (not shown) around point D with a radius equal to AB.



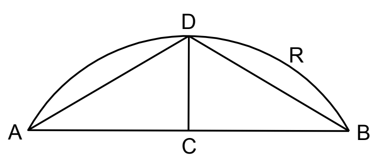
Thus	arc ECD = arc ARB	(arcs of equal chords are equal; Thm.22)
but	<u>arc CLD = arc ARB</u>	(that is given)
so	arc CLD = arc ECD,	

which is to say that the part is equal to the whole, which is *impossible*. So AB is *not* greater than CD. Likewise we can prove that neither is it less. Therefore

AB = CD.

2. Theorem 22 and its converse are also true about chords and arcs in *equal* circles:
* *If arc A in one circle equals arc B in an equal circle, then their chords are equal.*
* *If chord A in one circle equals chord B in an equal circle, then their arcs are equal.*

THEOREM 23: *How to bisect any circular arc.*



Suppose you have any arc of a circle, ARB. How do you cut it into two equal parts? Like this:

[1] Join AB.

[2] Bisect AB at C.

[3] Draw CD at right angles to AB.

[4] Join AD; join DB.

[5] Now, $\angle ACD = \angle BCD$ (since each is a right angle)
and AC = CB (since we bisected AB at C)
and <u>CD is common to ΔACD and ΔBCD</u>
so the corresponding sides of ΔACD and ΔBCD are equal (Side-Angle-Side)

[6] Thus AD = BD (being corresponding sides of ΔACD & ΔBCD)

[7] So arc AD = arc BD (Thm.22)

Q.E.F.

THEOREM 23 Remarks

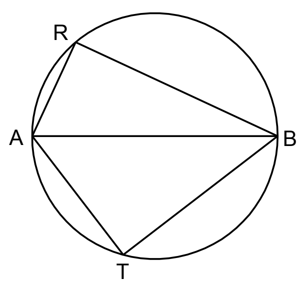
Have we done enough geometry now to cut any circular arc into 3 equal parts? No – that is a problem for more advanced geometry, and its simplest solution involves the use of conic sections. It is impossible to cut an arc of 60° into three equal parts, for example, using nothing but circles and straight lines, which are the only tools we are allowing ourselves in this book.

How about 4 equal parts? Of course. We simply bisect the arc once, and then bisect each of its halves.

How about 5 equal parts? No. That is even more difficult than 3.

Neither can we cut any given arc into 6 or 7 equal parts. But we can cut any given arc into 8 equal parts simply by repeated bisection (likewise with 16, 32, etc.).

THEOREM 24: *Any angle inside a semicircle is right.*



Take any circle with diameter AB cutting it into two semicircles, pick any point R along the circumference, and then join AR and RB. Angle ARB is a right angle.

To see it for yourself, start by picking any point T on the other semi-circumference, and then join AT and TB.

[1] Now $\angle ARB = \angle ATB$,

since they stand at the circumference and cut off equal arcs, namely half the whole circumference (Thm.20, Remarks)

[2] But $\angle ARB + \angle ATB$ = two right angles,

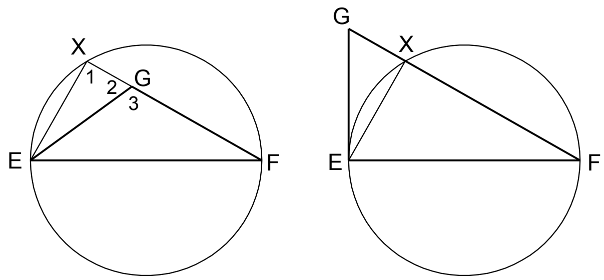
since they are opposite angles in a quadrilateral inscribed in a circle (Thm.21).

[3] So $\angle ARB$ = half of two rights,

i.e. $\angle ARB$ = one right angle.

Q.E.D.

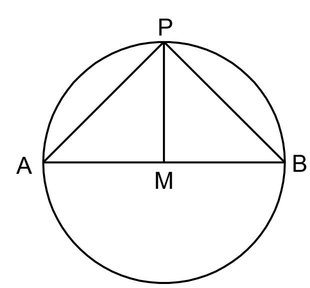
THEOREM 24 Remarks



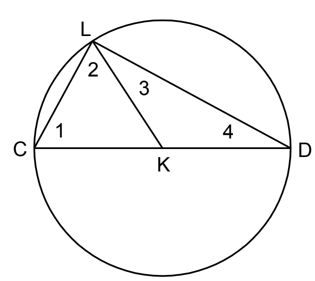
The converse of this Theorem is also true, namely *The circle whose diameter is the hypotenuse of a right triangle must pass through the vertex of the right angle*. If EGF is a right triangle, and ∠EGF is the right angle, then the circle having EF as its diameter passes through point G. If possible, assume it doesn't, but instead G winds up *inside* the circle. Then FG and the circle will intersect at some other point, X; join XE.

Thus ∠1 = one right angle (since it is an angle in a semi-circle; Thm.24)
but ∠2 = one right angle (since it is adjacent to ∠3, which is given as right)
so ΔEXG has *two* right angles in it, and thus its angle-sum is *greater* than two right angles, which is *impossible*. So it is impossible for G to fall inside the circle. Likewise it is impossible for it to fall outside it. Therefore G lies right on the circumference.

THEOREM 24 Questions



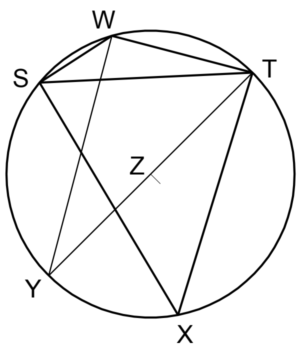
1. If M is the center of a circle, and MP is perpendicular to the diameter, prove that ∠APB is a right angle (independently of Theorem 24). Now use Theorem 20 to prove that *all* the angles at the circumference standing on diameter AB are right.



2. Let L be any point on the circumference of a circle and CD its diamter, K its center. To prove Theorem 24 in yet another way, start by establishing that

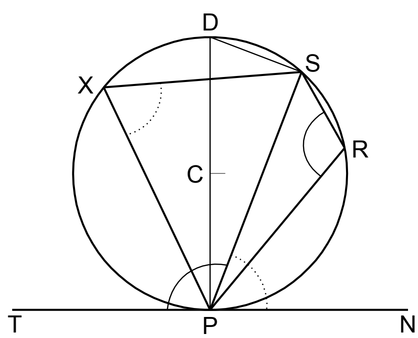
$$1 + 2 + 3 + 4 = 180°$$

and use the fact that ΔCKL and ΔLKD are both isosceles.



3. Prove that the angle in a segment of a circle that is *greater* than a semicircle is acute, and that the angle in a segment that is *less* than a semicircle is obtuse. Let ST divide a circle into two unequal segments, i.e. one greater than a semicircle, and one less. Find the center Z. Pick W and X at random on each of the two arcs. Draw SW, WT, SX, and XT. Draw the diameter TZY, and join WY. Start the proof by showing that ∠SWT is obtuse; then show that ∠SXT is acute by using Theorem 21.

THEOREM 25: *The angles between a tangent to a circle and any straight line drawn through the circle from the point of tangency are equal to the angles in the alternate segments of the circle.*



Conceive of a circle with a straight line TPN tangent to it at point P, and any straight line PS drawn through the circle and cutting it into two segments. Choose any point X on the arc of one segment, and any point R on the arc of the other. Then

∠SPN = ∠SXP

and ∠SPT = ∠SRP.

[1] Start the proof by joining center C to P and extending PC to D to complete diameter PD.

[2] Thus PD is at right angles to TPN, since TPN is tangent at P and C is the center of the circle (Thm.18).

[3] Now ∠DSP = a right angle (since it is an angle in a semicircle)

[4] So ∠SDP + ∠DPS = a right angle,
that is, the remaining two angles in ΔDSP must add up to a right angle, so that all three of its angles together will equal two right angles.

[5] But ∠SPN + ∠DPS = a right angle,
since these two angles make up ∠DPN, and DP is at right angles to PN (Step 2).

[6] So $\angle SDP = \angle SPN$,
because according to Steps 4 and 5, these two angles are complementary to the same angle, namely $\angle DPS$.

[7] But $\angle SDP = \angle SXP$,
since they both stand at the circumference and on the same arc SRP (Thm.20).

[8] So $\angle SPN = \angle SXP$ (putting together Steps 6 & 7)
And this is the first part of what we wanted to prove.

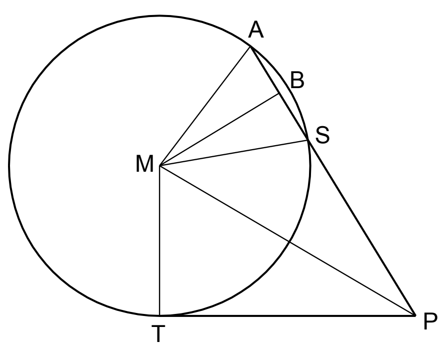
[9] Now $\angle SXP + \angle SRP$ = two rights,
since they are opposite angles of a quadrilateral inscribed in a circle (Thm.21)
So $\angle SPN + \angle SRP$ = two rights ($\angle SPN = \angle SXP$; Step 8)

[10] But $\angle SPN + \angle SPT$ = two rights (they are adjacent angles)

[11] So $\angle SPT = \angle SRP$,
since each is the supplement of angle SPN (Steps 9 & 10)

Q.E.D.

THEOREM 26: *If from a point on a circle's tangent a straight line is drawn cutting through the circle, then the rectangle contained by the whole cutting line and the part outside the circle is equal to the square on the tangent.*



Let PT be a straight line touching a circle with center M at point T, and take any point P on the tangent line. Draw any line PSA through the circle. Then ☐TP = AP · PS.

To prove it, start by bisecting AS at B, then join the following lines: MA, MB, MS, MP, MT.

[1] Now, MB is at right angles to AS, since MB is drawn from the center and it bisects AS (Thm.3). So MBS and MBP are right triangles.

[2] Also, MT is at right angles to TP, since MT is a radius and TP is tangent (Thm.18). So MTP is a right triangle.

[3] Now, $\square TP = \square MP - \square MT$ (ΔMTP; Pythagorean Theorem)

[4] But $\square MP = \square MB + \square BP$ (ΔMBP; Pythagorean Theorem)

[5] So $\square TP = (\square MB + \square BP) - \square MT$ (putting together Steps 3 & 4)

[6] But $\square MB = \square MS - \square BS$ (ΔMBS; Pythagorean Theorem)

[7] So $\square TP = (\square MS - \square BS) + \square BP - \square MT$ (Steps 5 & 6).
Reordering the terms, we can say
$\square TP = \square BP - \square BS + \square MS - \square MT$

[8] But $\square MS = \square MT$ (since MS = MT, being radii),

[9] So $\square TP = \square BP - \square BS$ (putting together Steps 7 & 8)

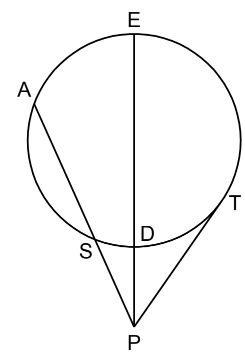
[10] Thus $\square TP = (BP + BS)\ (BP - BS)$,
because of Ch.2, Thm.4: the "difference between two squares" Theorem.

[11] Now $AB = BS$ (since we bisected AS at B)
so $AB + BP = BP + BS$ (adding equals to the same thing)
i.e. $AP = BP + BS$

[12] Thus $\square TP = AP\ (BP - BS)$ (putting together Steps 10 & 11)
i.e. $\square TP = AP \cdot PS$ (BP – PS *is* PS)

Q.E.D.

THEOREM 26 Remarks



1. A straight line drawn from outside a circle that *cuts* the circle is called a "secant" (from the Latin *secare*, to cut or divide). There is an interesting corollary to Theorem 26 about secants, namely that *If two secants are drawn to a circle from the same point, then the rectangles contained by the wholes of each and by their parts outside the circle are equal*. Consider secant PSA and secant PDE, and also draw in tangent PT. Then

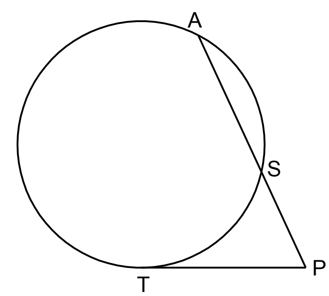
$AP \cdot PS = \square PT$ (Thm.26)

and $\underline{EP \cdot PD = \square PT}$ (Thm.26)

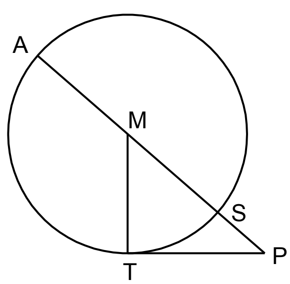
so $EP \cdot PD = AP \cdot PS$

2. The converse of Theorem 26 is also true. That is, if PSA is drawn through a circle, and PT is drawn to the circle such that $\square PT = AP \cdot PS$, then PT is tangent to the circle at T.

Proof: Two tangents from P can be drawn to the circle. And the square on each of them must equal rectangle $AP \cdot PS$ (Thm.26). Thus the square on each tangent equals the square on PT, and so each tangent from P equals PT. Since it is impossible for three equal lines to be drawn to a circle from one point outside it (Thm.8, Remark 2), therefore PT must *be* one of the tangents from P. *Q.E.D.*



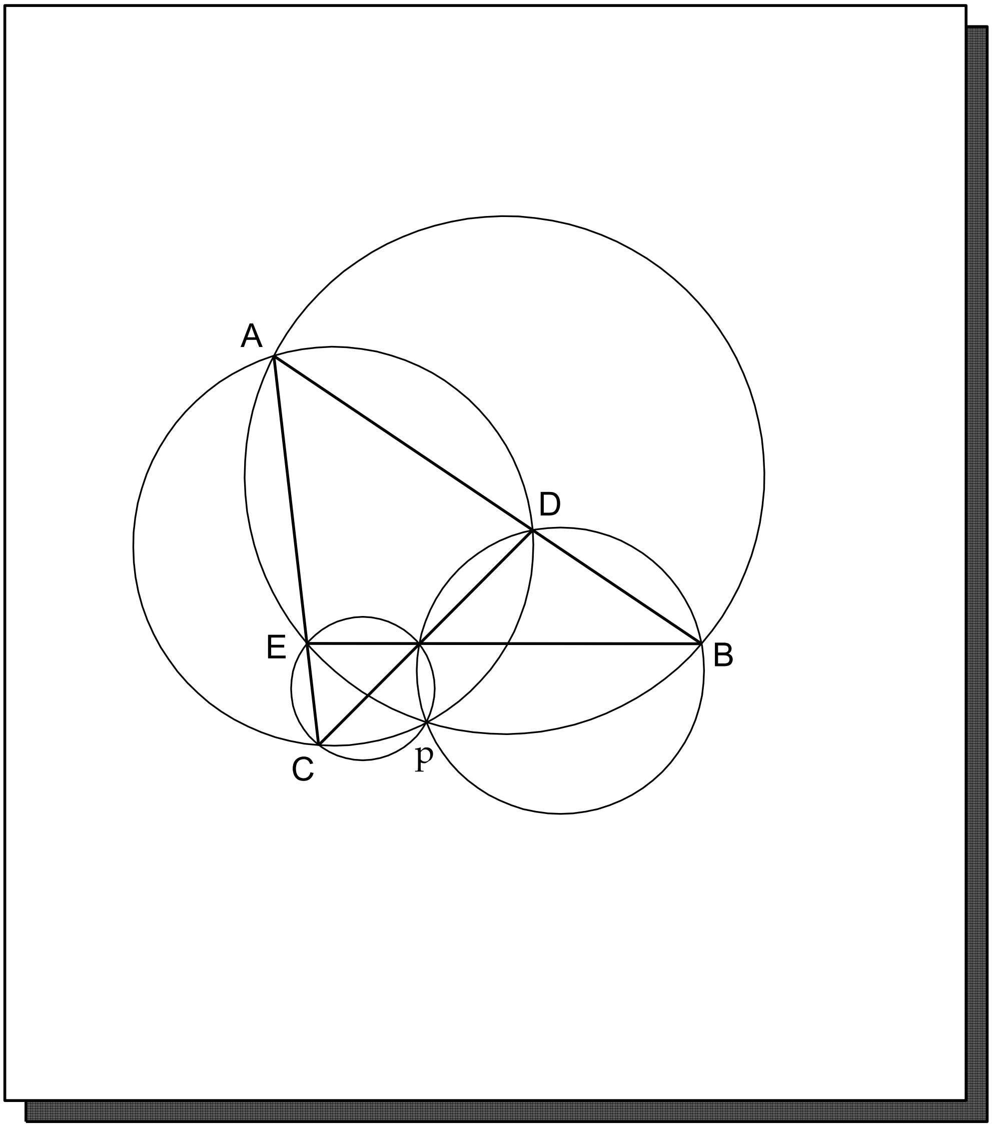
THEOREM 26 Question



What if PSA goes through the center M? How will that change the proof?

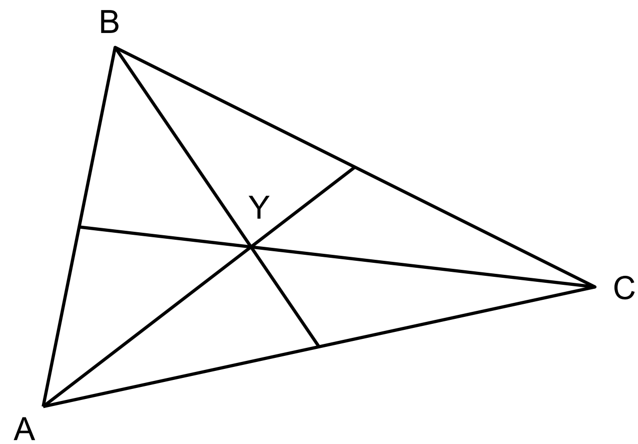
“HOOK”: THE MIQUEL POINT.

If ABC is any triangle, and E, D are any two points along AC, AB, and we join BE, CD, we get a “dented triangle,” or a “Menelaos Figure.” There are four triangles in this figure—two small ones and two larger, overlapping ones. If we circumscribe a circle around each of these four triangles, the four circles will all cut each other at one point, p.



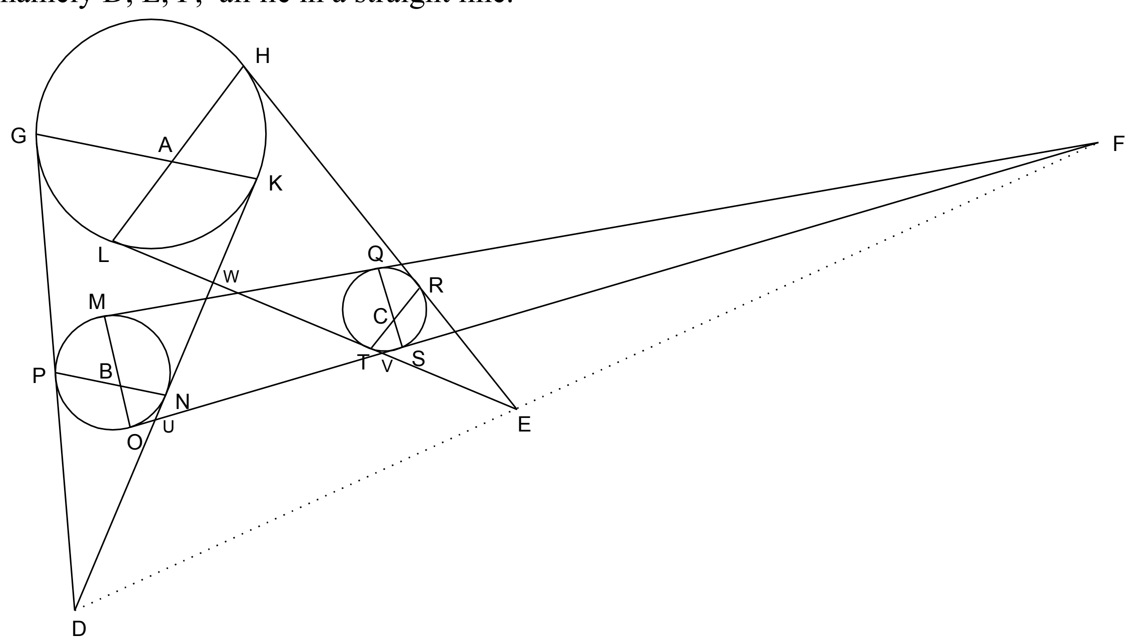
“HOOK”: CENTER OF WEIGHT OF A TRIANGLE.

If you take any triangle ABC, bisect its sides, and join each vertex to the midpoint of the opposite side, these three lines intersect at one point inside the triangle, Y. This point is the center of weight of the triangle. If you were to make such a triangle out of some rather uniformly dense material, and drill a hole through Y, then hang it on a small nail through Y, you could rotate the triangle into any orientation and it would always be balanced.



“HOOK”: COTANGENTS TO THREE CIRCLES.

Take any three circles in one plane, so long as no one of them is completely inside another one. Then it will be possible to take them in three pairs, and for each pair there will be a pair of straight lines tangent to both circles and which intersect—as DPG, DNK are each tangent to the two circles PN, GK. The three resulting points of intersection, namely D, E, F, all lie in a straight line.

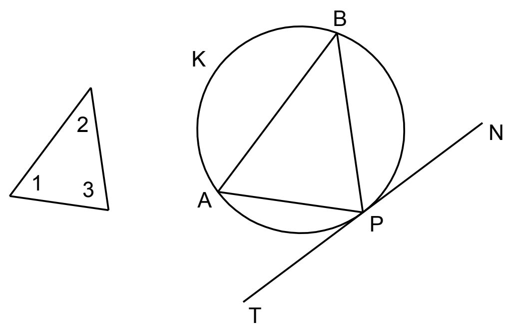


Chapter Four

Polygons

THEOREM 1: *How to inscribe in a circle a triangle that is equiangular with a given triangle.*

Take any triangle you like, and call its angles 1, 2, 3. Take any circle you like – call it K. Can we put inside circle K a triangle whose angles are equal to 1, 2, and 3? Of course.



[1] Draw a tangent TPN through any point P along circle K's circumference.

[2] Draw $\angle TPA = \angle 2$.
Draw $\angle NPB = \angle 1$

[3] Now $\angle 1 + \angle 2 + \angle 3$ = two right angles,
so $\angle NPB + \angle TPA + \angle 3$ = two right angles.

[4] But $\angle NPB + \angle TPA + \angle APB$ = two right angles

[5] So $\angle APB = \angle 3$ (putting together Steps 3 & 4)

[6] Join AB.

[7] Now $\angle TPA = \angle ABP$ (Ch.3, Thm.25)
But $\underline{\angle TPA = \angle 2}$ (Step 2)
So $\angle ABP = \angle 2$

[8] Since $\angle APB = \angle 3$ (Step 5)
and $\angle ABP = \angle 2$ (Step 7)
thus $\angle BAP = \angle 1$,
because whenever two angles in one triangle are equal to two angles in another, the third angle equals the third angle.

Therefore ΔAPB, the triangle inscribed in circle K, is equiangular with the given triangle.

Q.E.F.

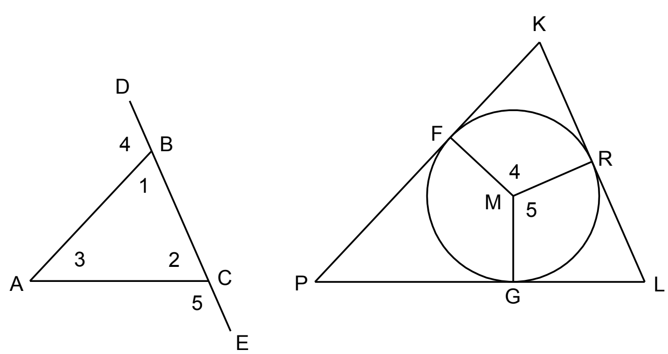
THEOREM 1 Questions

1. We can now inscribe a triangle *of any shape* inside a given circle. Obviously we cannot inscribe a triangle *of any size* inside a given circle – some triangles are too big or too small to be inscribed in a given circle. In fact, once we are given the three angles which the inscribed triangle is to have, and once we are given the circle to inscribe it in, the *size* of the inscribed triangle is already determined.

2. Try to prove that *If two triangles are inscribed in the same circle and they are equiangular, then they will be congruent, i.e. all their corresponding sides will also be equal and they will have the same area*. Consider using the converses of Theorems 20 and 22 in Chapter 3.

3. Is it possible to inscribe a *quadrilateral* inside a given circle? If we are given a quadrilateral and a circle, will we always be able to draw a quadrilateral inside the circle, whose corners are all on the circumference, and whose angles are the same as the given quadrilateral? What must be true about the quadrilateral for it to be possible?

THEOREM 2: *How to circumscribe about a circle a triangle that is equiangular with a given triangle.*



Say I give you a circle with center M and a random triangle ABC having angles 1, 2, 3. How can you circumscribe another triangle around circle M having angles equal to 1, 2, and 3? You can, as follows:

[1] Extend any side of ΔABC both ways, say BC (to D and E), forming the exterior angles 4 and 5.

[2] Choose point R at random on the circumference of circle M. Join MR.
Draw $\angle RMF = \angle 4$
Draw $\angle RMG = \angle 5$ (Ch.1, Thm.20)

[3] Draw lines at right angles to MF at F, to MR at R, and to MG at G.
Call the points where these three lines meet P, K, L.

[4] Since PFK is drawn at right angles to the end of radius MF, it is tangent to the circle at F (Ch.3, Thm.15). Likewise KRL is tangent at R, and PGL is tangent at G. Thus the triangle PKL is circumscribed about circle M (Ch.3, Def. 11).

[5] Now the angles of quadrilateral FMRK add up to a total of *four* right angles (since it is composed of two triangles, each of whose angles add up to *two* right angles).

But $\angle$KFM + $\angle$KRM = two rights, since each of these is a right angle.

Thus the remaining two angles in quadrilateral FMRK must add up to two rights,

i.e. $\angle$4 + $\angle$FKR = two rights,

[6] But $\angle$4 + $\angle$1 = two rights (since they are adjacent)

[7] So $\angle$FKR = $\angle$1 (putting together Steps 5 & 6)

Likewise $\angle$RLG = $\angle$2

[8] Since $\angle$FKR = $\angle$1 (Step 7)
and $\angle$RLG = $\angle$2 (Step 7)
thus $\angle$GPF = $\angle$3,
since whenever two angles in one triangle equal two angles in another triangle, the remaining angles are equal.

So triangle PKL, which has been circumscribed around circle M, is equiangular with the given triangle ABC.

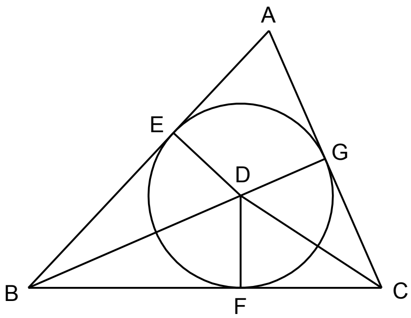
Q.E.F.

THEOREM 2 Questions

1. How do we know the lines drawn at right angles to MF, MR, and MG at F, R, and G will meet, as is asserted in Step 3?

2. What about a quadrilateral? If I give you a random quadrilateral and a circle, will you be able to circumscribe around the circle a quadrilateral whose angles are equal to the given quadrilateral? What must be true about the quadrilateral for it to be possible?

THEOREM 3: *How to inscribe a circle in any triangle.*



Now suppose I give you a triangle. Can you draw inside it a circle to which all three sides are tangent? You can, like this:

[1] Take your triangle ABC and bisect ∠ABC and ∠ACB (any two angles will do).
Let the bisectors meet at D.

[2] Drop DE perpendicular to AB.
Drop DF perpendicular to BC.
Drop DG perpendicular to AC.

[3] Now ∠BED = ∠BFD (both are right angles)
and ∠EBD = ∠FBD (BD bisects ∠EBF; Step 1)
and <u>BD is common to ΔDEB and ΔDFB</u>
so the corresponding sides of ΔDEB and ΔDFB are equal (Angle-Angle-Side)

[4] So DE = DF (being corresponding sides of ΔDEB & ΔDFB)

[5] In the same way, we can prove that
ΔDFC ≅ ΔDGC,
so DF = DG

[6] Thus DE = DF = DG (Steps 4 & 5),
so draw a circle with D as the center and DE as the radius, and its circumference will pass through E, F, and G.

[7] Since AEB is at right angles to the endpoint of radius DE,
AB is tangent to circle D at E (Ch.3, Thm.15)
Likewise BC is tangent to circle D at F,
and AC is tangent to circle D at G.

So circle D has been inscribed in triangle ABC (Ch.3, Def.11).

Q.E.F.

THEOREM 3 Remarks

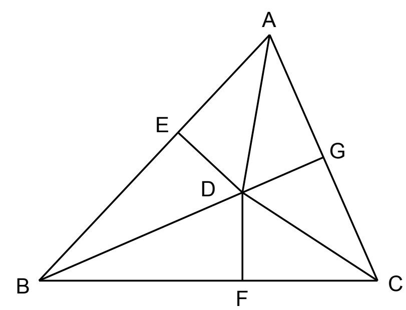
There is another important theorem lurking within Theorem 3, namely that *The bisectors of the three angles of any triangle all meet at one point*. BD and CD bisect angles ABC and ACB, and they meet at D. Join DA; if we can prove that it bisects angle BAC, then all three angle-bisectors meet at one point.

First,	DE = DG	(proved in Step 6 of Theorem 3)
so	☐DE = ☐DG	
so	☐AD – ☐DE = ☐AD – ☐DG	(subtracting equals from the same thing)
but	☐AD – ☐DE = ☐AE	(ΔADE; Pythagorean Theorem)
and	☐AD – ☐DG = ☐AG	(ΔAGD; Pythagorean Theorem)
so	☐AE = ☐AG	
i.e.	AE = AG	

thus ΔADE ≅ ΔAGD by the Side-Side-Side Theorem, since AE = AG, DE = DG, and AD is common to both triangles. Therefore their corresponding angles are equal, so

∠EAD = ∠GAD,

which means that AD is in fact the bisector of angle BAC.

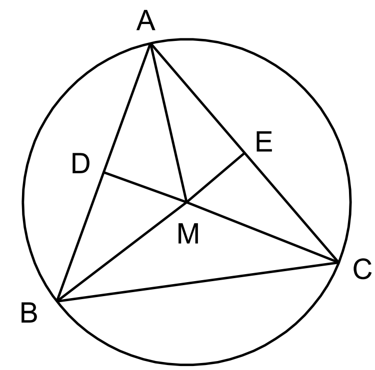


So the 3 angular bisectors of a triangle all meet in one point. You can illustrate this fact by carefully cutting out any triangle from a piece of paper and folding each of its angles in half. Do the three creases all pass through one point?

THEOREM 3 Question

Prove that the bisectors of angles ABC and ACB must in fact meet, as is asserted in Step 1, and that they must meet inside the triangle, not outside it. To prove these bisectors must meet, use the fact that any two angles of a triangle (such as △ABC) add up to less than 180°. To prove they must meet *inside* the triangle, assume for a moment that they meet outside, and see whether they can really bisect the angles of the original triangle anymore.

THEOREM 4: *How to circumscribe a circle about any triangle.*



If I give you a triangle ABC, how can you draw a circle that goes through A, B, and C? You can, as follows.

[1] Bisect AB at D, and bisect AC at E (any two sides will do).

[2] Draw perpendiculars to AB and AC at D and E, and let them meet at M.

[3] Draw AM, BM, CM.

[4] Now, ∠BDM = ∠ADM (both are right angles)
and BD = DA (we bisected AB at D)
and <u>DM is common to ΔBDM and ΔADM</u>
so the corresponding sides of ΔBDM and ΔADM are equal (Side-Angle-Side)

[5] Thus BM = AM (being corresponding sides of ΔBDM & ΔADM)

[6] Likewise we can prove that
ΔCEM ≅ ΔAEM,
so AM = CM

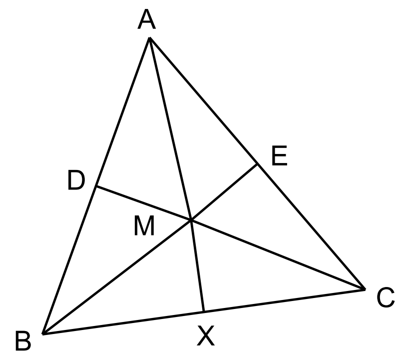
[7] Thus BM = AM = CM (Steps 5 & 6),
so draw a circle with M as center and AM as radius, and its circumference will pass through A, B, and C.
So a circle has been circumscribed about triangle ABC (Ch.3, Def. 10)

Q.E.F.

THEOREM 4 Remarks

1. Notice that, unlike quadrilaterals, all triangles are "cyclic"; *all* triangles can have a circle drawn through their three points.

2. There is an important theorem lurking within Theorem 4, namely this: *The three perpendicular bisectors of the sides of any triangle all meet at one point*. DM and EM meet at M, and these are the perpendicular bisectors of AB and AC. Drop MX perpendicular to BC, and if we can show that it bisects BC then we will have proved that the perpendicular bisectors of all three sides meet at M.



First	BM = CM	(proved in Step 7 of Theorem 4)
so	□BM = □CM	
so	□BM – □MX = □CM – □MX	(taking the same square from both sides)
but	□BM – □MX = □BX	(ΔBMX; Pythagorean Theorem)
and	□CM – □MX = □XC	(ΔCMX; Pythagorean Theorem)
so	□BX = □XC	
thus	BX = XC	

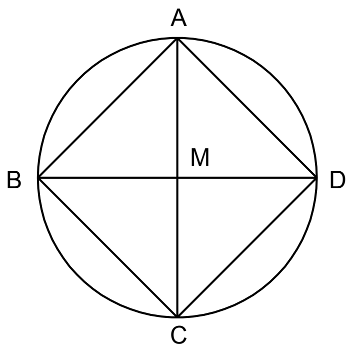
Therefore MX is not only perpendicular to BC, but bisects it at X, and so indeed the three perpendicular bisectors of the sides of ΔABC all meet at one point, M. This point M is called the *circumcenter* of a triangle.

THEOREM 4 Questions

1. What happens if ∠BAC is a right angle? What does BC become in relation to the circumscribed circle? Where will center M fall? What happens if ∠BAC is obtuse? Where will M fall then?

2. Prove that the two lines drawn perpendicular to AB and AC at D and E must in fact meet, as is asserted in Step 2. Start by joining DE, and you should see immediately that the perpendiculars at D and E must make angles less than two right angles on one side of DE.

THEOREM 5: *How to inscribe a square in any circle.*



Given a circle, how do we inscribe a square in it?

[1] Begin by finding its center, M. (Ch.3, Thm.1)
Draw any line through M, AMC; thus AC is a diameter.
Draw BMD at right angles to AMC; thus BD is another diameter.

[2] Draw AB, BC, CD, and DA, forming quadrilateral ABCD inscribed in circle M.

[3] Now, $\angle BMA = \angle BMC$ (both are right angles)
and $AM = MC$ (both are radii)
and <u>BM is common to ΔABM and ΔBCM</u>
so $\triangle ABM \cong \triangle BCM$ (Side-Angle-Side)

[4] Thus $AB = BC$ ($\triangle ABM \cong \triangle BCM$)

[5] We can prove in the same way that
$\triangle BCM \cong \triangle CDM$,
so $BC = CD$,
and again we can prove in the same way that
$\triangle CDM \cong \triangle DAM$,
so $CD = DA$

[6] So $AB = BC = CD = DA$ (Steps 4 & 5),
and so quadrilateral ABCD is *equilateral*.

[7] Since BMD is a diameter, therefore $\angle BAD$ is a right angle (Ch.3, Thm.24). For the same reason every other angle of quadrilateral ABCD is a right angle.

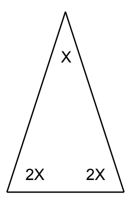
[8] Since ABCD is both *equilateral* (Step 6), and also *right angled* (Step 7), ABCD is a square.

Q.E.F.

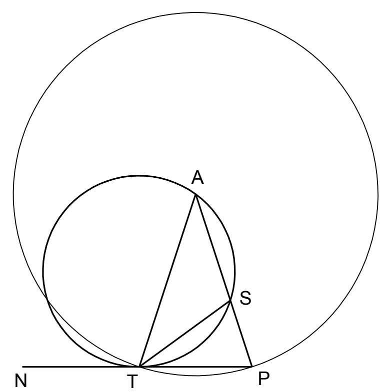
THEOREM 5 Questions

1. Can you see how to circumscribe a square around a given circle?
2. Can you see how to inscribe a circle in a given square?
3. Can you see how to circumscribe a circle around a given square?

THEOREM 6: *How to make an isosceles triangle whose base angles are each double its peak angle.*



Recall that in *every* isosceles triangle the base angles are equal to each other. But, for the sake of the upcoming Theorem 7, we want to make a particular kind of isosceles triangle in which *each* base angle is *double* the peak angle X. Here's how:



[1] Set out any straight line AS. Extend it to P so that ☐AS = AP · PS (Ch.2, Thm.12). Draw a circle with center A and radius AP. Mark off PT = AS by drawing a circle around P with a radius equal to AS (not shown). Join AT. Now AT = AP, being radii of circle A. Thus ATP is an isosceles triangle, and I say that ∠TPA is double ∠TAP.

[2] Circumscribe a circle about △AST. (Thm.4)
Since ☐AS = AP · PS (Step 1)
thus ☐PT = AP · PS (PT = AS; Step 1)
thus PT is tangent to the circumscribed circle (see Ch.3, Thm.26, Remark 2).

[3] Extend PT to N. Because PTN is tangent, therefore it makes angles with AT equal to the angles in the alternate segments of the circle (Ch.3, Thm.25).
Hence ∠ATN = ∠AST

[4] Hence the supplements of these angles are also equal to each other,
i.e. ∠ATP = ∠TSP

[5] But ∠TPA = ∠TPS (they are actually the same angle)
thus △APT is equiangular with △TSP (Ch.1, Thm.29)

[6] Hence △TSP is also isosceles, so

	PT = ST	
but	PT = AS	(Step 1)
so	AS = ST	

so △AST is also isosceles.

[7] Now	∠TSP = ∠STA + ∠TAS	(∠TSP is exterior to △AST)
or	∠TSP = 2 ∠TAS	(∠STA = ∠TAS, since △AST is isosceles)
so	∠TPS = 2 ∠TAS	
i.e.	∠TPA = 2 ∠TAP	

Q.E.F.

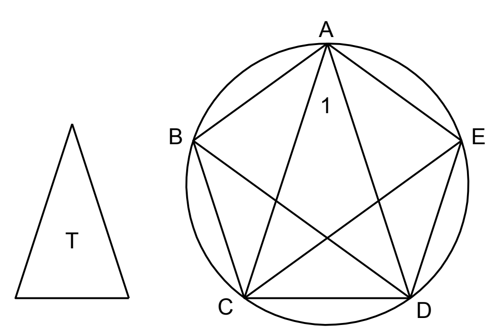
THEOREM 6 Remarks

1. The triangle we have constructed is called The Golden Triangle, for reasons we will understand later. Can you find another golden triangle in the construction diagram other than △ATP? Prove it.

2. What are the angles of a golden triangle in degrees? How many degrees is the peak angle? How many degrees is one of the base angles? If we call the peak angle X, each base angle is 2X, thus the sum of the angles in the whole triangle is 5X. But this must be 180°. Therefore X = 180° ÷ 5, which is 36°. So the peak angle is 36°, and each base angle is 72°.

THEOREM 7: *How to inscribe a regular pentagon inside any circle.*

Imagine you have a circle A, and you want to draw a regular pentagon inside it (that is, a five-sided polygon that is equilateral and equiangular). Follow these steps:



[1] Start by making an isosceles triangle T whose base angles are each double its peak angle (Thm.6).

[2] In the circle inscribe $\triangle ACD$ whose angles equal those of triangle T (Thm.1).
Thus $\frac{1}{2}\angle ADC = \angle 1$
and $\frac{1}{2}\angle ACD = \angle 1$

[3] Bisect $\angle ACD$ with CE.
Bisect $\angle ADC$ with DB.
Draw AB, BC, DE, EA, yielding pentagon ABCDE.

[4] Now, $\frac{1}{2}\angle ADC = \angle ADB$ (we bisected $\angle ADC$ with DB; Step 3)
but $\underline{\frac{1}{2}\angle ADC = \angle 1}$ (Step 2)
so $\angle ADB = \angle 1$

[5] In the same way, we can prove that
$\angle BDC = \angle 1$
$\angle DCE = \angle 1$
$\angle ECA = \angle 1$

[6] And all these equal angles are at the circumference. Therefore the arcs on which they stand are all equal to each other (Ch.3, Thm.20, Converse proved in Remarks),

i.e. arc AB = arc BC = arc CD = arc DE = arc EA

[7] Since all these arcs are equal, the chords joining their endpoints are all equal, too (Ch.3, Thm. 22, Converse proved in Remarks). That is

$$AB = BC = CD = DE = EA.$$

Therefore the pentagon ABCDE is *equilateral*.

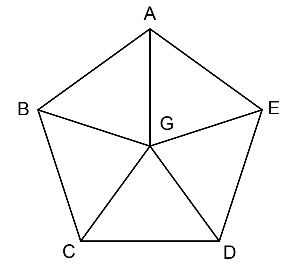
[8] Now $\angle ABC$ stands on three of the five equal arcs, namely on
arc CD + arc DE + arc EA.
Likewise $\angle BCD$, $\angle CDE$, $\angle DEA$, $\angle EAB$ each stand on three of the five equal arcs. Therefore these five angles stand on equal arcs.

[9] And all five of these angles standing on equal arcs are at the circumference. Therefore they are equal (Ch.3, Thm.20, proof in Remarks).
That is, $\angle ABC = \angle BCD = \angle CDE = \angle DEA = \angle EAB$.

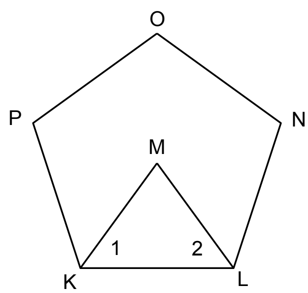
[10] Thus the pentagon ABCDE, inscribed in the circle, is both equilateral (Step 7) and equiangular (Step 9). Therefore it is a regular pentagon.

Q.E.F.

THEOREM 7 Remarks

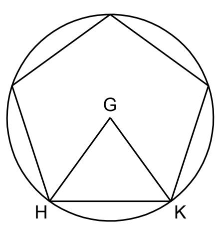


The base angle of a golden isosceles triangle is 72° (Thm.6, Remark 2). Now make an isosceles triangle whose *peak* angle is 72°. Make the equal sides CG and GD of whatever length you like. Since 72° × 5 = 360°, thus we can place the peak angles of 5 identical isosceles triangle of this kind around point G. Since the 5 triangles are identical isosceles triangles, their bases are equal, and therefore the pentagon ABCDE is equilateral. Again, since every angle of the pentagon, such as ∠ABC, consists of two of the identical base angles, pentagon ABCDE is equiangular. Therefore a regular pentagon consists of 5 identical isosceles triangles whose peak angles are each 72°.

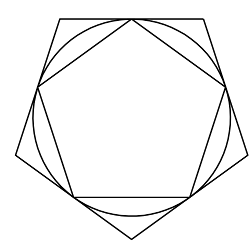


Clearly GA = GB = GC = GD = GE. The circle with center G and radius GA therefore circumscribes the pentagon. Given a regular pentagon OPKLN, then, it will be easy to circumscribe a circle about it. Choose any side KL, and draw angles 1 and 2 each equal to ∠GCD above. Therefore the lines drawn must meet inside the pentagon at a point, M, such that ∠KML = ∠CGD or 72°. Thus M is the point common to the 5 identical isosceles triangles composing pentagon OPKLN, and therefore M is the center of the circumscribing circle.

THEOREM 7 Questions



1. How many degrees is each angle of the regular pentagon? Start by recalling that ∠HGK is 72° in isosceles triangle HGK.

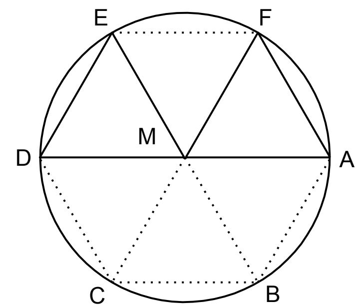


2. How would you circumscribe a regular pentagon around a circle? Start by inscribing one as we did in Theorem 7. Describe the construction and prove that the result is a regular pentagon.

3. Find a way to construct a regular pentagon on a given straight line as its side.

THEOREM 8: *How to inscribe a regular hexagon inside any circle.*

Take any circle; call its center M. We want to make a regular hexagon inside it, that is, we want to inscribe in it a six-sided polygon all of whose sides and angles are equal.



[1] Start by choosing any point A on the circumference, and join MA.

[2] Make an equilateral triangle on MA; thus its vertex F will lie on the circumference of the circle (Ch.1, Thm.1).

[3] Extend AM to D.
Make an equilateral triangle on MD; thus its vertex E will also lie on the circumference of the circle.

[4] Extend FM to C, EM to B. Draw AB, BC, CD, EF.

[5] Since all three angles of an equilateral triangle are equal, any one of its angles is one third of two right angles.

so $\angle FMA =$ one third of two rights
and $\angle EMD =$ one third of two rights.
and $\angle EMF$ must equal one third of two rights also, since these three angles add up to a straight line or 180°.

[6] Hence $\angle EMF$ is also the angle of an equilateral triangle
But $EM = MF$ (they are radii of the circle)
And therefore triangle EMF is also equilateral, and its side is the radius of the circle.

[7] Now $\angle DMC = \angle FMA$ (being vertical angles)
and $MC = MF$ (they are radii of the circle)
and $MD = MA$ (again, they are radii)
so $\Delta DMC \cong \Delta AMF$ (by Side-Angle-Side)
And therefore triangle DMC is also equilateral, and its side is the radius of the circle.

[8] In the same way, we can prove that ΔCMB and ΔMBA are equilateral triangles, too, whose side is the radius of the circle.

[9] Therefore the sides of hexagon ABCDEF are the sides of six identical equilateral triangles, and so all of its sides are equal. And each of the angles of hexagon ABCDEF is composed of two angles of an equilateral triangle, and so all of its angles are equal.
Therefore hexagon ABCDEF, inscribed in the circle, is a regular hexagon.

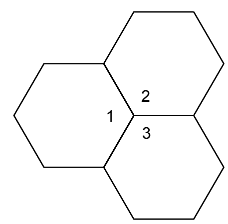
Q.E.F.

THEOREM 8 Remarks

It is amazing that the radius of a circle fits inside its circumference exactly six times.

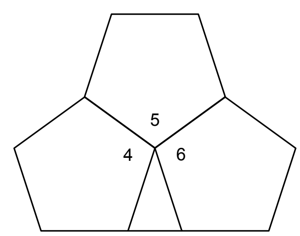
THEOREM 8 Questions

1. How many degrees are in one angle of an equilateral triangle? How many degrees are in one angle of a regular hexagon?



2. Prove that it is possible to cover a surface with identical regular hexagons, without leaving any gaps and without any overlap of the hexagons. Begin by looking at angles 1, 2, and 3.

We might put this little theorem this way: *it is possible to use regular hexagons as floor tiles*.

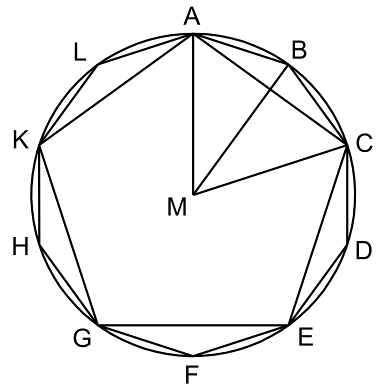


3. With the same sort of reasoning, show that it is *not* possible to cover a whole surface with identical regular pentagons – you will necessarily have gaps. Consider angles 4, 5, and 6 in your reasoning.

We might say *it is impossible to use regular pentagons as floor tiles*, since we leave gaps. Incidentally, can you identify the shape of the gaps left out by the pentagons?

THEOREM 9: *How to inscribe a regular decagon in any circle.*

Take any circle M, and you can make a regular decagon (a regular 10-sided polygon) inside it as follows.



[1] Inscribe a regular pentagon inside the circle, ACEGK (Thm.7).

[2] Bisect each of the five equal arcs cut off by its sides, namely arc AC at B, arc CE at D, arc EG at F, arc GK at H, and arc KA at L.

[3] Draw AB, BC, CD, DE, EF, FG, GH, HK, KL, LA.

[4] Since the five arcs cut off by the sides of the pentagon are equal, their halves are equal. And since those ten arcs are equal, the chords joining their endpoints are equal (Ch.3, Thm.22 Converse), i.e.

AB = BC = CD = DE = EF = FG = GH = HK = KL = LA

Thus the decagon ABCDEFGHKL is *equilateral*.

[5] Draw MA, MB, MC, thus forming isosceles triangles AMB and BMC.

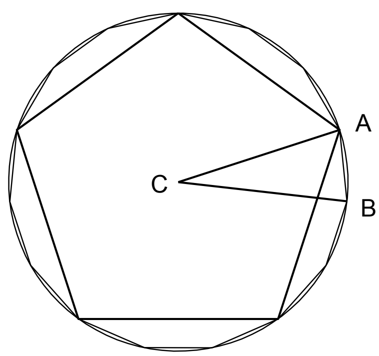
[6]	Now,	AB = BC	(Step 4)
	and	MA = MC	(being radii of the circle)
	and	MB is common to ΔMBA and ΔMBC	
	so	ΔMBA ≅ ΔMBC	(Side-Side-Side)

[7] Similarly we can prove that *all ten* of the isosceles triangles having a side of the decagon as base and having center M as vertex *are identical*. And therefore they all have the same base angle. But every angle of the decagon is equal to *two* of these identical base angles (e.g. ∠ABC = ∠MBA + ∠MBC), and so all of the angles of the decagon are equal.

[8] Since decagon ABCDEFGHKL, inscribed in the circle, is both equilateral (Step 4) and equiangular (Step 7), it is regular.

Q.E.F.

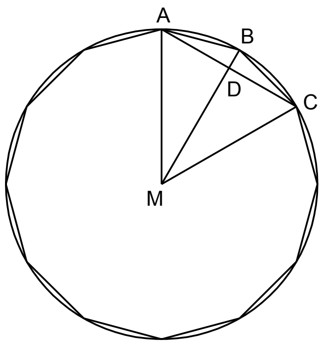
1. Now we have succeeded in making regular rectilineal figures of 3 sides, 4 sides, 5 sides, 6 sides, and 10 sides. What about 7, 8, and 9 sides? The 7 and 9-sided regular polygons require more sophisticated tools than straight lines and circles, but the regular octagon is easy – just bisect the four arcs of a circle around the four sides of the inscribed square. In fact, given any regular polygon that we can inscribe in a circle, we can always make another with twice as many sides just by bisecting the arcs of the circle. We cannot make the 11-sided regular polygon using only straight lines and circles. But the 12-sided regular polygon we can do. Just bisect the six arcs of a circle around the six sides of the inscribed regular hexagon. The 13 and 14-sided regular polygons transcend the powers of our elementary geometry.



But we can do the 15-sided regular polygon! If AB were the side of the regular 15-sided figure inscribed in a circle with center C, then ∠ACB would be one fifteenth of 360°, namely 24°. So if we can make a 24° angle, we can cut the circle into 15 equal parts and have the figure we seek. How? Well, the angle drawn from the center of a circle and standing on one side of the inscribed *decagon* is one tenth of 360°, i.e. 36°, so we can make that angle. And we can also make the 60° angle, since that is the angle of an equilateral triangle. So if we draw these two angles together with a common side, the difference between them will be an angle of 24°. Voilà.

Obviously, we can also do the 16-sided, by starting with the regular octagon (the “stop sign”) and bisecting.

Now what about the regular 17-sided figure? (This figure is called a regular “heptadecagon.”) Believe it or not, we actually can make it using nothing but circles and straight lines! This was not known until the great German mathematician Carl Friedrich Gauss (1777 – 1855) discovered it. He also found the general formula for determining when it is possible to make a regular polygon of a given prime number of sides using nothing but circles and straight lines, and when not.



2. Having mentioned the 12-sided regular polygon or dodecagon, it is worth adding here an interesting little theorem about it: Its area is exactly 3 times that of the square on the radius of the circle in which it is inscribed.

Given: Regular dodecagon ABC etc. in circle with center M.
Prove: The dodecagon's area = 3 □BM.

Join AC, and where it cuts radius BM call D.

To begin, note that the dodecagon consists of 12 identical isosceles triangles such as MAB. Thus

Area Dodecagon = 12 △MAB

Now by the regularity of the figure, AB = BC, and ∠ABD = ∠CBD, but BD is common to triangles ABD and CBD. Hence they are congruent, and so the adjacent angles ADB and CDB are right angles. Therefore AD is the height of △MAB, taking BM as base. Since △MAB is half the rectangle of that height and on that base, i.e. it is half rectangle MB · AD, therefore 12 triangles such as MAB equal 6 such rectangles. Therefore

Area Dodecagon = 6 MB · AD

But AC is double AD, and therefore MB · AC is double MB · AD, since if one rectangle has the same height as another, but double its base, it will be twice as big. And accordingly 6 rectangles such as MB · AD will equal only 3 rectangles such as MB · AC. Therefore

Area Dodecagon = 3 MB · AC

And since AC cuts off one sixth of the circumference of the circle, it is the side of the regular hexagon, which is equal to the radius of the circle. Therefore AC = MB, so that the rectangle MB · AC is the same as the square on MB, or the square on the radius. Hence

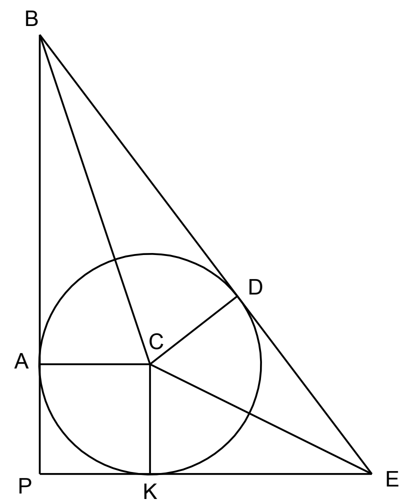
Area Dodecagon = 3 □MB.

THEOREM 9 Questions

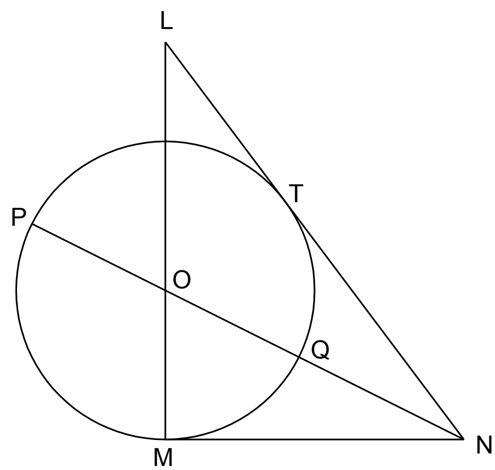
1. How many degrees are in one angle of the regular decagon?
2. In the diagram for Theorem 9, what kind of triangle is triangle AMB? Compare it to golden triangle AGC, and apply the theorem that says ∠AMC is double ∠AGC.

“HOOK”: THE 3-4-5 TRIANGLE AND THE CIRCLE.

If EP is 3 units of length, and PB is 4, and BE is 5, then EPB is a “3-4-5 right triangle.” If we inscribe the circle in it, center C, its radius will have a length of 1 unit.

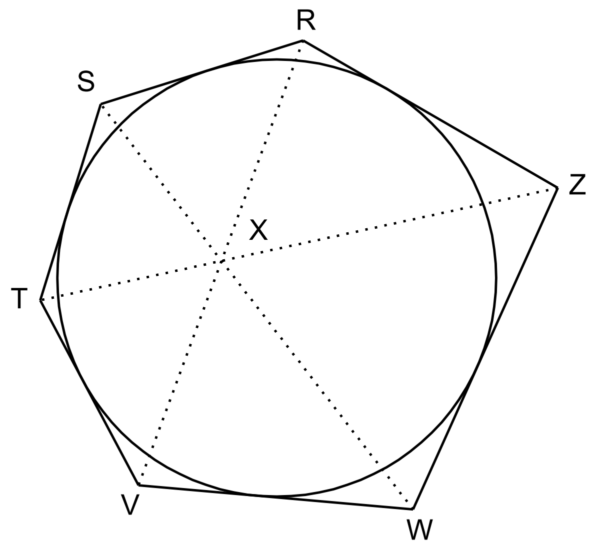


Also, if LMN is a 3-4-5 right triangle in which LM = 4 and LN = 5, and if we draw a circle whose diameter lies along LM and which is tangent to MN at M and also to LN at T, this will be the equivalent of inscribing a circle in the isosceles triangle which is made up of two triangles like LMN, symmetrical about LM. Let O be the center of this circle, and join NO, cutting the circle at P & Q. Then PQN is cut in “mean and extreme ratio,” also called “the golden ratio,” about which we will learn more in Chapters 5 and 6.



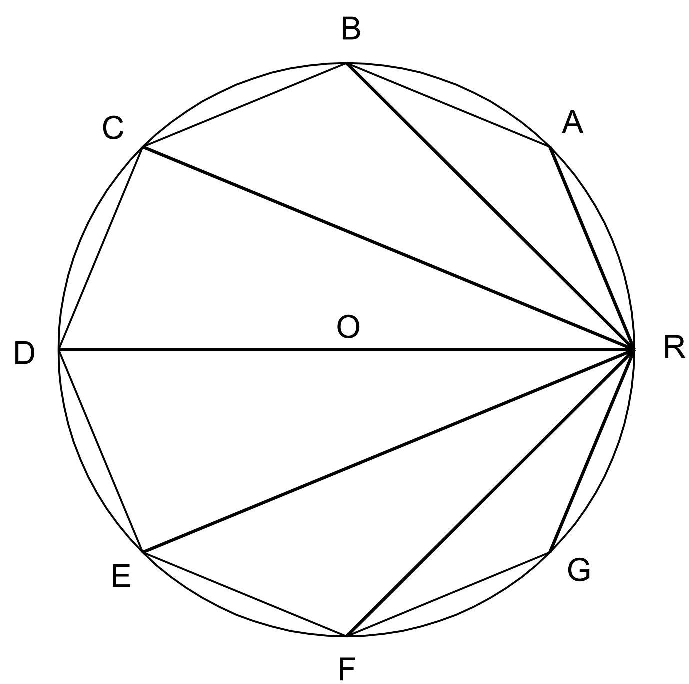
“HOOK”: BRIANCHON’S THEOREM.

Take any circle and draw a tangent to it, RS, of whatever length you please. From S, draw another tangent, ST. Continue doing this so that you end up with a hexagonal figure circumscribed about the circle, RSTVWZ. Join the “opposite” vertices of this figure: RV, SW, TZ. These three straight lines will intersect in a single point.



“HOOK”: CHORD PRODUCTS OF REGULAR POLYGONS.

In a circle with center O and a unit-long radius OR, draw a regular polygon of any number of sides you like, N, using R itself as the starting vertex. If we now join R to all the other vertices, we get chords (such as RA, RB, RC, etc.). Since we are calling the radius “1,” it is decided for us what we have to call each of the other chords. If we then multiply all the lengths of these chords together, what will the result be? It will always be N. For instance, if our polygon is a hexagon, the chord-product will be 6. If an octagon, 8. And so on.



Chapter Five

Proportion in General

DEFINITIONS

1. A lesser quantity is said TO MEASURE a greater one if it goes into it exactly a whole number of times.

For example, a foot measures a yard, since it goes into it exactly three times; but it does not measure a meter, since three feet is less than a meter, but four feet is more than a meter.

2. When a lesser quantity measures a greater one, the lesser is called A MEASURE of the greater, and the greater is called A MULTIPLE of the lesser.

For example, a foot is a measure of a yard, and a yard is a multiple of a foot. If B *is a multiple of* A, *say* 3 *times* A, *then the shorthand notation for this is* B = 3A, *which reads "* B *is three* A*", or "* B *is three times* A*".*

3. We form EQUIMULTIPLES of two quantities whenever we multiply each of them the same number of times.

For example, 12 *and* 15 *are not only multiples of* 4 *and* 5 *respectively, but equimultiples of* 4 *and* 5, *since* 12 *and* 15 *are three times* 4 *and* 5 *respectively*.

4. Whenever two quantities are capable of exceeding each other by being multiplied enough times, the relative size of one to the other is called a RATIO.

For example, a foot, when multiplied four times, exceeds a yard, and a yard, when multiplied two times, exceeds four feet. So the relation of a foot to a yard is a "ratio", namely the ratio of 1 *to* 3, *and the yard also has a ratio to the foot, namely that of* 3 *to* 1.

But a line does NOT have a ratio to an area, such as a rectangle. Even though each is a kind of quantity, the line has no area at all, and so no multiple of it can ever exceed the area of the rectangle. There is, then, no comparison of these two quantities with respect to size.

If a quantity A *has a ratio to another quantity* B, *the shorthand notation for their ratio is* A : B, *which reads "*A *to* B*". The order of the terms in a ratio makes a difference. For example, if* A *has to* B *the ratio of* 1 : 2 , *then* A *is HALF of* B. *On the other hand, if* A *has to* B *the ratio of* 2 : 1 , *then* A *is the DOUBLE of* B.

5. The first term in a ratio is called its ANTECEDENT, and the second term in a ratio is called its CONSEQUENT.

6. Given four quantities in two ratios, taking CORRESPONDING MULTIPLES of them means taking equimultiples of the antecedents, and again taking equimultiples of the consequents.

For example, given quantities A, B, C, D *in the ratios* A : B *and* C : D, *if we take multiples of* A *and* B *such as* 3A *and* 5B, *then the corresponding multiples of* C *and* D *are* 3C *and* 5D.

7. One pair of quantities in a ratio is said to COMPARE THE SAME WAY as another pair when each pair is the ratio of a greater to a lesser, or when each is the ratio of a lesser to a greater, or when both pairs are equalities.

For example, the numbers in the ratio 5 : 3 *and those in the ratio* 7 : 4 *compare the same way, since* $5 > 3$ *and* $7 > 4$.

Again, the numbers in the ratio 5 : 5 *and those in the ratio* 7 : 7 *compare the same way, since* $5 = 5$ *and* $7 = 7$.

But the numbers in the ratios 6 : 4 *and* 9 : 12 *do not compare the same way, since* $6 > 4$ *but* $9 < 12$.

8. If the multiples of two quantities always compare the same way as the corresponding multiples of two other quantities, then the first two quantities have the SAME RATIO as the other two.

For example, suppose you have two ratios A : B *and* C : D. *And suppose their corresponding multiples compare the same way, e.g.*

$$5A > 3B \qquad \textit{and} \qquad 5C > 3D$$
$$2A < 4B \qquad \textit{and} \qquad 2C < 4D$$

and so on. If this is true for all their corresponding multiples, then A *has to* B *the same ratio that* C *has to* D.

9. If one quantity has to a second the same ratio that a third has to a fourth, then the four quantities are said to be PROPORTIONAL.

If quantity A *has to* B *the same ratio that* C *has to* D, *the shorthand notation for that proportion is* A : B = C : D , *which reads "*A *is to* B *as* C *is to* D*", or, if you prefer, "the ratio of* A *to* B *is the same as the ratio of* C *to* D*".*

The four terms in a proportion need not all be different. For example, three terms X, Y, Z *can be in proportion like this:* X : Y = Y : Z.

10. By contrast, when it is possible to find a multiple of a 1st quantity *greater than* some multiple of a 2nd, but the corresponding multiple of a 3rd quantity is *not greater than* the corresponding multiple of a 4th, then the 1st has to the 2nd a GREATER RATIO than the 3rd has to the 4th.

For example, take 3, 5, 6, *and* 17 *in that order. Let's multiply* 3 *and* 5 *by* 4 *and* 2 *respectively, and likewise multiply* 6 *and* 17 *by* 4 *and* 2 *respectively*.

Notice that $4(3) > 2(5)$*, whereas* $4(6) < 2(17)$*. We took multiples of* 3 *and* 5*, and then corresponding multiples of* 6 *and* 17*, but the two pairs of multiples do not compare the same way. Since the multiple of* 3 *exceeded the multiple of* 5*, whereas the corresponding multiple of* 6 *fell short of the corresponding multiple of* 17*, therefore* 3 *has to* 5 *a greater ratio than* 6 *has to* 17*. The way to write this is* $3 : 5 > 6 : 17$.

Again, take 5, 2, 3 *and* 4 *in that order. Multiply* 5 *and* 2 *by* 8 *and* 6, *and also* 3 *and* 4 *by* 8 *and* 6.

Now $8(5) > 6(2)$ *but* $8(3) = 6(4)$

hence $5 : 2 > 3 : 4$.

11. If one term in a ratio is not greater than the other, then it is either less than or equal to it. The symbol for this is $\leq$. Again, if one term in a ratio is not less than the other, then it is either greater than or equal to it. The symbol for this is $\geq$.

Using these symbols, we can define "greater ratio" as follows.

If $A : B > C : D$,

then for some pair of numbers n *and* m *it will happen that*

$$nA > mB \quad \textit{but} \quad nC \leq mD.$$

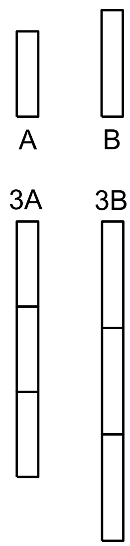
THEOREMS

THEOREM 1: *Given any number of quantities, the sum of their equimultiples is that same multiple of their sum. Again, given any number of quantities, the difference of their equimultiples is that same multiple of their difference.*

This is clear enough in examples.

Suppose you have two quantities which are able to be added together, namely A and B. Take the same multiple of each one, say 3 times, giving us equimultiples 3A and 3B.

Then the sum of these equimultiples is also 3 times the sum of the original quantities. That is:



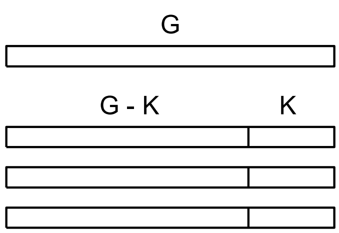
$$3A + 3B = 3(A + B)$$

This is because we can subtract A + B from 3A + 3B exactly three times.

And, for the same reason, no matter how many original quantities we have, and no matter what multiple we choose to take of them all, the sum of the multiples will always be that same multiple of the sum of the original quantities. So, for example,

$$5A + 5B + 5C = 5(A + B + C)$$

and so on.



The same goes for the differences between unequal quantities. Suppose G is greater than K, and you take any equimultiples of G and K, say 3G and 3K. Then

$$3G – 3K = 3(G – K).$$

If we set out three G's, as in the diagram, and subtract one K from each, there will be three remainders of G – K.

Q.E.D.

THEOREM 1 Remarks

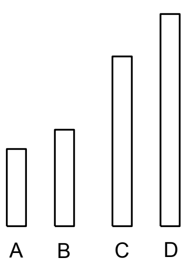
1. Confirm this Theorem with some numerical examples:

$3\times6 + 3\times5 + 3\times4 = 3(6 + 5 + 4)$

$7\times36 - 7\times19 - 7\times11 = 7(36 - 19 - 11)$

2. Another very basic truth about multiples is this: If we multiply a quantity by one number, and then the result by another number, it makes no difference which multiplier is used first. For example, if we double A and then triple its double, we get 6 times A, and again if we triple A and then double its triple, we still get 6 times A. In other words, 3(2A) = 2(3A).

THEOREM 2: *Proportional quantities are proportional inversely.*



Take any four quantities in proportion, say A : B = C : D.
I say they are also proportional inversely, that is B : A = D : C.

Here's why.

[1] Take any multiples of A and B, say 5A and 3B.
Take the corresponding multiples of C and D, namely 5C and 3D.

[2] Since A : B = C : D, therefore
if 5A > 3B, then 5C > 3D
but if 5A < 3B, then 5C < 3D (Def. 8),
and that is true for *any* multiples of A and B and the corresponding multiples of C and D.

[3] Restating the information in Step 2, but putting B & D first in each case, we have
if 3B < 5A, then 3D < 5C
but if 3B > 5A, then 3D > 5C,
and that will be true for *any* multiples of B and A, and the corresponding multiples of D and C.

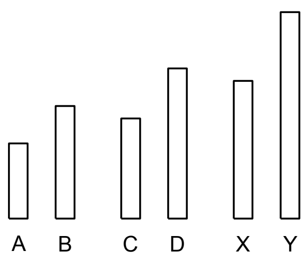
[4] So taking any multiples of B and A, the corresponding multiples of D and C must compare the same way. Therefore
B : A = D : C (Def. 8).

Q.E.D.

THEOREM 2 Remarks

Here is a numerical example of the Theorem:
since 3 : 4 = 6 : 8,
so too 4 : 3 = 8 : 6.

THEOREM 3: *Two ratios that are the same with a third ratio are the same as each other.*



Take any three pairs of quantities A and B, C and D, X and Y, which are such that
A : B = X : Y
and C : D = X : Y.
Then A : B = C : D as well.

Here's the proof.

[1] Take any multiples of A and B, say 5A and 3B.
Take the corresponding multiples of X and Y, namely 5X and 3Y.
Take the corresponding multiples of C and D, namely 5C and 3D.

[2] Suppose 5A > 3B.
Then, since A : B = X : Y, (given)
therefore 5X > 3Y. (Def. 8)

[3] Now C : D = X : Y (given)
But 5X > 3Y,
therefore 5C > 3D. (Def. 8)

[4] So if 5A > 3B, then 5C > 3D (Steps 2 & 3)
Likewise we can prove that
if 5A < 3B, then 5C < 3D.

[5] So, taking any multiples of A and B, it turns out that they must compare the same way as the corresponding multiples of C and D.
Therefore A : B = C : D. (Def. 8)

Q.E.D.

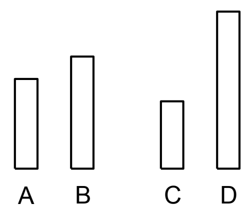
THEOREM 3 Remarks

1. That two quantities equal to the same thing are equal to each other is a self-evident axiom. But the fact that two ratios the same as a third ratio are also the same as each other needs to be proved, because "same ratio" has a rather involved definition.

2. Notice that it is not necessary for all 4 quantities to have a ratio in order for the Theorem to be true. For example, two lines may have the same ratio as two numbers (for example, the ratio of being double), and two areas may also have that same ratio. Then it will follow that the two lines and the two areas have the same ratios, even though neither line has any ratio to either area.

THEOREM 4: *No ratio can be greater than another ratio, and at the same time less than it.*

If possible, suppose that $A : B > C : D$
and yet at the same time $A : B < C : D$.
Watch what follows from each supposition.



For some pair of numbers, say 7 and 3,

$7A > 3B$ but $7C \leq 3D$ (since $A : B > C : D$; Def. 11)

Again, for some pair of numbers, call them N and M,

$NA \leq MB$ but $NC > MD$ (since $A : B < C : D$; Def.11)

Now, multiply both sides of the top two inequalities by N, and multiply both sides of the bottom two inequalities by 7, and we get four new inequalities:

$$\begin{array}{llll} & 7NA > 3NB & & 7NC \leq 3ND \\ \text{and} & \underline{7NA \leq 7MB} & & \underline{7NC > 7MD} \\ \text{so} & 7MB > 3NB & \text{and} & 3ND > 7MD, \end{array}$$

in each case taking the first and last terms among three unequal things.

Hence $7M > 3N$ and $3N > 7M$,
because in each case the greater multiple (whether of B or of D) must have a greater multiplier. But the number 7M cannot be both greater than and less than the number 3N. So it is also impossible that one ratio should be both greater than and less than another ratio.

Q.E.D.

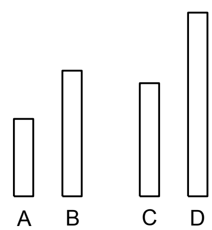
THEOREM 4 Remarks

We can now see that just as any two comparable quantities must either be equal, or one greater than the other, likewise any two ratios must either be the same or one must be greater than the other. Consider any two ratios, A : B and C : D. Either all multiples of C and D corresponding to multiples of A and B compare in the same way, or some don't. If all do, then the two ratios are the same (Def. 8). If some do not, then one ratio will be greater, and by the present Theorem, the other must be less.

THEOREM 5: *If the multiples of two quantities are equal, and the corresponding multiples of two other quantities are also equal, then the four original quantites are proportional.*

Given: Quantities A, B, C, D.
Numbers n and m such that
nA = mB and nC = mD

Prove: A : B = C : D



[1] Take any random multiples of A and B, say xA and yB, where x and y are any numbers you please. Suppose first that xm > ny.

[2] Then xmB > nyB and xmD > nyD
since the greater multiplier yields a greater multiple of the same thing.

[3] But xmB = xnA and xmD = xnC
since we are given that mB = nA and mD = nC, and we have multiplied them all by x.

[4] Hence xnA > nyB and xnC > nyD
putting together Steps 2 and 3.

[5] Dividing all by n, we have
xA > yB and xC > yD
Likewise if we assume instead that xm < ny, then
xA < yB and xC < yD
and if we assume instead that xm = ny, then
xA = yB and xC = yD

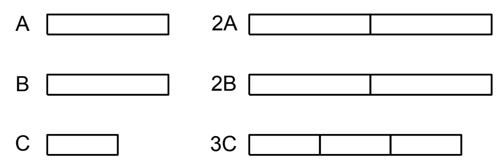
So, taking random multiples of A and B, the corresponding multiples of C and D must compare in the same way. Hence
A : B = C : D (Def. 8)

Q.E.D.

THEOREM 5 Remarks:

Verify the theorem concretely by finding a numerical example.

THEOREM 6: *Equal quantities have the same ratio to any one quantity.*



Take any two equal quantities A and B, and any third quantity C to which they have a ratio. I say that A : C = B : C.

[1] Take any multiples of A and C, say 2A and 3C.
Take the corresponding multiple of B, namely 2B.

[2] Since A = B (given)
thus 2A = 2B (equimultiples of equal things are equal)

[3] So if $2A > 3C$, then also $2B > 3C$ (since 2A = 2B)
but if $2A < 3C$, then also $2B < 3C$ (since 2A = 2B)
and if 2A = 3C, then also 2B = 3C (since 2A = 2B)

[4] Thus any multiples of A and C must compare the same way as the corresponding multiples of B and C. Therefore

A : C = B : C (Def. 8).

Q.E.D.

THEOREM 6 Remarks

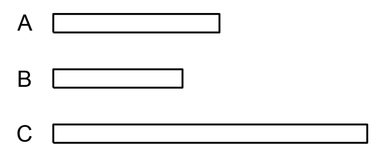
1. We used the numbers 2 and 3 in this Theorem, but there is nothing special about them – if you look at the proof carefully, you will see that no Step depends on those specific numbers. We could have used general symbols for any two numbers (such as n and m) just as easily.

2. With numbers, this Theorem is an especially trivial piece of obviousness, i.e. equal numbers have the same ratio to the same number:

5 : 7 = 5 : 7.

3. It should also be clear that the same quantity has the same ratio to equal quantities: If A = B, then C : A = C : B. For, by the present Theorem, A : C = B : C, but by Theorem 2, these are proportional inversely, i.e. C : A = C : B.

THEOREM 7: *How to find equimultiples of two unequal quantities such that the multiple of the greater will be greater than some multiple of a third quantity, while the multiple of the lesser will be less.*



Suppose A > B, and C is any third quantity comparable to them.

Is it possible to multiply both A and B by the same number N, and C by some number R, such that

NA > RC

but NB < RC ?

Believe it or not, *no matter how close* A *and* B *are to being equal*, no matter how small the difference between them is, we can always do it. Here's how:

[1] There must be a first multiple of (A – B) that is greater than C.
Let it be 9(A – B).
So 9(A – B) > C.

[2] There must also be a first multiple of C that is greater than 9B.
Let it be 4C.
Since 4C is the *first* multiple of C that exceeds 9B,
hence 9B is greater than or equal to 3C
so C + 9B is greater than or equal to 4C (adding C to each)

[3]	Now	9(A – B) > C	(Step 1)
	so	9A – 9B > C	(Thm.1)
	so	9A > C + 9B	(adding 9B to each)
	but	C + 9B ≥ 4C	(Step 2)
	so	9A > 4C	

[4]	So	9A > 4C	(Step 3)
	but	9B < 4C	(Step 2)

We did it!

Q.E.F.

THEOREM 7 Remarks:

1. As with Theorem 6, there is nothing special about the numbers 9 and 4 occurring in this Theorem. Use *n* and *m*, or some other pair of numbers, if you prefer.

2. Let's try a numerical example. Let

$A = 20$
$B = 19$
$C = 23$

We want to multiply 20 and also 19 by the same number, so that the multiple of 20 is greater than some multiple of 23, while the multiple of 19 is less.

Following what we did in the Theorem, $(A - B) = 1$. And the first multiple of this that is greater than 23 is 24. That is Step [1]. Following Step [2], we next find the first multiple of 23 that is greater than 19×24. Well,

$19 \times 24 = 456$

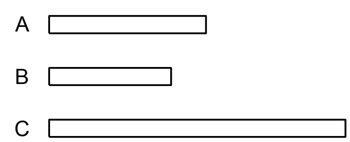
and $20 \times 23 = 460$, so that is the first multiple of 23 that is greater than 19×24.

According to the Theorem, the multiples satisfying the requirements are:

$24 \times A = 480$
$24 \times B = 456$
$20 \times C = 460$

3. Find a way to multiply 36 and 35 by the same number so that the multiple of 36 exceeds some multiple of 57, but the multiple of 35 does not.

THEOREM 8: *Of two unequal quantities, the greater one has a greater ratio to any other quantity than the lesser one has to it.*



Let A and B be unequal quantities, A the greater one, and C any other quantity comparable with them. Then

$A : C > B : C.$

The proof is as follows.

[1] Since $A > B$, therefore we can find a multiple of A that is greater than some multiple of C, while the same multiple of B is less (Thm.7). Suppose then that

$5A > 3C$

but $5B < 3C$.

[2] Therefore it is possible to take a multiple of A that exceeds a multiple of C (i.e. $5A > 3C$), while the corresponding multiple of B does not exceed that same multiple of C (i.e. $5B < 3C$). Therefore

$A : C > B : C$ (Def. 9)

Q.E.D.

THEOREM 8 Remarks

1. Here is a numerical example. Since 7 > 3, then if we take any third number, such as 5, it follows that 7 : 5 > 3 : 5.

2. Also, *of two unequal quantities, any other quantity has to the lesser one a greater ratio.* Say A and B are unequal, A the greater, and C is any third quantity comparable to them. Then C : B > C : A. Why?

Since A > B, we can find a multiple of A that is greater than some multiple of C, while the same multiple of B is less (Thm.7). Say

3C > 5B

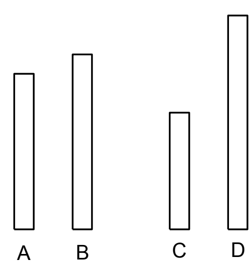
but 3C < 5A.

Therefore it is possible to take multiples of C and B (namely 3C and 5B), such that the multiple of C is greater than that of B, while the same multiple of C is less than the corresponding multiple of A. Therefore

C : B > C : A (Def. 10).

THEOREM 9: *A first quantity has to a second quantity a greater ratio than a quantity less than the first has to a quantity greater than the second.*

Suppose A and B are any two quantities having a ratio,
and C < A
and D > B.
Then A : B > C : D.



Here's why:

[1] Since A > C, (given)
therefore it is possible to find some multiple of A that exceeds a certain multiple of B, while the same multiple of C falls short of it (Thm.7). Suppose, then, that

5A > 2B
while 5C < 2B

[2] Since D > B, (given)

thus	2D > 2B	(doubling each)
but	$\underline{5C < 2B}$	(Step 2)
so	2D > 5C	

[3] So $5A > 2B$ (Step 1)
but $5C < 2D$ (Step 2)

And therefore it is possible to take a multiple of A that is greater than some multiple of B whereas the corresponding multiple of C is *less* than the corresponding multiple of D.

Therefore $A : B > C : D$ (Def.10)

Q.E.D.

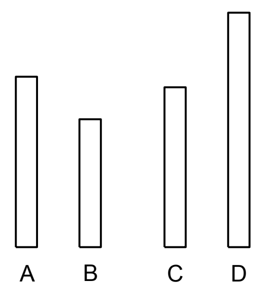
THEOREM 9 Remarks

1. Here is a numerical example.
$4 < 9$
and $12 > 3$,
so $9 : 3 > 4 : 12$.

2. Also, if $A > B$ and C is not greater than D, then $A : B > C : D$. For if C is not greater than D, it is either equal to it, or less than it.

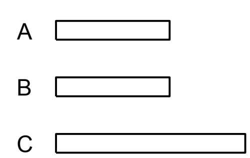
First suppose that $C = D$. Then $3A > 3B$, but $3C = 3D$; and thus a multiple of A is greater than a multiple of B, while the corresponding multiple of C is not greater than the corresponding multiple of D. Therefore $A : B > C : D$ (Def. 10).

Now suppose that $C < D$. Then $3A > 3B$, but $3C < 3D$, and so again $A : B > C : D$ (Def. 10).



THEOREM 10: *Quantities having the same ratio to the same quantity are equal.*

Suppose two quantities A and B have the same ratio to C, that is $A : C = B : C$.
Then $A = B$.



Proof:

[1] If A were *greater* than B, then $A : C > B : C$ (Thm.8)
But in fact $A : C = B : C$ (given)
Therefore A is *not* greater than B.

[2] If A were *less* than B, then $A : C < B : C$ (Thm.8)
But in fact $A : C = B : C$ (given)
Therefore A is *not* less than B.

[3] Since A is not greater than B, nor is it less than B, therefore it is equal to B.

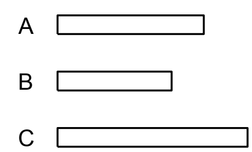
Q.E.D.

THEOREM 10 Remarks

1. This Theorem is the converse of Theorem 6.

2. This Theorem is an important tool for proving the equality of all kinds of quantities. Whenever we can derive a proportion such as this: $X : Z :: Q : Z$, we can conclude from this Theorem that $X = Q$.

THEOREM 11: *Of two quantities, the one having a greater ratio to any third quantity is greater.*



Suppose you have three quantities A, B, C, and $A : C > B : C$.
Then $A > B$.

Proof:

[1] If A were *equal* to B, then $A : C = B : C$ (Thm.6)
But in fact $A : C > B : C$ (given)
Therefore A is *not* equal to B.

[2] If A were *less* than B, then $A : C < B : C$ (Thm.6)
But in fact $A : C > B : C$ (given)
which is impossible, since two ratios cannot be greater than each other (Thm. 4).
Therefore A is *not* less than B.

[3] Since A is not equal to B, nor is it less than B, therefore it is greater than B.

Q.E.D.

THEOREM 11 Remarks

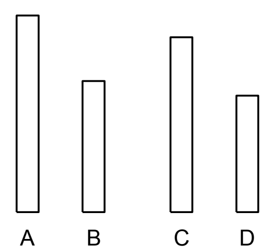
Also, *of two quantities, the one to which any third quantity has a lesser ratio is greater*.
Suppose that C : B > C : A. Then it has to be that A > B. Why?

(1) If A = B, then C : B = C : A (Thm.6, Remark 3)
but C : B > C : A (given)
Therefore A is not equal to B.

(2) If A < B, then C : A > C : B (Thm.8, Remark 2)
but C : B > C : A (given)
and two ratios cannot be greater than each other (Thm.4).
Therefore A is not less than B.

(3) Since A is not equal to B (Step 1), and A is not less than B (Step 2),
therefore A is greater than B.

THEOREM 12: *If the first term in a proportion is greater than the third, then the second will be greater than the fourth (but if less, less, and if equal, equal).*



Suppose A : B = C : D
and A > C.
Then also B > D.

Proof:

[1] First try to assume that B = D.

So	B = D	(assumed)
But	A > C	(given)
so	A : B > C : B	(Thm.8)
or	A : B > C : D	(assuming B = D)
But	A : B = C : D	(given)
So	B is not equal to D.	

[2] Next try to assume that B < D.

So	B < D	(assumed)
But	A > C	(given)
so	A : B > C : D	(Thm.9)
But	A : B = C : D	(given)

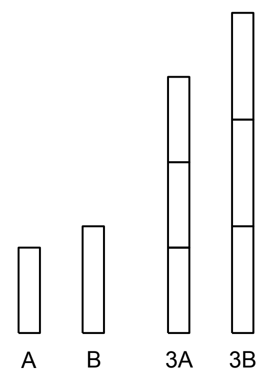
So B is not less than D.

[3] Since B is not equal to D (Step 1)
and B is not less than D (Step 2)
thus B > D.

[4] Similarly, we can prove that
if A < C, then also B < D,
and if A = C, then also B = D.

Q.E.D.

THEOREM 13: *Any two quantities have the same ratio as their equimultiples, taken in corresponding order.*



Let A and B be any two quantities in a ratio, and take any equimultiples of them, say 3A and 3B.
I say that A : B = 3A : 3B.

[1] Take any multiples of A and B, such as 2A and 5B.
Take the corresponding multiples of 3A and 3B, namely 2(3A) and 5(3B).

[2] Now, if 2A > 5B
then 3(2A) > 3(5B)
since equimultiples of unequals are unequal in the same order.
Thus 2(3A) > 5(3B)
since the order of multiplication makes no difference (as we noted in the Remarks to Theorem 1). In sum,
if 2A > 5B then 2(3A) > 5(3B).

[3] Likewise, if 2A < 5B then 2(3A) < 5(3B),
and if 2A = 5B then 2(3A) = 5(3B).

[4] So, taking any multiples of A and B, it turns out that the corresponding multiples of 3A and 3B must compare the same way. Therefore A : B = 3A : 3B (Def. 8).

Q.E.D.

THEOREM 13 Remarks

1. It follows also that two quantities have the same ratio as their corresponding measures. For example: $A : B = \frac{1}{3}A : \frac{1}{3}B$.

The reason is that A and B are *equimultiples* of $\frac{1}{3}A$ and $\frac{1}{3}B$, and so, by the present Theorem, the four of them are proportional.

2. Clearly, too, if $A : B = C : D$, then any equimultiples of A and B will be proportional to any multiples of C and D. For example, $5A : 5B = 3C : 3D$.

For	$5A : 5B = C : D$	(Thm.13)
but	$\underline{3C : 3D = C : D}$	(Thm.13)
hence	$5A : 5B = 3C : 3D$	(Thm.3)

3. Also, *Corresponding multiples of proportional quantities are proportional.*

i.e. if $A : B = C : D$

then $3A : 2B = 3C : 2D$

(I chose 3 and 2 for my multiples, here, but any numbers would do).

To prove this, simply take any multiples of $3A$ and $2B$, say $5(3A)$ and $6(2B)$.

Take also the corresponding multiples of $3C$ and $2D$, i.e. $5(3C)$ and $6(2D)$.

Now if	$5(3A) > 6(2B)$	
then	$15A > 12B$	(simply by multiplying)
thus	$15C > 12D$	(since $A : B = C : D$)
so	$5(3C) > 6(2D)$	

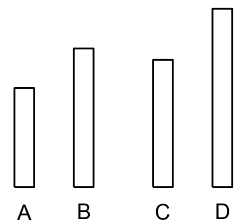
So if $5(3A) > 6(2B)$ then $5(3C) > 6(2D)$.

In short, any random multiples being taken of 3A and 2B, the corresponding multiples of 3C and 2D must compare the same way. Therefore

$$3A : 2B = 3C : 2D.$$

THEOREM 14: *Alternating the terms in a proportion makes a new proportion.*

Suppose that A : B = C : D.
Then also A : C = B : D.



Here's why:

[1] Take any multiples of A and C, say 7A and 2C.
Take the corresponding multiples of B and D, namely 7B and 2D.

[2]	Now, since	A : B = C : D,	(given)
	then	7A : 7B = 2C : 2D	(Thm.13, Remark 2)
[3]	So	if 7A > 2C, then 7B > 2D	(Thm.12)
	but	if 7A < 2C, then 7B < 2D	(Thm.12)
	and	if 7A = 2C, then 7B = 2D	(Thm.12)

[4] So, taking any multiples of A and C, it turns out that the corresponding multiples of B and D must compare in the same way. Therefore

A : C = B : D (Def. 8)

Q.E.D.

THEOREM 14 Remarks

1. Of course, the proportion will not alternate unless all four quantities are comparable. For example, if two lines have the same ratio as two areas, we cannot alternate the proportion, since then we would be saying that a line has to an area the same ratio that another line has to another area, whereas, in fact, a line does not have *any* ratio to an area at all.

2. The new ratios resulting from alternating the terms in the original proportion do not have to be the same as the ratios in the original proportion. For example, given

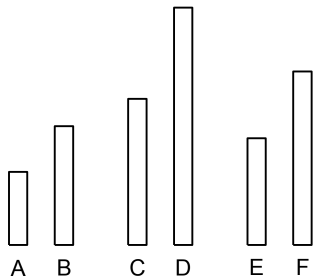
2 : 4 = 3 : 6,

if we alternate that proportion, we get a new proportion:

2 : 3 = 4 : 6.

But these two new ratios are not the same as the two original ratios.

THEOREM 15: *If any two ratios are the same as a third, then the sums of their corresponding terms also have that same ratio.*



Suppose that	$A : B = C : D$
and also that	$A : B = E : F$
Then	$A : B = C + E : D + F$

[1] Take any multiples of A and B, say 2A and 3B.
Take the corresponding multiples of C, D, E, F, namely 2C, 3D, 2E, 3F.

[2]	If	$2A > 3B$,	
	then	$2C > 3D$	(since $A : B = C : D$)
	and	$2E > 3F$	(since $A : B = E : F$).
	But then	$2C + 2E > 3D + 3F$,	(adding)
	that is	$2(C + E) > 3(D + F)$	(Thm.1)

[3] So if	$2A > 3B$,	then	$2(C + E) > 3(D + F)$.
Likewise if	$2A < 3B$,	then	$2(C + E) < 3(D + F)$.
and if	$2A = 3B$,	then	$2(C + E) = 3(D + F)$.

[4] Therefore, taking any multiples of A and B, we find that whenever they are unequal, the corresponding multiples of (C + E) and (D + F) are also unequal and in the same order. Therefore $A : B = C + E : D + F$.
Q.E.D.

THEOREM 15 Remarks

1. Obviously C D E F must be the same kinds of quantities so we can *add* them together. As an example, let's use numbers:

$1 : 2 = 3 : 6$
and $1 : 2 = 5 : 10$
so $1 : 2 = 3 + 5 : 6 + 10$.

2. Another similar and important Theorem is this:
Given: $A : B = C : D$
Prove: $A + B : B = C + D : D$

Take any multiples of A + B and B, e.g. 2(A + B) and 3B.
Take the corresponding multiples of C + D and D, i.e. 2(C + D) and 3D.

Now, if	$2(A + B) > 3B$	
then	$2A + 2B > 3B$	(Thm.1)
so	$2A > B$	(subtracting 2B from each side)
thus	$2C > D$	(since $A : B = C : D$)
so	$2C + 2D > 3D$	(adding 2D to each side)
i.e.	$2(C + D) > 3D$	(Thm.1)

So	if $2(A + B) > 3B$	then $2(C + D) > 3D$.
Likewise	if $2(A + B) < 3B$	then $2(C + D) < 3D$.
and	if $2(A + B) = 3B$	then $2(C + D) = 3D$.

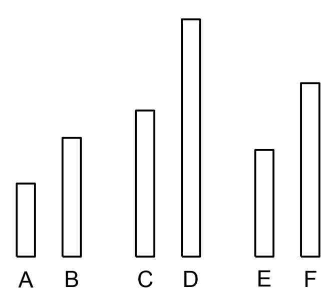
That is, taking any random multiples of (A + B) and B, they must compare the same way as the corresponding multiples of (C + D) and D. Therefore $A + B : B = C + D : D$.

Q.E.D.

Notice that this Theorem does not depend on A, B, C, and D being all comparable quantities. A and B must be comparable, and C and D also, but A and B might be volumes, and C and D might be areas.

THEOREM 16: *If any two ratios are the same as a third, then the differences of their corresponding terms also have that same ratio.*

Suppose that $A : B = C : D$
and also that $\underline{A : B = E : F}$
then $A : B = (C - E) : (D - F)$



We will start by proving that (C – E) and (D – F) have the same ratio as E and F, and from there it will be easy to show they have the same ratio as A and B. Accordingly,

[1] Take any multiples of (C – E) and E, say 3(C – E) and 5E.

[2] Now, $C : D = E : F$,
as is clear from what is given. Therefore, alternately

	$C : E = D : F$			(Thm.14)
So if	$3C > (5 + 3)E$	then	$3D > (5 + 3)F$	(Def. 8)
i.e if	$3C > 5E + 3E$	then	$3D > 5F + 3F$	(Thm.1)
so if	$3C - 3E > 5E$	then	$3D - 3F > 5F$	(subtracting 3E and 3F)
so if	$3(C - E) > 5E$	then	$3(D - F) > 5F$	(Thm.1)

[3] By the same kind of argument,

if	$3(C - E) < 5E$	then	$3(D - F) < 5F$
and if	$3(C - E) = 5E$	then	$3(D - F) = 5F$

[4] So, taking random multiples of $(C - E)$ and E, they must compare the same way as the corresponding multiples of $(D - F)$ and F. Therefore

$(C - E) : E = (D - F) : F$ (Def. 8)

[5] Alternating this proportion, we have

$(C - E) : (D - F) = E : F$ (Thm.14)

[6] But since $E : F = A : B$ (given), it follows that

$A : B = (C - E) : (D - F)$ (Thm.3)

Q.E.D.

THEOREM 16 Remarks

1. Here is a numerical example:

$1 : 2 = 7 : 14$
and $\underline{1 : 2 = 3 : 6}$
so $1 : 2 = 7 - 3 : 14 - 6$

2. In Step 5 we alternated a proportion. How do we know that C, D, E, and F are all comparable quantities, such that we *can* alternate the proportion? We must presume this in what we are given: if they were not all comparable, then we could not even subtract E from C, for example.

3. A similar and important Theorem is this:

Given: $A : B = C : D$.
Prove: $A - B : B = C - D : D$.

Take any multiples of $A - B$ and B, e.g. $2(A - B)$ and $3B$.
Take the corresponding multiples of $C - D$ and D, i.e. $2(C - D)$ and $3D$.

Now, if $2(A - B) > 3B$

then	$2A - 2B > 3B$	(Thm.1)
so	$2A > 5B$	(adding 2B to each side)
thus	$2C > 5D$	(since $A : B = C : D$)
so	$2C - 2D > 3D$	(subtracting 2D from each side)
i.e.	$2(C - D) > 3D$.	(Thm.1)

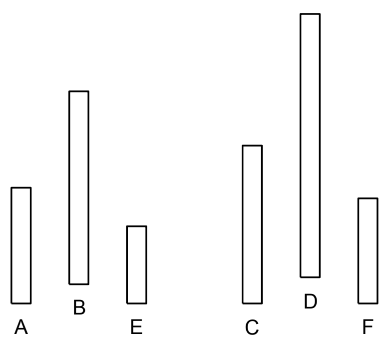
so if	$2(A - B) > 3B$	then $2(C - D) > 3D$.
Likewise if	$2(A - B) < 3B$	then $2(C - D) < 3D$
and if	$2(A - B) = 3B$	then $2(C - D) = 3D$

That is, taking any random multiples of $(A - B)$ and B, we find they must compare the same way as the corresponding multiples of $(C - D)$ and D.
Therefore $A - B : B = C - D : D$ (Def. 8)

Q.E.D.

FORMING A PROPORTION "EX AEQUALI"

THEOREM 17: *If the antecedents in one proportion are also the antecedents in a second proportion, then a new proportion arises by taking the consequents of the first proportion as antecedents, and the consequents of the second proportion as consequents.*



Suppose	$A : B = C : D$
and also	$\underline{A : E = C : F}$
then	$B : E = D : F$

Take any multiples of B and E, say 2B and 3E.
Take the corresponding multiples of D and F, i.e. 2D and 3F.

[1]	Now, if	$2B > 3E$	
	then	$A : 2B < A : 3E$	(Thm.8, Remark 2)
	but	$\underline{A : 2B = C : 2D}$	(since $A : B = C : D$; & Thm.13 Remark 3)
	thus	$C : 2D < A : 3E$	

[2]	So	$C : 2D < A : 3E$	(Step 1)
	but	$\underline{A : 3E = C : 3F}$	(since $A : E = C : F$; & Thm.13 Remark 3)
	thus	$C : 2D < C : 3F$	
	therefore	$2D > 3F$	(Thm.11 Remarks)

[3]	Thus if	$2B > 3E$	then	$2D > 3F$	(Steps 1-2)
	Likewise if	$2B < 3E$	then	$2D < 3F$	
	and if	$2B = 3E$	then	$2D = 3F$	

[4] That is, taking any random multiples of B and E, they must compare the same way as the corresponding multiples of D and F.
Therefore $B : E = D : F$. (Def. 8)

Q.E.D.

THEOREM 17 Remarks

1. Here is a numerical example:

$3 : 6 = 4 : 8$
and $\underline{3 : 9 = 4 : 12}$
so $6 : 9 = 8 : 12$

2. According to the Theorem, whenever two proportions have equal antecedents, their consequents form a proportion (taken in proper order). We are actually forming a proportion just by "dropping out" the 2 pairs of identical antecedents in the original two proportions, leaving us with just the consequents. Ancient mathematicians called this procedure "forming a proportion *ex aequali.*"

3. It is not necessary for the antecedents of the two proportions to be the *same* in order for the Theorem to work; they need only be *proportional*. Thus

if $A : B = C : D$
and $E : F = G : H$
and $\underline{A : E = C : G}$ (i.e. the antecedents of the first two proportions are proportional)
then $B : F = D : H$ (i.e. the consequents of the first two proportions are proportional).

If you don't believe it, here is the proof:

$A : B = C : D$ (given)
$\underline{A : E = C : G}$ (given)
so $B : E = D : G$ (Thm.17)
so $E : B = G : D$ (Thm.2)

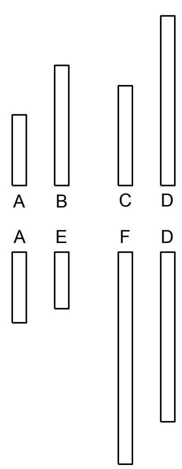
but E : F = G : H (given)
so B : F = D : H (Thm.17). *Q.E.D.*

4. If B and D happen to be comparable quantities (as opposed to being a line and an area, for example), the proof for Theorem 17 is much simpler:

	A : C = B : D	(Alternate of the first given proportion)
	A : C = E : F	(Alternate of the second given proportion)
so	B : D = E : F	(Thm.3)
thus	B : E = D : F	(Alternating)

FORMING A PROPORTION BY TRACING THE LETTER "U"

THEOREM 18: *If the extreme terms in one proportion are also the extreme terms in a second proportion, a new proportion arises by taking the middle terms of the first proportion as extremes, and the middle terms of the second proportion as middles.*



Suppose that	A : B = C : D
and also that	A : E = F : D
then	B : E = F : C

Let's adopt the usual method, and try to show that B : E = F : C by proving that random multiples of B and E must compare the same way as the corresponding multiples of F and C.

Take any multiples of B and E, say 3B and 5E.
Take the corresponding multiples of F and C, namely 3F and 5C.

[1]	Now if	3B > 5E	
	then	A : 5E > A : 3B	(Thm.8, Remark 2)
	but	A : 3B = C : 3D	(A : B = C : D; Thm.13 Remark 3)
	so	A : 5E > C : 3D	
	but	A : 5E = F : 5D	(A : E = F : D; Thm.13 Remark 3)
	thus	F : 5D > C : 3D	

[2]	So	$C : 3D < F : 5D$	(Step 1)
	but	$\underline{F : 5D = 3F : 15D}$	(Thm.13; multiplying by 3)
	thus	$C : 3D < 3F : 15D$	

[3]	So	$C : 3D < 3F : 15D$	(Step 2)
	but	$\underline{C : 3D = 5C : 15D}$	(Thm.13; multiplying by 5)
	thus	$5C : 15D < 3F : 15D$	
	therefore	$3F > 5C$	(Thm.11)

[4]	Thus if	$3B > 5E$	then	$3F > 5C$	(Steps 1 – 3)
	Likewise if	$3B < 5E$	then	$3F < 5C$	
	And if	$3B = 5E$	then	$3F = 5C$	

[5] That is, taking any random multiples of B and E, they must compare to each other the same way as the corresponding multiples of F and C compare to each other.

Therefore $B : E = F : C$ (Def. 8)

Q.E.D.

THEOREM 18 Remarks

1. Here is a numerical example:

	$4 : 3 = 16 : 12$
and	$\underline{4 : 2 = 24 : 12}$
so	$3 : 2 = 24 : 16$

2. As a mnemonic device for remembering the proportion that arises out of the original two proportions, think of it as forming a "letter U": if you look at B, E, F, and C, in the original two given proportions, you are tracing out a letter "U". And that is the order in which they form a proportion: B : E = F : C. So we might call this "The U-Theorem", and we might call the procedure of forming a proportion in this way "Tracing the U". Ancient geometers called this a "perturbed proportion."

3. For the Theorem to work, the extremes need not be the *same*, but only *proportional* in the order of the letter U. Thus

if	$A : B = C : D$	
and	$E : F = G : H$	
and	$\underline{A : E = H : D}$	(i.e. extremes of the 1st two proportions are proportional)
then	$B : F = G : C$	(i.e. the middles of the 1st two proportions are proportional)

The proof for it is as follows.

First	$A : B = C : D$	(given)
and	$\underline{A : E = H : D}$	(given)
so	$B : E = H : C$	(Thm.18)
so	$E : B = C : H$	(inverting)
but	$\underline{E : F = G : H}$	(given)
so	$B : F = G : C$	(Thm.18)

Q.E.D.

"HOOK": THREE MEANS.

There are many different kinds of "means" or "middles" between any two comparable magnitudes, e.g. between two line-lengths. Three of the most interesting are the arithmetic mean (or average), the geometric mean (or mean proportional), and the harmonic mean:

ARITHMETIC MEAN

Definition: $m_a - a = b - m_a$

Formula: $m_a = (a + b)/2$

GEOMETRIC MEAN

Definition: $a : m_g = m_g : b$

Formula: $m_g = \sqrt{ab}$

HARMONIC MEAN

Definition: $(m_h - a) : a = (b - m_h) : b$

Formula: $m_h = (2ab)/(a+ b)$

There are many interesting theorems relating these means. Here is one of them:

THEOREM: The product of two extremes is equal to the product of their arithmetic and harmonic means.

Given: extremes a and b,
their arithmetic mean: $(a + b)/2$
their harmonic mean: $(2ab)/(a + b)$

Proof:

Multiply the two formulas as given, and the $(a + b)$ terms cancel out, and so do the 2s, leaving just (ab), the product of the extremes.

Q.E.D.

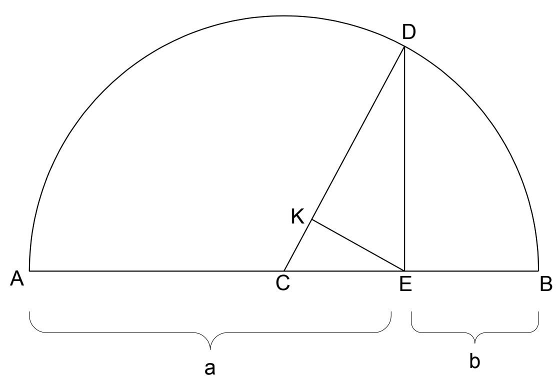
PORISM 1: *The geometric mean* m_g *is the geometric mean not only between the original extremes* a *and* b, *but also between the arithmetic and harmonic means between* a *and* b.

For $\quad ab = m_a m_h \quad$ (Theorem above)

so $\quad (\surd a\cdot b)(\surd a\cdot b) = m_a m_h$

so $\quad m_a : m_g = m_g : m_h$

Q.E.D.



PORISM 2:

In the accompanying figure, if a and b are our extremes,
then CD is their arithmetic mean,
DE is their geometric mean,
DK is their harmonic mean.

Since CD = CA = (a + b)/2, the radius is the arithmetic mean.

And DE = $\sqrt{a \cdot b}$ = the geometric mean between a and b.

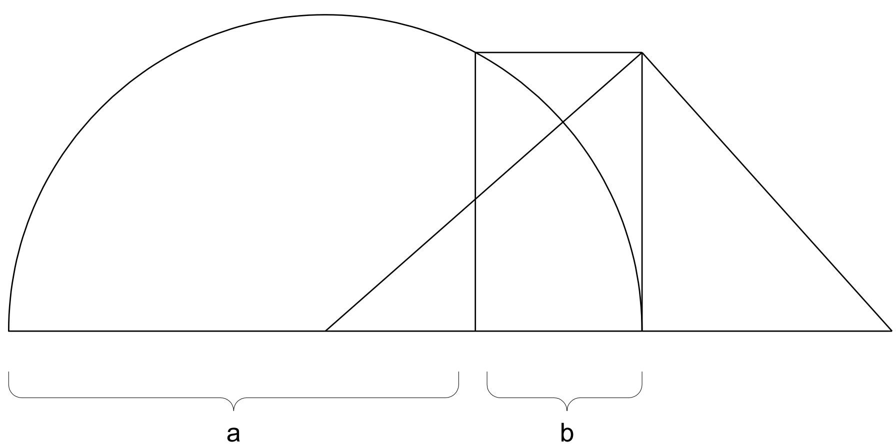
And DK is the 3rd proportional such that

CD : DE = DE : DK

i.e. it is the 3rd proportional from the arithmetic and geometric means between a and b. But only the harmonic mean between a and b can be that (according to Porism 1 above). Hence DK is the harmonic mean between a and b.

Q.E.D.

Again, it is plain that the following figure exhibits the three means between a and b:

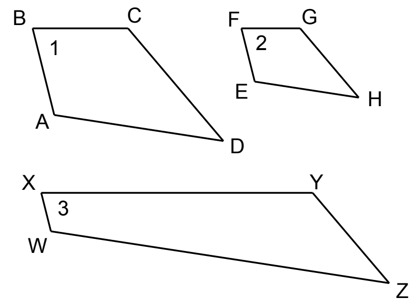


Chapter Six

Proportions in Plane Geometry

DEFINITIONS

1. SIMILAR RECTILINEAL FIGURES are those with corresponding angles equal and the sides about the equal angles proportional.



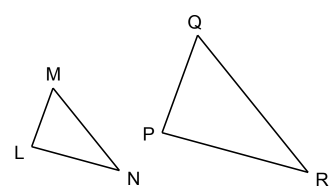
For example, quadrilateral ABCD *is similar to quadrilateral* EFGH, *because all their corresponding angles are equal* (e.g. $\angle 1 = \angle 2$) *AND the sides about the equal angles are proportional when taken in corresponding order, i.e.* AB : BC = EF : FG.

On the other hand, quadrilateral WXYZ, *although equiangular with* ABCD, *is NOT similar to it, because the sides about its angles have ratios different from the ratios of the sides about the angles in* ABCD. *For example*, $\angle 3 = \angle 1$, *but* AB : BC *is not the same as* WX : XY.

Also, the sides of a rhombus are proportional to the sides of a square, since they are all equal, but a rhombus is not similar to a square, since its angles differ from those of a square.

From these examples, it should be clear that similar figures are the same shape, but can have different sizes.

2. If ΔLMN is similar to ΔPQR, we can use the symbol ~ to designate that relationship. So "ΔLMN ~ ΔPQR" means "Triangle LMN *is similar to* triangle PQR".



3. Quantities are CONTINUOUSLY PROPORTIONAL when they form a proportion in which the consequent of the first ratio is the antecedent of the next ratio.

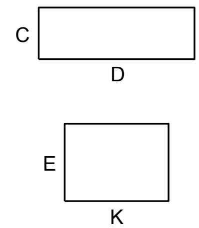
For example, 1, 2, 4 *are continuously proportional, because* 1 : 2 = 2 : 4. *Also,* 3, 6, 12, 24 *are continuously proportional, because* 3 : 6 = 6 : 12 = 12 : 24.

4. To DUPLICATE a ratio means to find a third term in continuous proportion with the two terms in the original ratio, and to form a new ratio of the first term to the third term.

For example, to duplicate the ratio of 1 : 2, *we find a third term in continuous proportion with them, namely* 4, *since* 1 : 2 = 2 : 4, *and so* 1 : 4 *is the duplicate ratio of* 1 : 2.

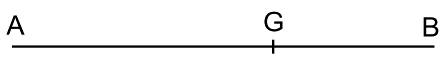
Similarly, if we find two ratios in continuous proportion with the original ratio, as in 1 : 2 = 2 : 4 = 4 : 8, *then the extreme terms are said to have the triplicate ratio of the original terms, i.e.* 1 : 8 *is the triplicate ratio of* 1 : 2.

5. Two figures have RECIPROCALLY PROPORTIONAL SIDES if a side in the first figure is to a side in the second figure as another side in the second figure is to another side in the first figure.
For example, the sides of rectangles C · D *and* E · K *are reciprocally proportional if* C : E = K : D.



6. A straight line is CUT IN MEAN AND EXTREME RATIO if the whole of it has to the greater part of it the same ratio that the greater part of it has to the remaining part of it.

For example, AB *is cut in mean and extreme ratio at* G *if* BA : AG = AG : GB.



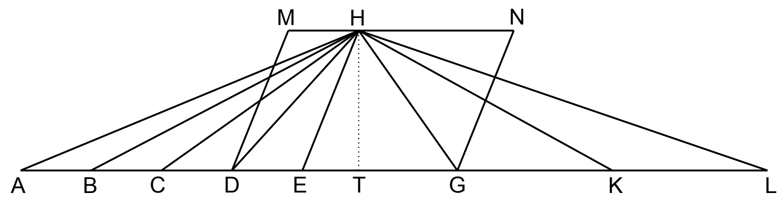
AG is called the GREATER SEGMENT of AB, and GB is called the LESSER SEGMENT.

Because of the beauty of this ratio (which is used in art and architecture and is approximated by some parts of the human body and other things in nature), it is called the GOLDEN RATIO. So a GOLDEN RECTANGLE is one whose sides are to each other in the golden ratio.

THEOREMS

THEOREM 1: *Triangles having the same height are to each other as their bases. Parallelograms having the same height are to each other as their bases.*

Triangles DHE and EHG have the same height, HT.
I say that △DHE : △EHG = DE : EG, that is, their areas have the same ratio that their bases do.



[1] Extend the bases both ways.
Cut off any number of parts equal to DE, say DC, CB, BA.
Cut off any number of parts equal to EG, say GK, KL.

[2] Since AB = DE, therefore △AHB = △DHE (Ch.1, Thm.33)
Likewise △BHC and △CHD also equal △DHE.

[3] Since GK = EG, therefore △GHK = △EHG (Ch.1, Thm.33)
Likewise △KHL = △EHG.

[4] So △EHA = 4 △DHE, and EA = 4DE
and △EHL = 3 △EHG, and EL = 3EG
and, in general, when we multiply the *base* of either of the original triangles, the triangle on that multiple base, and having height HT, also multiplies the *area* of the original triangle the same number of times.

[5] Suppose △EHA > △EHL (i.e. 4 △DHE > 3 △EHG),
then also EA > EL (i.e. 4 DE > 3 EG),
since the base of the greater triangle will be greater, just as the bases of equal triangles are equal, for triangles under the same height.

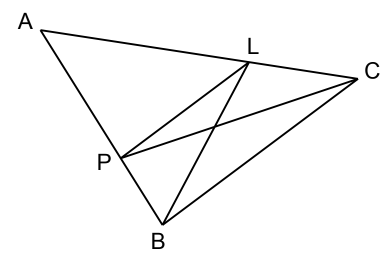
So if	4 △DHE > 3 △EHG,	then	4 DE > 3 EG.
Likewise, if	4 △DHE < 3 △EHG,	then	4 DE < 3 EG
and if	4 △DHE = 3 △EHG,	then	4 DE = 3 EG

[6] And so *whatever multiples* of △DHE and △EHG we take, however they compare, the corresponding multiples of their bases must compare the same way.
Therefore △DHE : △EHG = DE : EG (Ch.5, Def. 8)

[7] And since these triangles have the same ratio as their doubles (Ch.5, Thm.13), it follows that the parallelograms DMHE and EHNG, being the doubles of these triangles (Ch.1, Thm.33), also have the same ratio as the bases DE and EG.

Q.E.D.

THEOREM 2: *If a straight line is drawn inside a triangle parallel to one of the sides, it will cut the remaining sides proportionally; and if a straight line cut two sides of a triangle proportionally, it will be parallel to the remaining side.*



Take any triangle ABC. Draw PL parallel to BC.
I say that AP : PB = AL : LC.

Draw PC and BL.

[1] AP : PB = △APL : △PBL (Thm.1)

[2] △PBL = △LCP (same base, same height)

[3] AP : PB = △APL : △LCP (putting together Steps 1 and 2)

[4] △APL : △LCP = AL : LC (Thm.1)

[5] So AP : PB = AL : LC,
since each of these ratios is the same as △APL : △LCP (by Steps 3 and 4), and since two ratios the same with a third ratio are the same as each other (Ch.5, Thm.3).

And so the sides of the triangle have been cut proportionally. *Q.E.D.*

Again, suppose that in some triangle ABC the sides have been cut proportionally, such that AP : PB = AL : LC. I say that PL is parallel to BC.

Draw PC and BL.

[1]	AP : PB = AL : LC	(given)
	<u>AP : PB = △APL : △PBL</u>	(Thm.1)
so	AL : LC = △APL : △PBL	(Ch.5, Thm.3)
but	<u>AL : LC = △APL : △LCP</u>	(Thm.1)
so	△APL : △PBL = △APL : △LCP	

[2] Thus △PBL and △LCP have the same ratio to the same thing (namely to △APL).

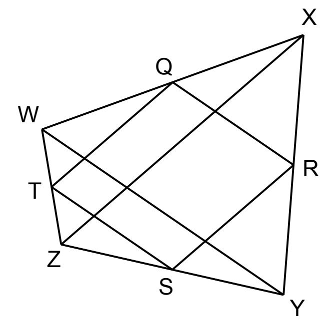
Thus △PBL = △LCP (Ch.5, Thm.10)

[3] But △PBL and △LCP are on the same base LP. Since they are equal, therefore they must also be under the same height (Ch.1, Thm.33 converse). Therefore P and L must each be the same height above BC, i.e. PL is parallel to BC. *Q.E.D.*

THEOREM 2 Remarks

1. Notice we have just proven a theorem and its converse together.

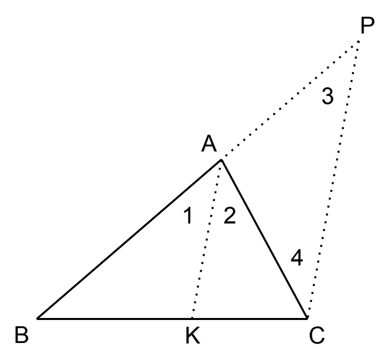
2. Using the second part of this Theorem, you can now easily prove the following interesting theorem: *If you bisect the sides of ANY quadrilateral WXYZ at the points Q, R, S, T, then QRST is a parallelogram*. Draw QR, RS, ST, TQ, WY, and XZ. Since WQ = QX and WT = TZ, therefore WQ : QX = WT : TZ. Thus QT is parallel to XZ in △WXZ. Finish the proof.



THEOREM 3: *The line bisecting an angle of a triangle cuts the base into two parts having the same ratio as the two sides.*

Conversely, if the base of a triangle is cut into two parts having the same ratio as the two sides, then the line joining the point of section to the opposite angle bisects the angle.

Take any triangle ABC. Bisect ∠BAC with line AK. I say BK : KC = BA : AC.
Draw CP || AK, cutting BA extended at P.



[1] Now $\angle 2 = \angle 4$ (CP || AK)
and $\underline{\angle 2 = \angle 1}$ (given)
so $\angle 1 = \angle 4$
but $\underline{\angle 1 = \angle 3}$ (CP || AK)
so $\angle 3 = \angle 4$

[2] Thus AP = AC (since $\angle 3 = \angle 4$ in Step 1)

[3] But BK : KC = BA : AP (since CP || AK; Thm.2)

[4] So BK : KC = BA : AC (putting together Steps 2 and 3)

Q.E.D.

Conversely, take any triangle ABC, and suppose that BK : KC = BA : AC.
I say that AK bisects ∠BAC, i.e. ∠1 = ∠2.
Draw CP || AK, cutting BA extended at P.

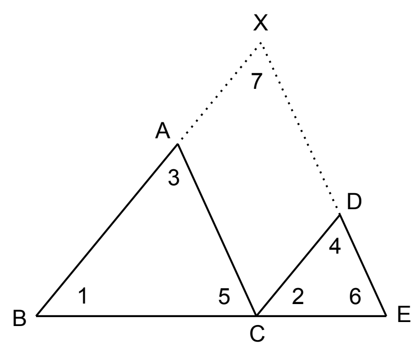
[1] Now BK : KC = BA : AC (given)
but BK : KC = BA : AP (since CP || AK, Thm.2)
so BA : AC = BA : AP (Ch.5: 2 ratios the same as a 3rd are the same)
so AC = AP (Ch 5: 2 quantities with the same ratio to the same quantity must be equal to each other).

[2] Thus ∠3 = ∠4 (angles opposite equal sides in a triangle are equal)
But ∠3 = ∠1 (since CP || AK)
so ∠1 = ∠4
but ∠2 = ∠4 (since CP || AK)
so ∠1 = ∠2

Q.E.D.

THEOREM 4: *Equiangular triangles are similar.*

Suppose ΔABC and ΔDCE are equiangular, i.e.
∠1 = ∠2
∠3 = ∠4
∠5 = ∠6
Then I say that the sides about their equal angles are proportional when taken in corresponding order, and thus △ABC ~ △DCE.



[1] Place BC and CE in a straight line.

[2] Extend BA and ED till they meet at X.

[3] Since ∠5 = ∠6 (given)
thus CA || EX. (Ch.1, Thm.24 Remark 1)

[4] Since ∠1 = ∠2 (given)
thus CD || BX (Ch.1, Thm.24 Remark 1)

[5]	So ACDX is a parallelogram.		
[6]	And	BA : AX = BC : CE	(CA $\parallel$ EX; Thm.2)
	But	AX = CD	(opposite sides of a parallelogram)
	so	BA : CD = BC : CE	
	thus	AB : BC = CD : CE	(alternating the proportion)
[7]	Now	BC : CE = XD : DE	(CD $\parallel$ BX; Thm.2)
	But	XD = AC	(opposite sides of a parallelogram)
	so	BC : CE = AC : DE	
	thus	BC : AC = CE : DE	(alternating the proportion)
[8]	So	AB : BC = CD : CE	(Step 6)
	and	BC : AC = CE : DE	(Step 7)
	so	AB : AC = CD : DE	(Ch.5, Thm.17, *ex aequali*)
[9]	So	AB : BC = DC : CE	(Step 6)
	and	BC : CA = CE : ED	(Step 7)
	and	BA : AC = CD : DE	(Step 8)

Therefore the sides about the equal angles in the two triangles are proportional. Therefore $\triangle$ABC ~ $\triangle$DCE.

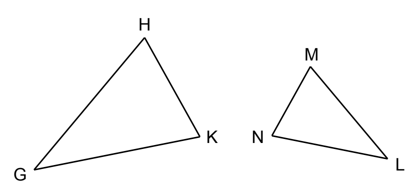
Q.E.D.

THEOREM 4 Remarks

1. All we need is two angles, of course, because if two angles in a triangle are equal to two angles in another triangle, they must be equiangular.

2. Can you prove the assumption in Step 2 that BA and ED must meet at some point?

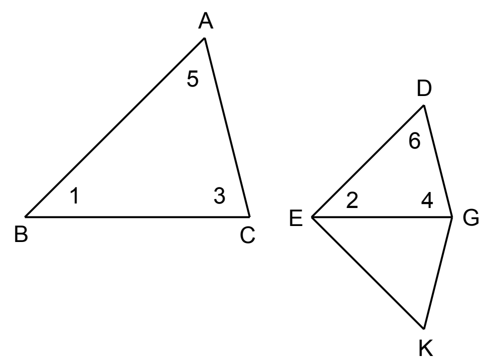
3. Notice that similar triangles need not be oriented the same way; for example, one can be "flipped" compared to the other, as $\triangle$GHK and $\triangle$LMN.



THEOREM 5: *Triangles with proportional sides are similar.*

Suppose △ABC and △DEG have proportional sides, i.e.

AB : BC = DE : EG
BC : CA = EG : GD
CA : AB = GD : DE



Then I say the angles contained by the proportional sides are equal, i.e.

1 = 2
3 = 4
5 = 6

and thus △ABC ~ △DEG. Here's an easy proof:

[1] Make △EGK on EG, such that it is equiangular with △ABC.
Since △EGK is equiangular with △ABC, therefore its sides are proportional to the sides about the equal angles in △ABC.

e.g.	AB : BC = KE : EG	
but	AB : BC = DE : EG	(given)
so	KE : EG = DE : EG	

thus KE = DE, since we learned in Ch.5 that quantities having the same ratio to the same quantity must be equal to each other.

[2] Similarly, we can prove that
KG = DG.
And EG is common to both triangles.

[3] Thus △EGK ≅ △DEG (Side-Side-Side)
But △EGK is equiangular with △ABC, by the construction in Step 1.
Thus △DEG is also equiangular with △ABC.
But these two triangles are given as having the sides about those equal angles proportional. Therefore they are also similar.

Q.E.D.

THEOREM 5 Remarks

There are 3 proportions given for this Theorem. Even if there were only two given, the Theorem will still hold, because the third one would follow automatically from the other two *ex aequali*, as illustrated in Theorem 4, Step 8.

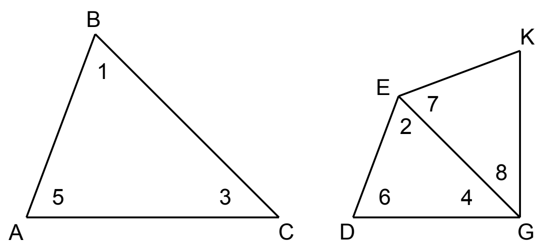
THEOREM 6: *Triangles with one angle of one equal to one angle of the other, and with the sides about the equal angles proportional, are similar.*

Suppose you have two triangles, $\triangle ABC$ and $\triangle DEG$, in which

$\angle 1 = \angle 2$

and AB : BC = DE : EG.

Then $\triangle ABC \sim \triangle DEG$.



The proof is nice and easy:

[1] Draw $\angle 7 = \angle 1$

Draw $\angle 8 = \angle 3$

Thus $\triangle KEG$ is equiangular with $\triangle ABC$.

Thus $\triangle KEG$ is similar to $\triangle ABC$ (Thm.4), and so the sides about the equal angles are proportional.

[2] So AB : BC = KE : EG

but AB : BC = DE : EG (given)

so KE : EG = DE : EG (these ratios are both the same as AB : BC)

so KE = DE (each has the same ratio to EG)

[3] But $\angle 7 = \angle 2$ (each being equal to $\angle 1$)

and GE is common to $\triangle DEG$ and $\triangle KEG$

so $\triangle DEG \cong \triangle KEG$ (Side-Angle-Side)

[4] But $\triangle KEG \sim \triangle ABC$ (Step 1)

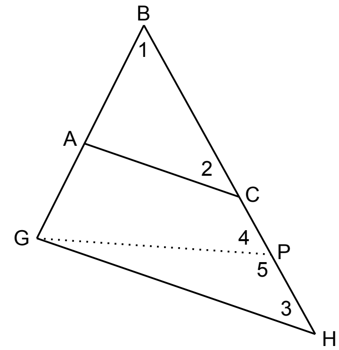
Thus $\triangle DEG \sim \triangle ABC$

Q.E.D.

THEOREM 6 Remarks

1. Notice that in Step 4 we take it as obvious that if a first triangle is similar to a second one, then any triangle congruent to the first one is also similar to the second one. It is also evident that *if two triangles are similar to a third, then they are similar to each other*, since if they are both equiangular with the same triangle, they must be equiangular with each other, and thus they must be similar to each other.

2. Here is a related Theorem: *If two triangles have one angle in one equal to one angle in the other, and another pair of corresponding angles both acute, and a pair of sides in one proportional to a corresponding pair in the other, then the triangles are similar.*



Given: △ABC and △GBH sharing ∠1, with ∠2 and ∠3 both acute, and

BA : AC = BG : GH

Prove: △ABC is similar to △GBH

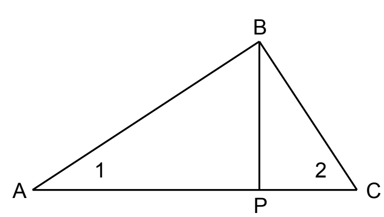
If possible, assume GH is not parallel to AC. Draw GP parallel to GH.
Thus △ABC and △GBP are equiangular, and hence similar. Thus

	BA : AC = BG : GP	(△ABC and △GBP are similar)
but	BA : AC = BG : GH	(given)
so	BG : GP = BG : GH	
so	GP = GH	(each having the same ratio to BG)
thus	∠3 = ∠5	(△GPH being isosceles)
now	∠4 = ∠2	(since △ABC and △GBP are similar)
and	∠2 is acute	(given)
thus	∠ 4 is acute	
hence	∠5 is obtuse	(being the supplement of ∠4)

but ∠3 = ∠5, and therefore △GPH has two obtuse angles, which is impossible. Therefore our initial assumption is impossible, and it is necessary that GH be parallel to AC. But then it follows that △ABC is similar to △GBH. *Q.E.D.*

THEOREM 7: *The perpendicular dropped from the right angle to the hypotenuse in a right triangle divides it into two right triangles similar to each other and to the whole.*

Let ABC be a right triangle, AC its hypotenuse, and drop BP perpendicular to AC.
Then △APB ~ △BPC ~ △ABC.
Is that just my opinion? No, it's a fact. Here's why:



[1]		∠APB = ∠ABC	(they are both right angles)
	and	∠1 is common to △APB and △ABC	
	so	△APB and △ABC are equiangular	
	so	△APB ~ △ABC	(Thm.4)

[2] Now $\angle BPC = \angle ABC$ (they are both right angles)
and $\angle 2$ is common to $\triangle BPC$ and $\triangle ABC$
so $\triangle BPC$ and $\triangle ABC$ are equiangular (Thm.4)
so $\triangle BPC \sim \triangle ABC$

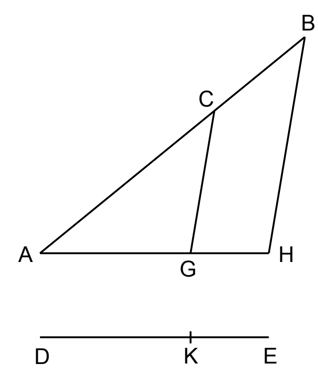
[3] So $\triangle APB$ and $\triangle BPC$ are equiangular,
each being equiangular with $\triangle ABC$ (Steps 1 and 2).
Thus $\triangle APB \sim \triangle BPC$ (Thm.4)

Q.E.D.

THEOREM 7 Remarks

Prove in particular that $\angle ABP = \angle 2$, and that $\angle CBP = \angle 1$.

THEOREM 8: *How to cut a given straight line similarly to a given cut straight line; also, how to cut off any fraction of a straight line.*



Suppose you have a straight line AB, and also another straight line DE cut at some point K. How can you cut AB at a point C so that AC : CB = DK : KE ?

Like this:

[1] Draw AH at any angle to AB, and make AH = DE.

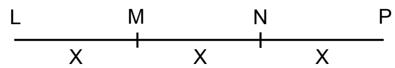
[2] Cut off AG = DK.

[3] Join HB.
Draw GC || HB.
Then C is the point we are looking for!

Proof:

[4] Since GC || HB in triangle ABH,
thus AC : CB = AG : GH (Thm.2)
but <u>AG : GH = DK : KE</u> (since AG = DK and GH = KE)
so AC : CB = DK : KE

Q.E.F.



Suppose now you want to cut off some particular fraction of AB, say *two thirds* of AB. How do you do it? Lay out any straight line X, and lay it out three times in a straight line. Thus LN is *two thirds* of LP.

Using the construction above, cut AB at a point C so that
AC : AB = LN : LP.
Since 2 : 3 = LN : NP
thus AC : CB = 2 : 3
i.e. AC is *two thirds* of AB, and so we have cut AB in the required fraction.
Similarly we can cut off any other fraction of straight line AB.

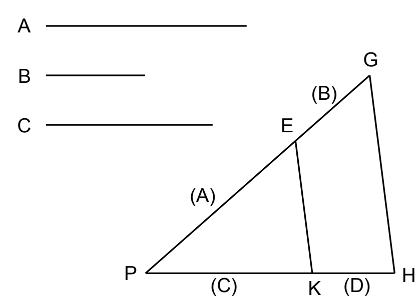
Q.E.F.

THEOREM 9: *How to find a fourth proportional to three straight lines.*

Suppose you have three lines A, B, C.
How can we find a fourth line D such that A : B = C : D ?
Like this:

[1] Place A and B in a straight line such as PEG, in which
PE = A
EG = B

[2] Draw PK = C at any angle to PEG.



[3] Join EK.
Draw GH || EK, cutting PK (extended) at H.
Then KH is the line D we are looking for;
i.e. A : B = C : KH.

If you are skeptical, here's the irrefutable proof:

[4] Since EK || GH in triangle PGH,
thus PE : EG = PK : KH (Thm.2)
but PE = A, EG = B, PK = C (Steps 1 and 2)
so A : B = C : KH

Q.E.F.

THEOREM 9 Remarks

We can now also find a third proportional to two given straight lines. For example, if we are given A and B, we can find a third line D such that A : B = B : D. All we have to do is use the same construction we just used, letting C = B, or PK = EG. For by the construction

A : B = C : D.

But since C = B, then we have

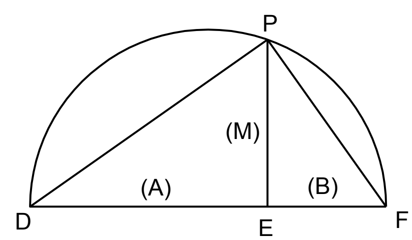
A : B = B : D.

THEOREM 10: *How to find the mean proportional between two straight lines.*

If you have two straight lines A and B, how do you find a straight line M such that

A : M = M : B ?

Like this:



[1] Place A and B in a straight line such as DEF, in which DE = A and EF = B.

[2] Draw a semicircle on DF as diameter, and draw EP perpendicular to DF, cutting the circumference at P. Call EP "M".

[3] Join DP. Join PF.

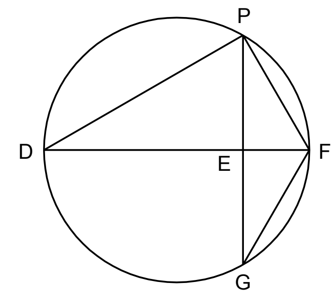
[4] Now, ∠DPF is a right angle (since it is in a semicircle).
And PE is perpendicular to DF (Step 2)
So △DEP ~ △PEF (Thm.7)
Thus the sides about the equal angles in these triangles are proportional when taken in corresponding order.

[5] But ∠DEP and ∠PEF are equal angles (both being right) in these two triangles. Therefore the sides about them are proportional, i.e.

DE : EP = EP : EF.

[6] But DE = A, EP = M, EF = B (Steps 1 and 2)
So A : M = M : B (substituting these terms in Step 5)

Q.E.F.



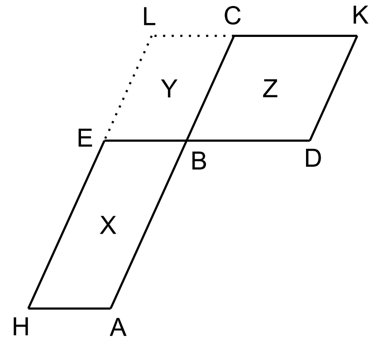
1. How do we know in Step 5 that we have taken the sides about the equal angles *in corresponding order*? As follows. If we complete the circle and extend PE down to G, clearly PE = EG, since PEG is perpendicular to a diameter.

Thus $\triangle$PEF ≅ $\triangle$GEF (Side Angle Side)
so PF = GF
and so the arcs cut off by these chords are also equal,
i.e. arc PF = arc GF.
Therefore the angles standing on these arcs from the circumference are also equal,
i.e. $\angle$PDF = $\angle$GFP,
or $\angle$PDE = $\angle$EPF.
Thus these are corresponding angles in $\triangle$DEP and $\triangle$PEF, and so the opposite sides PE and EF are corresponding sides. We already know hypotenuses DP and PF correspond. Hence DE and EP, the remaining sides, correspond.

2. We have a nice corollary with this Theorem: *The perpendicular dropped from the circumference to the diameter of a circle is a mean proportional between the segments into which it cuts the diameter*. Likewise the perpendicular dropped from the right angle of a right triangle to its hypotenuse is a mean proportional between the segments into which it cuts the hypotenuse.

3. We already know that the square on EP is equal to the rectangle contained by DE · EF (from Ch.2). Now we see that the side of a square equal to any rectangle is a *mean proportional* between the two sides of that rectangle.

THEOREM 11: *Equal and equiangular parallelograms have reciprocally proportional sides; conversely, equiangular parallelograms with reciprocally proportional sides are equal.*



Let CBDK and EBAH be equiangular parallelograms (i.e. ∠EBA = ∠CBD, ∠BEH = ∠BCK, etc.). And let them also have equal areas. I say that their sides are reciprocally proportional, that is AB : BC = DB : BE.
The proof is very simple:

[1] Place EB and BD in a straight line.

Thus	∠EBC + ∠CBD = two rights	
But	<u>∠EBA = ∠CBD</u>	(given)
so	∠EBC + ∠EBA = two rights	

and thus AB and BC are also in a straight line.

[2] Extend HE and KC till they meet at L, completing parallelogram ELCB, having the same angles as the two other parallelograms.

[3] Looking at the areas of the parallelograms,

	Z : Y = DB : BE	(Thm.1)
But	<u>Z = X</u>	(given)
so	X : Y = DB : BE	
But	<u>X : Y = AB : BC</u>	(Thm.1)
So	AB : BC = DB : BE	

Q.E.D.

Conversely, let CBDK and EBAH be equiangular parallelograms in which AB : BC = DB : BE. Then I say that EBAH = CBDK, i.e. they have the same area.

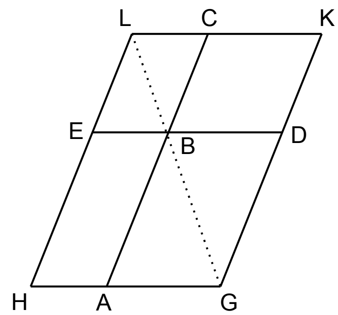
[1] Let them be placed as before.

[2]

Now	AB : BC = DB : BE	(given)
but	<u>AB : BC = X : Y</u>	(Thm.1)
so	X : Y = DB : BE	
but	<u>Z : Y = DB : BE</u>	(Thm.1)
so	X : Y = Z : Y	
so	X = Z	
i.e.	EBAH = CBDK	

Q.E.D.

THEOREM 11 Remarks:



1. Complete the parallelogram in $\angle HLK$ by extending HA and KD to G. Look familiar? The diagonal LG passes through B and parallelograms EBAH and CBDK are none other than the equal "complements" about the diagonal. To prove it is easy enough.

Since	$AB : BC = DB : BE$,	
thus	$AB : BD = CB : BE$	(alternating the proportion)
so	$BA : AG = LE : EB$	(since $BD = AG$ and $CB = LE$)
but	$\angle BAG = \angle LEB$	(by the parallels)
hence	$\triangle BAG$ is similar to $\triangle LEB$	(Thm.6)
so	$\angle ABG = \angle ELB$	
and	$\underline{\angle EBA = \angle BEL}$	(by the parallels)
so	$\angle ABG + \angle EBA = \angle ELB + \angle BEL$	(adding equals to equals)

Now if we add $\angle EBL$ to both sides, we get

$$\angle ABG + \angle EBA + \angle EBL = \angle ELB + \angle BEL + \angle EBL$$

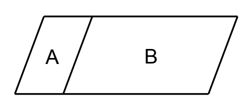
But the right side of this equation is all the angles in $\triangle EBL$, and so they add up to 180°.

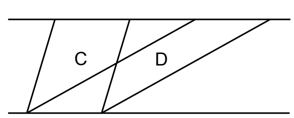
So $\quad \angle ABG + \angle EBA + \angle EBL = 180°$

from which it follows that LBG is a straight line.

You should be able to prove now that parallelograms EBAH and CBDK are similar to each other and to HLKG.

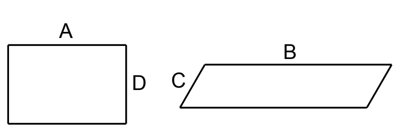
2. This Theorem is about equal and equiangular parallelograms. Obviously it is possible for parallelograms to be equiangular without having equal areas, as A and B.





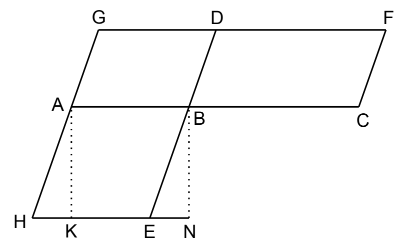
Or again, they can have equal areas without being equiangular, as C and D.

Or again, parallelograms can be equiangular without having proportional (or reciprocally proportional) sides, as a square S and a rectangle R that has a different area.



Also, nothing prevents parallelograms from having reciprocally proportional sides without being equiangular, as in the case of a rectangle of sides A and D and a tilted-over parallelogram in sides B and C, where $A : B = C : D$.

3. It is also true, conversely, that if two parallelograms are equal in area and have reciprocally proportional sides, then they are equiangular. Suppose ABNK and CBDF are equal in area, but not equiangular, so that ABC is one straight line, but NBD is not. Extend DB to E, and complete parallelograms ABDG and ABEH.



Clearly ABEH = ABNK

so also ABEH = CBDF.

And since these last two are also equiangular, therefore, by the Theorem above,

AB : BC = DB : BE

Now since CBDF is equiangular with ABEH, but not with ABNK, thus ABEH and ABNK are not equiangular.

So $\angle AHE \neq \angle BNK$ ($\neq$ means "is not equal to")

or $\angle BEN \neq \angle BNE$.

Hence BE $\neq$ BN in $\triangle BEN$.

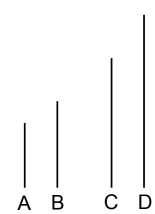
Now since AB : BC = DB : BE

It follows that AB : BC $\neq$ DB : BN, since BE $\neq$ BN.

That is, the sides of ABNK and CBDF are *not* reciprocally proportional. Therefore, equal but not equiangular parallelograms do not have reciprocally proportional sides, and so equal parallelograms that *do* have reciprocally proportional sides must also be equiangular. *Q.E.D.*

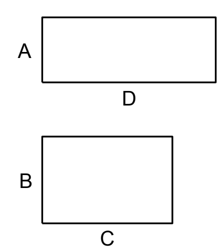
THEOREM 12: *If four straight lines are proportional, then the rectangle contained by the means is equal to the rectangle contained by the extremes.*

Conversely, if two rectangles are equal, then their sides are reciprocally proportional.



Let A, B, C, D be proportional straight lines, i.e. A : B = C : D.
I say that A · D = B · C, i.e. that these rectangles are equal in area.

[1] Make rectangle A · D and rectangle B · C.



[2] Since both are rectangles, they are equiangular parallelograms.

[3] And also their sides are reciprocally proportional, since we are given that

A : B = C : D.

[4] Therefore they have equal areas (Thm.11), that is A · D = B · C. *Q.E.D.*

Conversely, let A · D = B · C, i.e. suppose these are any two rectangles having equal areas. Then I say that A : B = C : D.

[1] Since both are rectangles, they are equiangular parallelograms.
[2] Also, they are equal (we are given that).
[3] Therefore their sides are reciprocally proportional (Thm.11), i.e.

A : B = C : D.

Q.E.D.

THEOREM 12 Remarks

If it happens that B = C, then B · C is actually a *square*. And since

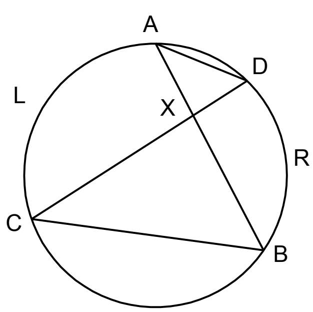
A : B = C : D,

we can also say (since B = C in this case) that

A : B = B : D.

So, when a square is equal to a rectangle in area, its side is a mean proportional between the sides of the rectangle. And, conversely, when three straight lines are in continuous proportion, the square on the mean is equal in area to the rectangle contained by the extreme terms in the proportion.

THEOREM 13: *If two lines inside a circle cut each other, then their segments contain equal rectangles.*



Let AB and CD cut each other inside a circle at X.
I say that AX · XB = CX · XD, i.e. these rectangles have the same area.

[1] Join AD. Join CB.

[2] Now, ∠CXB = ∠AXD (these are vertical angles)
and <u>∠XCB = ∠XAD</u> (both stand on arc BRD from the circumference)
so △CXB and △AXD are equiangular

[3] Thus △CXB ~ △AXD (Thm.4)

So the sides about their equal angles are proportional (when taken in corresponding order).

[4] Since $\angle XCB = \angle XAD$, therefore the sides opposite these angles are corresponding sides, i.e. XB and XD are corresponding sides.

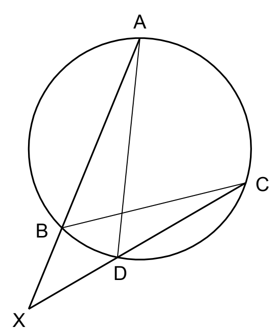
Likewise $\angle CBX = \angle ADX$, therefore the sides opposite these angles are corresponding sides, i.e. CX and AX are corresponding sides.

[5] Thus CX : XB = AX : XD, these being corresponding sides about the equal angles CXB and AXD (Steps 3 and 4).

[6] So $AX \cdot XB = CX \cdot XD$ (Thm.12).

Q.E.D.

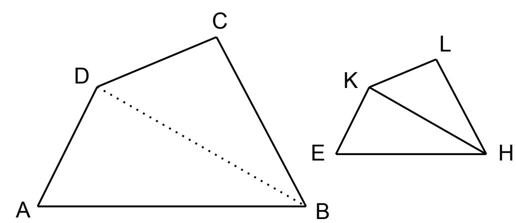
THEOREM 13 Remarks



If AB and CD meet *outside* the circle, i.e. if X is outside the circle, is the theorem still true? Is it still true that the rectangles $AX \cdot XB$ and $CX \cdot XD$ have the same area? Start by trying to prove that the triangles CXB and AXD are still similar.

THEOREM 14: *How to make, on a given straight line, a rectilineal figure similar and similarly situated to a given rectilineal figure.*

Take any rectilineal figure ABCD, and any straight line EH. How can we make a figure on EH similar to ABCD, in which EH is the side corresponding to AB? As follows.



[1] Divide ABCD into triangles (in this case two) ABD and DBC.

[2] Draw $\angle$HEK = $\angle$BAD.
Draw $\angle$EHK = $\angle$ABD.
So $\triangle$EHK is equiangular with $\triangle$ABD.
So $\triangle$EHK is similar to $\triangle$ABD (Thm.4)

[3] Draw $\angle$HKL = $\angle$BDC.
Draw $\angle$KHL = $\angle$DBC.
So $\triangle$KHL is equiangular with $\triangle$DBC.
So $\triangle$KHL is similar to $\triangle$DBC (Thm.4)

[4] Thus figure EHLK is equiangular with figure ABCD, each being composed of an equal number of correspondingly arranged equiangular triangles.

[5] In order for EHLK and ABCD to be similar, though, the sides about their equal angles must be proportional. Is that the case?

Obviously	DA : AB = KE : EH	($\triangle$EHK ~ $\triangle$ABD; Step 2)
And	BC : CD = HL : LK	($\triangle$KHL ~ $\triangle$DBC; Step 3)

But can we say AB : BC = EH : HL ?

[6] Yes, since	AB : BD = EH : HK	($\triangle$EHK ~ $\triangle$ABD; Step 2)
And also	<u>BD : BC = HK : HL</u>	($\triangle$KHL ~ $\triangle$DBC; Step 3)
Thus	AB : BC = EH : HL	(*Ex Aequali*, Ch.5, Thm.17)

Likewise CD : DA = LK : KE.

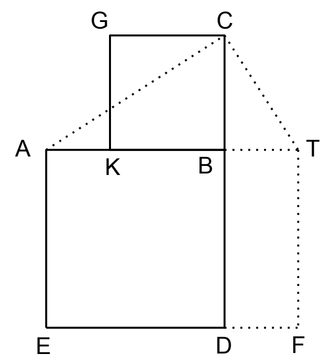
[7] Since EHLK and ABCD are equiangular (Step 4), and since the sides about their equal angles are proportional (Steps 5 and 6), therefore they are similar to each other, and AB corresponds to EH.

Q.E.F.

THEOREM 14 Remarks

Since we *made* a rectilineal figure similar to a given one by putting together triangles that are similar to those into which the given figure was divided (and by arranging them similarly), it is an obvious corollary to this Theorem that *Any two similar rectilineal figures can be divided into an equal number of similar triangles, similarly arranged*.

THEOREM 15: *If a third proportional is found to the sides of two squares, then the first square is to the second square as the side of the first square is to the third proportional.*



Take any square ABDE, and place any other square KBCG on it so that two sides are in line, as DB and BC. Join AC, and draw CT at right angles to AC, meeting AB (extended) at T. Complete the rectangle DBTF.

Now, since ACT is a right triangle, and CB is dropped from the right angle to the hypotenuse, therefore AB : BC = BC : BT (Thm.10, Remark 2). Thus BT is a third proportional to the sides of the two squares. I say that

□AB : □BC = AB : BT.

[1] For, the square ABDE and the rectangle DBTF are parallelograms under the same height, and so they are as their bases (Thm.1),

i.e. □AB : DB · BT = AB : BT
or □AB : AB · BT = AB : BT (AB = DB)

[2] But AB : BC = BC : BT (by construction)
so AB · BT = □BC (Thm.12 Remarks)

[3] Thus, if substituting □BC for AB · BT in Step 1, we have

□AB : □BC = AB : BT

And so the first square is to the second square as the side of the first square is to the third proportional line.

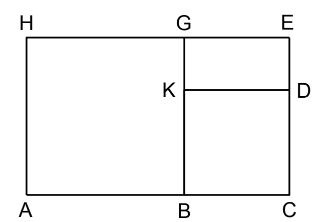
Q.E.D.

THEOREM 15 Remarks

1. Here is a numerical example:
   Square A has a side that is 2 feet long.
   Square B has a side that is 4 feet long.

According to the Theorem, Square A is not to Square B as 2 to 4, but rather if we find X such that 2 : 4 = 4 : X, then Square A is to Square B as 2 is to X. Of course, in this case X = 8. So the square A is to square B as 2 is to 8, or more simply as 1 is to 4.

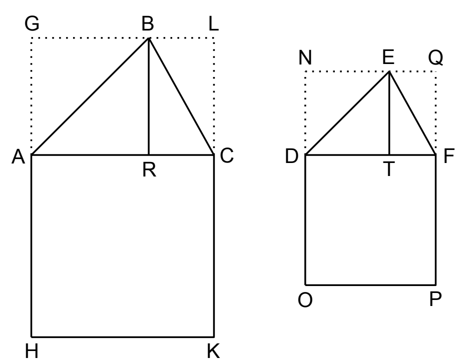
2. Unequal squares are never in the same ratio as their sides. Suppose square ABGH is greater than square BCDK. Line up sides AB and BC. If we extend HG and CD till they meet at E, BCEG will be a rectangle of the same height as the square on AB.



Hence ☐AB : BCEG = AB : BC
Accordingly, since ☐BC is less than BCEG,
☐AB : ☐BC > AB : BC.
So the greater square always has to the lesser a ratio greater than the corresponding ratio between the sides.

3. Looking back to Definition 4, we can also now say that *Squares are to each other in the duplicate ratio of their sides*.

THEOREM 16: *Similar triangles have the same ratio as the squares on their corresponding sides.*



Take any two similar triangles, ABC and DEF, and build squares ACKH and DFPO on corresponding sides AC and DF. I say △ABC : △DEF = ☐AC : ☐DF.

[1] Drop BR perpendicular to AC. Drop ET perpendicular to DF.

Since ∠BAR = ∠EDT
(since △BAC is similar to △EDF)
thus △BAR is equiangular with △EDT,
so BR : RA = ET : ED
but <u>BA : AC = ED : DF</u> (since △BAC is similar to △EDF)
hence BR : AC = ET : DF (*ex aequali*, Ch.5, Thm.17)

[2] Complete the rectangle of height BR, base AC, namely ACLG.
Complete the rectangle of height ET, base DF, namely DFQN.

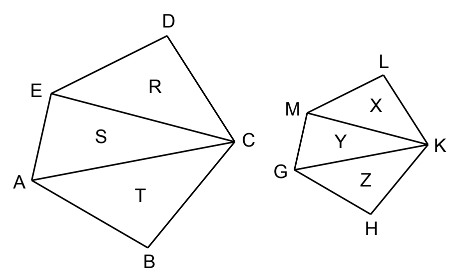
Now	ACLG : ☐AG = LC : CK	(Thm.1)
Or	ACLG : ☐AC = BR : AC	(LC = BR, CK = AC)
and	DFQN : ☐DF = ET : DF	(for similar reasons)
but	BR : AC = ET : DF	(Step 1)
so	ACLG : ☐AC = DFQN : ☐DF	

[3]

So	ACLG : DFQN = ☐AC : ☐DF	(alternating)
but	ACLG : DFQN = △ABC : △DEF	(wholes are as their halves)
so	△ABC : △DEF = ☐AC : ☐DF	

Q.E.D.

THEOREM 17: *Similar rectilineal figures have the same ratio as the squares on their corresponding sides.*



Take any two similar rectilineal figures ABCDE and GHKLM.
I say that ABCDE : GHKLM = ☐AB : ☐GH.

[1] By the corollary to Theorem 14, we know we can divide ABCDE and GHKLM into similarly arranged similar triangles, equal in number. For brevity, let the triangles in ABCDE be called R, S, T, and let those in GHKLM be called X, Y, Z.

[2] Since S and Y are similar triangles,

thus	S : Y = ☐AC : ☐GK	(Thm.16)
But	T : Z = ☐AC : ☐GK	(Thm.16)
So	T : Z = S : Y	
But	T : Z = ☐AB : ☐GH	(Thm.16)
So	S : Y = ☐AB : ☐GH	

And likewise we can prove that *every* pair of corresponding triangles in the two figures has the ratio of ☐AB : ☐GH.

[3] That is to say,

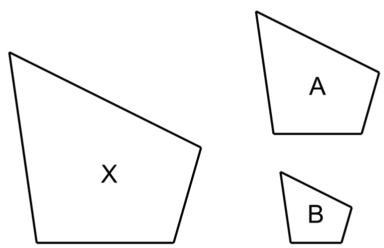
	T : Z = ☐AB : ☐GH	
and	S : Y = ☐AB : ☐GH	
and	R : X = ☐AB : ☐GH	
Thus	R + S + T : X + Y + Z = ☐AB : ☐GH	(Ch.5, Thm.15)
i.e.	ABCDE : GHKLM = ☐AB : ☐GH	

Q.E.D.

THEOREM 17 Remarks:

Looking back at Theorem 15, Remark 2, we can now see that similar rectilineal figures are to each other in the duplicate ratio of their corresponding sides.

THEOREM 18: *Figures similar to the same rectilineal figure are similar to each other*.



Suppose A is similar to X, and B is also similar to X. Then A is similar to B. Why? Because …

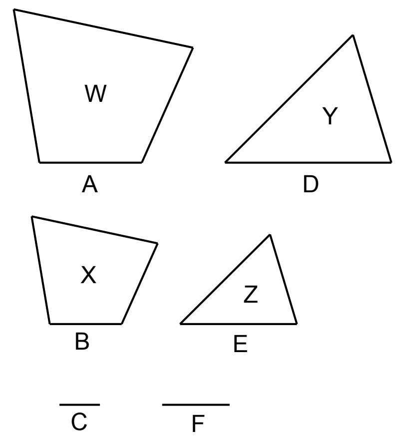
[1] The angles in A equal the angles in X (Since A is similar to X)
<u>The angles in B equal the angles in X</u> (Since B is similar to X)
So the angles in A equal the angles in B (since things equal to the same thing are themselves equal).
So A and B are *equiangular*.

[2] The sides about the angles in A have the same ratios as the sides about the equal angles in X.
The sides about the angles in B also have the same ratios as the sides about the equal angles in X
So the sides about the angles in A have the same ratios as the sides about the equal angles in B (since two ratios that are both the same with another ratio are the same as each other).

[3] So A and B are equiangular *and* the sides about the equal angles are proportional. Therefore A is similar to B (Def. 1).

Q.E.D.

THEOREM 19: *If four straight lines are proportional, the similar figures similarly situated on them will be proportional.*



Suppose you have 4 straight lines A, B, D, E and $A : B = D : E$.
Then place any similar figures W and X on A and B, similarly situated. Place any two other similar figures Y and Z on D and E, similarly situated.

I say that $W : X = Y : Z$.

[1] Find a third proportional line to the straight lines A and B, namely C, so that

$A : B = B : C$ (Thm.9)

Find a third proportional line to the straight lines D and E, namely F, so that

$D : E = E : F$ (Thm.9)

[2] So $A : C = \square A : \square B$ (Step 1 and Thm.15)

And $D : F = \square D : \square E$ (Step 1 and Thm.15)

[3] But the similar figures are also in the ratios of these squares on their corresponding sides (Thm.15),

so $W : X = A : C$

and $Y : Z = D : F$

[4] Now $A : B = D : E$ (given)

and $\underline{B : C = E : F}$ ($A : B = B : C$, and $D : E = E : F$, Step 1)

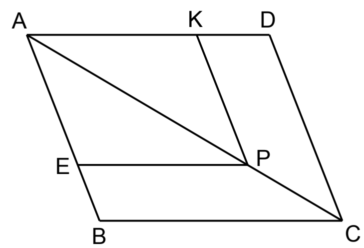
so $A : C = D : F$ (*ex aequali*, Ch.5, Thm.17)

[5] But each of the ratios in this last proportion is the same as one of the ratios of similar figures (see Step 3).

So $W : X = Y : Z$

Q.E.D.

THEOREM 20: *If two parallelograms share an angle and are about the same diagonal, then they will be similar.*



Take any parallelogram ABCD, and choose any point P along its diagonal, completing parallelogram AEPK. I say that AEPK is similar to ABCD.

[1] Now $\angle AEP = \angle ABC$ (since EP $\parallel$ BC)

[2] And $\angle KAE$ is common to both parallelograms.

[3] Since the opposite angles in any parallelogram are equal, it follows that the remaining angles in AEPK and ABCD are correspondingly equal. So they are equiangular.

[4] Now since $\angle AEP = \angle ABC$ (Step 1), and $\angle EAP$ is common to $\triangle EAP$ and $\triangle BAC$, hence these triangles are equiangular and therefore similar.

So AE : EP = AB : BC

[5] Since the opposite sides in any parallelogram are equal, it follows that the remaining sides in AEPK and ABCD about their equal angles (and taken in corresponding order) are proportional.

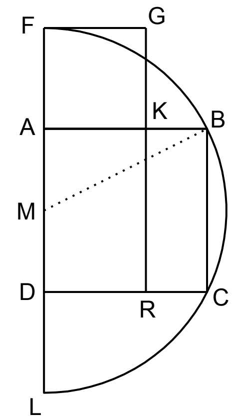
Thus AEPK is similar to ABCD. So all parallelograms about the diagonal and sharing an angle with the whole parallelogram ABCD will be similar to the whole and therefore to each other.

Q.E.D.

THEOREM 20 Remarks:

Conversely, if two similar parallelograms share an angle they will also share a diagonal. Using the same diagram as in the Theorem, suppose that AEPK and ABCD are similar parallelograms sharing the angle at A. I say that P lies on diagonal AC. From the similarlity of the parallelograms, AE : EP = AB : BC. Since EP is parallel to BC, it must cut AC at some point, say Q. And since EPQ is parallel to BC in triangle ABC, therefore AE : EQ = AB : BC. From the two proportions, it is clear that EQ = EP, and therefore Q and P are the same point. So P lies on AC.

THEOREM 21: *How to cut a line in mean and extreme ratio.*



Take any straight line AB.
How can we cut it at a point K so that BA : AK = AK : KB ?
Thus:

[1] Draw square ABCD.

[2] Bisect AD at M and join MB.

[3] Draw a circle with center M and radius MB (thus passing through C, and cutting AD extended at F and L).

[4] Draw square AFGK. Extend GK to R.

K is the point we sought. How do we know? As follows:

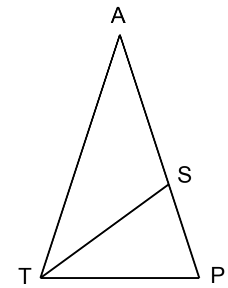
[5]	Now	☐AB = FA · AL	(AB is ⊥ to diameter FAL; Ch.2, Thm.6)
	But	FA = GF	(being sides of ☐AFGK)
	and	AL = FD	(each is a radius plus half of AD)
	so	☐AB = GF · FD	
[6]	Or	ABCD = FGRD	(renaming them)
	Now subtract AKRD from each and we have		
		KBCR = FGKA	
	or	CB · BK = ☐AK	(renaming them)
[7]	Thus	CB : AK = AK : KB	(Thm.12)
	or	BA : AK = AK : KB	(CB = BA)

Q.E.F.

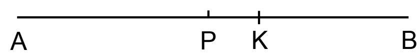
THEOREM 21 Remarks:

1. Do you recognize the diagram? Compare it to Ch.2, Theorem 12. There we extended a straight line AS to B so that ☐AS = AB · BS. By the present Theorem, we see that ASB was cut in mean and extreme ratio (BA : AS = AS : SB).

2. We used Ch.2, Theorem 12 to make a "Golden Triangle" in Ch.4, Theorem 6. This was the isosceles triangle TAP, whose base angles were each double its peak angle. Recall that in the course of making it, we made line ASP such that $\square AS = AP \cdot PS$; thus $PA : AS = AS : SP$, so that AP was divided at S in mean and extreme ratio or the "Golden" ratio. But we had also made $PT = AS$, so that $AP : PT$ was also the Golden ratio. For this reason, $\triangle TAP$ is called a "Golden Triangle."



3. We used a Golden Triangle to construct a regular pentagon in Ch.4, Theorem 7. So the Golden ratio is the key to producing that figure as well.

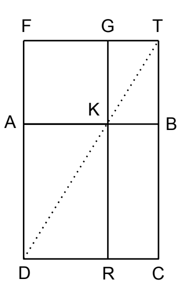


4. Can we cut AB at another point besides K, say P, such that $BA : AP = AP : PB$? No. It is impossible. K is unique. If possible, suppose P is on AK,

and	$BA : AP = AP : PB$.	
Thus	$AB \cdot BP = \square AP$	(Thm.12)
Now	$\underline{AB \cdot BP > AB \cdot BK}$	(since $BP > BK$)
so	$\square AP > AB \cdot BK$	
but	$\underline{AB \cdot BK = \square AK}$	(since $AB : AK = AK : KB$)
so	$\square AP > \square AK$	
Thus	$AP > AK$, which is impossible.	

It is true, of course, that if we cut off $AP = BK$, then $AB : BP = BP : PA$, because P is simply the mirror image of K on the other side of AB's midpoint. Thus it does not divide AB in an essentially new way.

5. It should be clear, too, that any two lines cut in mean and extreme ratio are cut in the same ratio. To cut any other line besides AB in mean and extreme ratio, we would use the same construction again, and the similarity of all the figures in both constructions would make it obvious that the lines are cut similarly.



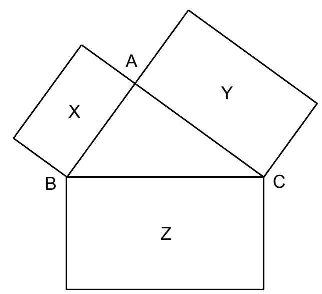
6. Looking back at the diagram for this Theorem, complete the rectangle FDCT. Notice that square FK and rectangle KC are equal (Step 6), and they are parallelograms drawn in opposite corners of FDCT and they are both equiangular with rectangle FDCT. They also have a common corner K. Therefore K lies on the diagonal of rectangle FDCT (Ch.1, Thm.34 Questions). Accordingly rectangles AKRD and GTBK are similar to the whole rectangle FDCT and to each other (Thm.20).

7. Since AK : KB is the golden ratio,
and GK : KB = AK : KB,
hence GK · KB is called a “Golden Rectangle.”

Since, by the preceding remark, GK · KB is similar to AD · DR and FD · DC, these are also golden rectangles.

8. Note that if we draw a square inside a Golden Rectangle on its lesser side, the remaining rectangle is another Golden Rectangle. e.g. FDCT is a Golden Rectangle, and when we take away square ABCD, what remains is BA · AF.
But BA : AF = BA : AK (since AF = AK),
and BA : AK is the Golden ratio.
Hence BA : AF is also the Golden ratio, and thus BA · AF is a Golden Rectangle. And when we subtract square FGKA from it, we are left with Golden Rectangle GK · KB.

THEOREM 22: *In right triangles the rectilineal figure on the hypotenuse is equal to the sum of the similar (and similarly situated) figures on the other two sides.*



Imagine a right triangle ABC, whose hypotenuse is BC. Imagine further some rectilineal figure Z on hypotenuse BC. If X and Y are similar to Z, and similarly situated on AB and AC, then X + Y = Z.

Here's the proof:

[1] First, □AB : □AC = X : Y, since similar figures are as the squares on their corresponding sides (Thm.17).

[2] Thus □AB : X = □AC : Y (alternating the proportion)
so □AB + □AC : X + Y = □AC : Y (Ch.5, Thm.15)

[3] Now □BC : □AC = Z : Y, since similar figures are as the squares on their corresponding sides (Thm.17).
Thus □BC : Z = □AC : Y (alternating the proportion)

[4] So □BC : Z = □AC : Y (Step 3)
and □AB + □AC : X + Y = □AC : Y (Step 2)
thus □AB + □AC : X + Y = □BC : Z

[5] So □AB + □AC : □BC = X + Y : Z (alternating Step 4)
but □AB + □AC = □BC (Pythagorean Theorem)
thus X + Y = Z.

Q.E.D.

THEOREM 22 Remarks:

1. The diagram depicts rectangles, but they could just as well be irregular (but similar) pentagons, or any other kind of rectilineal figure.

2. Do you think that the Theorem would hold for similar curvilineal figures, such as semicircles?

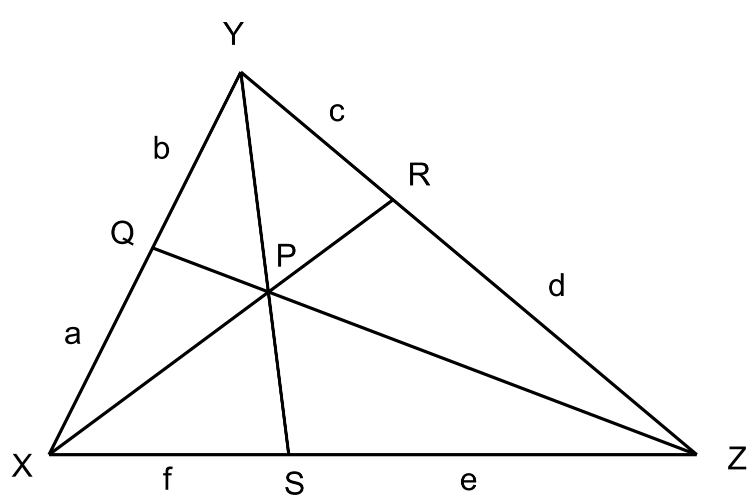
3. This Theorem is in some ways a generalized version of the Pythagorean Theorem. The Pythagorean Theorem proved a relationship between *squares* on the sides of a right triangle, whereas this Theorem proves that the same relationship exists between *any* similar rectilineal figures similarly situated on the sides of a right triangle.

"HOOK": CEVA'S THEOREM.

If in any triangle XYZ lines drawn from the three vertices to the opposite sides (XR, YS, ZQ) are concurrent at P, cutting the sides XY, YZ, ZX into a & b, c & d, e & f (labeling all these clockwise from X),

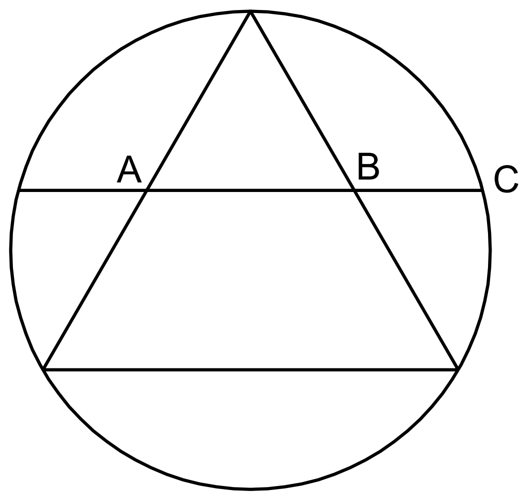
then $$\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f} = 1$$

and conversely.



“HOOK”: THE EQUILATERAL TRIANGLE AND THE GOLDEN RATIO.

If the midpoints of two sides of an equilateral triangle inscribed in a circle be joined and extended to meet the circle, this line is cut in mean and extreme ratio.



Chapter Seven

Numbers

DEFINITIONS

1. When the same kind of thing is taken repeatedly, the result is a MULTITUDE.

For example, if you take a chair, and another chair, you have a multitude of chairs. Again, if you have a pear, a banana, and an orange, although you do not have a multitude of pears, you do have a multitude of fruits.

2. The UNIT is what is indivisible, constituting a multitude by repetition of itself.

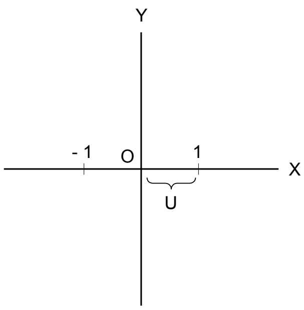
For example, if you have a multitude of chairs, the unit constituting this multitude is one chair. Suppose you have 24 chairs: then the repetition of a dozen chairs also constitutes it, but a dozen is not the unit of this multitude, since the dozen is itself a multitude constituted by repetition of something – a dozen is still itself a multitude of chairs. The unit, on the other hand, is the elementary beginning of multitude; it is not itself a multitude, and hence it is indivisible. The unit is like the atom of a multitude.

If we consider concrete multitudes such as our 24 chairs, we see that being a unit can sometimes happen to a divisible thing: one chair can be sawn in half. But does this belong to the chair because it is a unit in a multitude, or because it is a chair? If a chair were divisible precisely because it is a unit, then every *unit in* every *multitude would be divisible. And that is false. What is the unit in a multitude of geometric points? One geometric point. Is that divisible? No. Consequently, it does not belong to a unit as such to be divisible, but rather one apple is divisible because it is an apple, and one line is divisible because it is a line, and so on. So no unit is as such divisible.*

Mathematics ignores what the quantities it studies are made of. For example, it studies circles without considering what they might be made of, and so the circle studied by the geometer is neither heavy nor light, neither hot nor cold, neither hard nor soft. Circularity as such does not have such properties, although it might be found in things having such properties. Similarly, mathematics studies multitudes without considering what they might be multitudes of, and so the units it studies cannot be divisible: unity itself does not have divisibility, although it might belong to something that is divisible.

Multitude might also be found in things that have position, orientation, and location, such as apples. But differences in position, orientation, and location make no difference with respect to multitude; 10 apples remain 10 apples regardless of changes in their relative position or arrangement. Hence the mathematical unit in itself has no position, location, or orientation.

Nothing prevents us from choosing something divisible or something having position or orientation as a kind of conventional unit to measure other things of like kind, such as a unit length or a unit weight or a unit time. For example, we might lay out straight lines X *and* Y *at right angles to each other at point* O, *to use as references for designating any point in the plane. Choose any point on* X *to the right of* O *and label it* 1. *Thus the line length between* O *and* 1 *(call it* "U"*) is our chosen unit length. To distinguish the same length in the opposite direction, we call it* – 1 *("negative one"). Notice that* U *is something divisible, having position, and also direction (it is to the right of* O*). All this proves to be very useful elsewhere in mathematics, of course, but note that* U *is not a pure unit. It is not "the unit," the "one" by which we count all things, but a unit* length*, having an arbitrary length and orientation (none of which belong to "one" as such). It is no more identical with "one" than "two apples" is identical with "two."*



3. A NUMBER is a multitude measured by the unit.

Some multitudes might be infinite, but a number is a measured multitude, and hence finite. It is a multitude that can be comprehended by counting.

Such things as "three and a half", and other things such as "negative six", and still other things such as "the square root of two", have come to be called "real numbers" with reference to a conventional unit that is divisible and has position (such as U *in the discussion of the last definition). And yet* – 6 *and* + 6 *do not differ in multitude, but only in direction (or some other extraneous thing). And* 3 ½ *divides its own unit, which must therefore be a divisible thing, and not simply the unit considered entirely apart from certain divisible things out of which a multitude might be made. Prior to all such compound notions, applying the concept of number to various things and combining it with such things as direction and length, there is the pure concept of number as a kind of multitude.* 6 *is simply and purely a kind of multitude, but* – 6 *or* + 6 *is a multitude* plus an additional idea of direction *(or something similar).* 3 *is simply and purely a multitude, but* 3 ½ *is a multitude* of divisible things*. This chapter is about pure numbers*.

Is 1 *a number? If you ask me to hand you "a number of nails", and I hand you one nail, you might not be satisfied. One is not a multitude or a plurality, and therefore it is not a number in the same sense that* 5 *or* 6 *are numbers. Still, since* 1 *is the beginning and measure of all the numbers, and since it has a ratio to every number, it is also called a "number."*

4. An EVEN NUMBER is a number divisible into two equal numbers.

For example, 2, 4, 6 are even numbers.

5. An ODD NUMBER is a number not divisible into two equal numbers.

For example, 3, 5, 7 are odd numbers.

6. A FACTOR of a number is any number that measures it, i.e. goes into it exactly some number of times.

For example, the factors of 12 are 1, 2, 3, 4, 6, 12 (but not 5, 7, 8, 9, 10, 11). A factor of a number is also called a "divisor" of it, since it divides it exactly, and it is also called a "measure" of it since it fits into it exactly.

7. A PRIME NUMBER is a number with no factors other than 1 and itself.

For example, 2 and 3 are prime numbers, since neither has any factors other than the number 1 and itself.

8. Two different numbers are PRIME TO EACH OTHER if they have no factor in common except 1.

For example, 8 and 15 are prime to each other, having no factor in common other than 1. But 12 and 15 are not prime to each other, since they share a factor besides 1 (namely 3).

9. A COMPOSITE NUMBER is a number with factors other than 1.

For example, 14 is a composite number, since it has factors besides 1, namely 2 and 7. Obviously no prime number is composite, and no composite number is prime, since their definitions are opposed.

10. TO MULTIPLY one number N by another number M means to find the sum of as many N's as there are units in M. The resulting number is called the PRODUCT of M times N.

For example, to multiply 3 by 5 means to add together as many threes as there are units in 5, i.e. $3 + 3 + 3 + 3 + 3 = 15$. *So 15 is the product of 5 times 3.*

The symbolic notation for multiplying 5 times 3 is this: $5 \times 3 = 15$. *We can also write it this way:* $5 \cdot 3 = 15$.

11. TO DIVIDE one number N by another number M means to count the number of times M can be subtracted from N. The resulting number is called the QUOTIENT of N divided by M.

For example, to divide 12 by 4 means to count the number of times 4 can be subtracted from 12:

(1) $12 - 4 = 8$
(2) $8 - 4 = 4$
(3) $4 - 4 = 0$

It can be subtracted 3 times. So 3 is the quotient of 12 divided by 4.

The symbolic notation for dividing 12 by 4 is this: $12 \div 4 = 3$.

Obviously a greater number is not always exactly divisible by a lesser one, for example $16 \div 3$. 3 *can be subracted from* 16 *five times, but one unit of* 16 *is left over.*

Note that multiplication and division are opposite operations, and they undo each other. If

(a) $M \times N = P$

then by definition N *fits into* P *exactly* M *times. But that means we can subtract* N *from* P *exactly* M *times, i.e.*

(b) $P \div N = M$

Now by equation (a) *we know that* $P = M \times N$, *so let us substitute* $M \times N$ *for* P *in equation* (b):

$$(M \times N) \div N = M$$

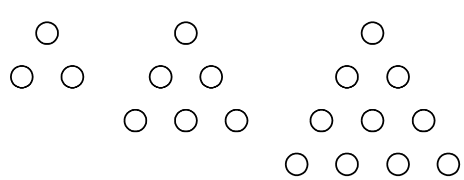
which is to say that any number M *multiplied by any number* N, *and then again divided by* N, *leaves us with* M *once more. Or, by equation* (b), $M = P \div N$, *so let us substitute* $P \div N$ *for* M *in equation* (a):

$$(P \div N) \times N = P$$

so any number P *divided by a number* N, *and then again multiplied by that number* N, *leaves us with* P *once more (at least, so long as we suppose* N *goes into* P *some exact number of times; it is also true more generally, but that goes beyond the point made here).*

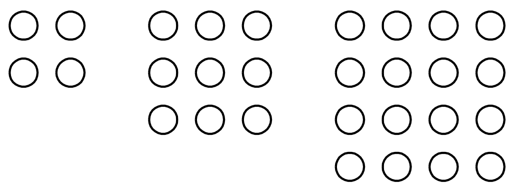
12. A TRIANGULAR NUMBER is the sum of any number of consecutive numbers beginning with 1; the last number added is called the BASE of the triangular number.

For example, 3 is the first triangular number, since 3 = 1 + 2, and 2 is its base. And 6 is the second triangular number, since 6 = 1 + 2 + 3, and 3 is its base. Such numbers are called "triangular" because the numbers which add up to them can be stacked on top of each other, and the result is a triangular form.



13. A SQUARE NUMBER is the product of two equal numbers. The number which, thus multiplied by itself, yields a square number, is the SQUARE ROOT of the square number.

For example, 4 is a square number, since 4 = 2 × 2, and 2 is the square root of 4. And 9 is the next square number, since 9 = 3 × 3, and 3 is the square root of 9. The symbolic notation $5^2 = 25$ *reads "five squared equals twenty five", and the notation* $5 = \sqrt{25}$ *reads "five equals the square root of twenty five". Square numbers are called "square" because their units can be arranged in a square pattern. Accordingly, the square root is also called the "side" of the square, as in "3 is the side of the square number 9".*



14. A CUBE NUMBER is the product of three equal numbers. The number which, thus multiplied by itself, yields a cube number, is the CUBE ROOT of the cube number.

For example, 8 is the first cube number, since 8 = 2 × 2 × 2, and 2 is the cube root of 8. The symbolic notation $2^3 = 8$ *reads "two cubed equals eight", and the notation* $2 = \sqrt[3]{8}$ *reads "two equals the cube root of eight".*

15. A POWER of a number is the product of that number times itself any number of times.

For example, the fourth power of 5 is 5 × 5 × 5 × 5, and the notation $5^4 = 625$ *reads "five to the fourth power equals six hundred and twenty five".*

16. A FACTORIAL NUMBER is the product of any number of consecutive numbers beginning with 1.

For example, 24 is a factorial number, since 24 = 1 × 2 × 3 × 4. The notation $4! = 24$ *reads "four factorial equals twenty four".*

17. A PERFECT NUMBER is a number equal to the sum of all its factors less than itself.

For example, 6 is the first perfect number, since, other than 6 itself, all its factors are 1, 2, 3, and it also happens that 6 = 1 + 2 + 3. With most numbers the sum of all their factors (other than themselves) falls short or exceeds the number itself. For example, the factors of 8 less than itself are 1, 2, 4 which add up to 7. The factors of 12 less than itself are 1, 2, 3, 4, 6 which add up to 16.

PRINCIPLES OF NUMBER THEORY

1. No number is both even and odd.

2. If one is added to or subtracted from an even, the result is odd.

3. If one is added to or subtracted from an odd, the result is even.
A possible exception occurs if the odd number you start with is 1. *Is* 1 – 1 *"even"? Perhaps, but it is not an even "number" insofar as a "number" means "a multitude of units." No units at all, or zero, is not a multitude of units. Still,* 1 + 1 *is even.*

4. Any two numbers can be added together, and the result is a number.
Any number can have numbers added to it without limit.

5. Any two numbers can be multiplied together, and the result is a number.
Any number can be multiplied without limit.

6. Any lesser number can be subtracted from a greater number, and the result is a number.

7. Any number can be divided by any one of its factors, and the result is a number.

8. A number that measures two numbers also measures their sum.
For example, if 2 goes into 6, and it also goes into 10, then 2 also goes into 6 + 10.

9. A number that measures two unequal numbers also measures their difference.
For example, if 3 goes evenly into 15, and also into 9, then 3 also goes evenly into 15 – 9.

10. A number that measures a number also measures any of its products.
For example, if 5 goes into 35, then 5 also goes into 26 × 35, and also into 327 × 35, and in general 5 must go into N × 35, no matter what number N is.

THEOREMS

THEOREM 1: *The unit has to any number the same ratio that any second number has to the product of the two numbers.*

Take any two numbers, say N and M. Then I say that the following proportion holds:

$1 : M = N : N \times M$

Start off with the ratio of 1 to M. Any equimultiples of these two have the same ratio that they have to each other (Ch.5, Thm.13). So let's take each of them N times, giving us $N \times 1$ and $N \times M$. Thus we have

$1 : M = N \times 1 : N \times M.$

But $N \times 1$ is taking N once, which is just N. So we have, in fact,

$1 : M = N : N \times M.$

Q.E.D.

THEOREM 1 Remarks

1. Let's just confirm this with a few concrete examples, for the sake of clarity.
   $2 \times 3 = 6$, and lo and behold, it is also true that $1 : 2 = 3 : 6$.
   Also, $4 \times 5 = 20$, but also it is true that $1 : 4 = 5 : 20$.

2. Can you see how this Theorem is actually just a different way of stating Definition 10?

3. We can formulate a similar Theorem for the division of numbers, namely that *The unit has to any number the same ratio which its quotient with a second number has to the second number*. That is, for any two numbes M and N, it will be true that

$1 : M = N \div M : N.$

Start by noting that $M \times (N \div M) = N$ (see Def.11). Thus, by Theorem 1, the above proportion follows. Here is an example:

$12 \div 4 = 3$, and it is also true that $1 : 4 = 3 : 12$.

THEOREM 2: *The order of multiplication makes no difference.*

Given: Numbers A and B
Prove: $A \times B = B \times A$

This is no big surprise. But it is still worth noting that "three fives" equals "five threes"; although the result is the same, the process of adding together three fives is not identical to the process of adding together five threes. So *why* must the result be the same?

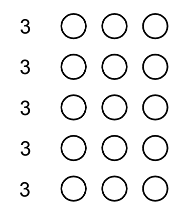
[1]	Well,	$1 : A = B : B \times A$	(Thm.1)
	so	$1 : B = A : B \times A$	(alternating)
	but	$\underline{1 : B = A : A \times B}$	(Thm.1)
	so	$A : B \times A = A : A \times B$	

[2] Since $B \times A$ and $A \times B$ have the same ratio to A, therefore they must be equal.

Q.E.D.

THEOREM 2 Remarks:

1. The truth of this Theorem can be manifested visually: 5 rows of 3 is necessarily at the same time 3 columns of 5, i.e. 5 threes must equal 3 fives, since they are just two different ways of looking at the same number.



2. The Theorem proves the case for two numbers, but it is just as true for any number of numbers we multiply together. The final product is the same regardless of the order in which we multiply them. For example, $3 \times 4 \times 5 = 5 \times 4 \times 3$. For, by the Theorem

$$4 \times 5 = 5 \times 4 \qquad \text{(each is equal to 20)}$$

but again, by the Theorem

$$3 \times 20 = 20 \times 3$$

and so

$$3 \times 4 \times 5 = 5 \times 4 \times 3.$$

THEOREM 3: *For any three numbers* A, B, C, *as long as* B *is exactly divisible by* C, *then* $(A \times B) \div C = A \times (B \div C)$.

Whether we multiply $A \times B$ first, and *then* divide the product by C, or instead divide B by C first, and *then* multiply the result by A, we get the same number. For example,

$(4 \times 6) \div 3 = 4 \times (6 \div 3)$.

or $24 \div 3 = 4 \times 2$

Must it always work out, regardless of the numbers we choose? Naturally. And here's proof:

[1] Let $(A \times B) \div C = K$
thus $A \times B = K \times C$ (multiplying both sides by C)

[2] Let $B \div C = N$
thus $B = C \times N$ (multiplying both sides by C)

[3] Now $A \times B = K \times C$ (Step 1)
or $A \times (C \times N) = K \times C$ ($B = C \times N$; Step 2)
so $A \times N = K$ (dividing both sides by C)
or $A \times N = (A \times B) \div C$ ($K = (A \times B) \div C$; Step 1)
hence $A \times (B \div C) = (A \times B) \div C$ ($N = (B \div C)$; Step 2)

Q.E.D.

THEOREM 3 Remarks:

1. Such a Theorem is not extremely surprising, but it is extremely useful to know when we can change the order of operations without affecting the final result. Verify the Theorem yourself with other numerical examples.

2. Here is another similar rule: suppose you have four numbers A, B, C, D and A is exactly divisible by C and B is exactly divisible by D. Then I say that

$(A \times B) \div (C \times D) = (A \div C) \times (B \div D)$

For example $(6 \times 4) \div (3 \times 2) = (6 \div 3) \times (4 \div 2)$.

Why must this always work out?

[1] Let $A \div C = M$
and $B \div D = N$

[2] Since multiplication is the reverse of division, hence it is also true that
$M \times C = A$
and $N \times D = B$

[3] So $(A \div C) \times (B \div D) = M \times N$ (see Step 1)

[4] But $(A \times B) = [M \times C] \times [N \times D]$ (see Step 2)
Hence, dividing both sides by $(C \times D)$, we have
$$(A \times B) \div (C \times D) = ([M \times C] \times [N \times D]) \div (C \times D)$$
and since the order of multiplication makes no difference, we can write
$$(A \times B) \div (C \times D) = (M \times N) \times (C \times D) \div (C \times D)$$
But since on the right we first multiply and then divide by $(C \times D)$, this leaves us with the number $(M \times N)$, and so
$$(A \times B) \div (C \times D) = M \times N$$

[5] Comparing Steps 3 and 4, we see that
$$(A \div C) \times (B \div D) = (A \times B) \div (C \times D).$$

THEOREM 4: *If four numbers are proportional, the product of the means equals the product of the extremes. Conversely, if among four numbers the product of the means equals the product of the extremes, then they are proportional.*

Given: $A : B = C : D$
Prove: $A \times D = B \times C$

We know that	$A : B = C : D$	(given)
and so	$A : C = B : D$	(alternating)

Now if we multiply the first two terms by B, we maintain the proportion, and again if we multiply the last two terms by A, we maintain the proportion (Ch.5, Thm.13).

Hence	$B \times A : B \times C = A \times B : A \times D$	
so	$B \times A : A \times B = B \times C : A \times D$	(alternating)
but	$B \times A = A \times B$	(Thm.2)
hence	$B \times C = A \times D$	

Q.E.D.

Now, conversely,

Given: $A \times D = B \times C$
Prove: $A : B = C : D$

We know that	$A \times D = B \times C$	(given)

But since it is also true that $B \times A = A \times B$, we can say that

	$B \times A : A \times B = B \times C : A \times D$	
so	$B \times A : B \times C = A \times B : A \times D$	(alternating)

But equimultiples have the same ratio has the terms they multiply (Ch.5, Thm.13),

and so	$A : C = B : D$	
or	$A : B = C : D$	(alternating)

Q.E.D.

THEOREM 5: *Beginning with any two unequal numbers, the last number produced by reciprocal subtraction is a common factor of them both.*

Of course you are wondering what "reciprocal subtraction" means. It's not hard. Take unequal numbers, say 9 and 7, and take the lesser from the greater:

$9 - 7 = 2$

Now compare the subrtracted number to the difference, and take the lesser from the greater:

$7 - 2 = 5$

Now compare the subtracted number to the difference, and take the lesser from the greater:

$5 - 2 = 3$

You get the idea. This process cannot go on forever, since numbers are finite and the number from which we are subtracting is smaller every time. So we must eventually end up with nothing. But the only subtraction that leaves us with nothing is when something is subtracted from itself. Hence the *last* subtraction in such a process must be

[1] $X - X = 0$

Now the Theorem states that X is a factor of both original numbers, whatever they may be. Why? Note that after the first step, the two numbers in the subtraction for any step are (a) the number that was subtracted in the previous step, and (b) the difference in the previous step. So, given our *last* step, the *second to last step* must be

[2] $Z - X = X$

and so $Z = 2X$; so X is a factor of Z. Again, the *third to last step* must be

[3] $Q - X = Z$

(or else $Q - X = Z$; it doesn't matter which). Thus $Q = X + Z$, or $Q = 3X$ (since $Z = 2X$ by Step 2). So X is now a factor of Q as well. Accordingly, X must be a factor of *every* number in the whole process, right back to our original numbers A and B.

Q.E.D.

THEOREM 5 Remarks:

1. Here is a numerical example. Start with 27 and 12:

$$
\begin{aligned}
&27 - 12 = 15\\
&\quad 15 - 12 = 3\\
&\qquad 12 - 3 = 9\\
&\qquad\quad 9 - 3 = 6\\
&\qquad\qquad 6 - 3 = 3\\
&\qquad\qquad\quad 3 - 3 = 0
\end{aligned}
$$

So the last number produced was 3, and indeed it is a factor of both 27 and 12. Notice, too, that the last step is of the form X – X = 0, and the second to last step is of the form Z – X = X, as asserted in the proof.

2. What happens if we choose numbers that are prime to each other, which have no common factor except the unit? Then the process must end by producing the unit, since we proved that the process ends with a common factor of the two original numbers. Try an example. Start with 9 and 4, which are prime to each other:

$$
\begin{aligned}
&9 - 4 = 5\\
&\quad 5 - 4 = 1\\
&\qquad 4 - 1 = 3\\
&\qquad\quad 3 - 1 = 2\\
&\qquad\qquad 2 - 1 = 1\\
&\qquad\qquad\quad 1 - 1 = 0
\end{aligned}
$$

Notice that 1 showed up before the process finished – the Theorem does not forbid that, but demands that if 1 is the only common factor of the two original numbers, the last number in the process must be 1.

THEOREM 6: *Given any pair of numbers that are prime to each other, to find multiples of them that differ by a unit.*

Suppose you have a pair of numbers A and B which are prime to each other. Can you multiply each of them so that the multiples differ by 1? Consider 12 and 5, which are prime to each other. By Theorem 5, we know that their reciprocal subtraction must produce the number 1, at least at the end of the process if not sooner. Let's go through the steps, then:

[1] $12 - 5 = 7$
[2] $7 - 5 = 2$
[3] $5 - 2 = 3$
[4] $3 - 2 = 1$

Since we hit 1 (as we must), let's stop on this last Step [4]:

$$3 - 2 = 1$$

Now, by Step [3] we see that $3 = 5 - 2$. So replace:

$$(\mathbf{5 - 2}) - 2 = 1$$

And by Step [2] we see that $2 = 7 - 5$. So replace:

$$[5 - (\mathbf{7 - 5})] - (\mathbf{7 - 5}) = 1$$

And by Step [1] we see that $7 = 12 - 5$. So replace:

$$[5 - \{ (\mathbf{12 - 5}) - 5 \}] - [(\mathbf{12 - 5}) - 5] = 1$$

Now the expression is exclusively in terms of the original numbers 12 and 5. Simplifying, we have

$$(5 + 5 + 5 + 5 + 5) - (12 + 12) = 1$$

That is, we have a multiple of 5 and a multiple of 12 which differ by 1. Since this process does not depend on 5 and 12 in particular, but presumes only that the original pair of numbers are prime to each other, we have found a way to multiply any pair of numbers that are prime to each other so that their multiples differ by 1.

Q.E.F.

THEOREM 6 Remarks:

1. Using the process described in the Theorem, find a pair of multiples of 16 and 7 that differ by 1.

2. Sometimes, of course, a pair of numbers that are prime to each other *themselves* differ by 1. In fact, any two consecutive numbers are prime to each other, such as N and N + 1. Why? Because any number that measures them both must also measure their difference, namely 1. But the only thing that measures 1 is 1. Therefore the only thing that measures both is 1, i.e. N and N + 1 are prime to each other.

3. The Theorem stipulates that the original pair of numbers must be prime to each other. Must we use numbers that are prime to each other, or can we use any pair of numbers? We must use numbers that are prime to each other. If instead we start with a pair of numers that have a common factor greater than 1, then it will be impossible to find multiples of them that differ by 1. Proof:

Take any two numbers, A and B, and any unequal multiples of them, NA and MB. Now any number that measures both A and B must also measure both their multiples NA and MB. And whatever measures both NA and MB must also measure their difference. Hence, if A and B have a common factor greater than 1, then since this common factor must measure the difference between any unequal multiples of A and B, it follows that such a difference must be greater than 1. In general *all unequal multiples of any two numbers must differ by at least the greatest common factor of those two numbers*. For example, since 4 and 6 have a greatest common factor of 2, all multiples of 4 and 6 that differ must differ by 2 or more.

THEOREM 7: *Two numbers proportional to two numbers that are prime to each other are equimultiples of them.*

Given:	A is prime to B
	A : B = C : D
Prove:	A and B measure C and D the same number of times.

[1] Since A is prime to B, therefore find multiples of them that differ by 1 (Thm.6).

Say	nA – mB = 1	

[2]

Now	A : B = C : D	(given)
so	nA : mB = nC : mD	(corresponding multiples)

Subtracting the consequent from the antecedent in each case, we will still have a proportion (Ch.5, Thm.16, Remark 3).

So	nA – mB : mB = nC – mD : mD	
or	1 : mB = nC – mD : mD	(nA – mB = 1; Step 1)

[3] Hence $mD = mB(nC - mD)$ (products of means and extremes)
thus $D = B(nC - mD)$ (dividing both sides by m)
and so B measures D exactly $(nC - mD)$ times.

[4] Now looking at the given proportion, and multiplying the means and extremes,

$$A \cdot D = B \cdot C$$

or $A \cdot B \cdot (nC - mD) = B \cdot C$ ($D = B \cdot (nC - mD)$, by Step 3)

and so $A \cdot (nC - mD) = C$ (dividing both sides by B)
and so A measures C exactly $(nC - mD)$ times.

[5] Looking at Steps 3 and 4, we see that B measures D and A measures C, and the same number of times in each case.

Q.E.D.

THEOREM 7 Remarks:

1. How do we know that we can subtract $nC - mD$? This will not be a number at all unless nC is greater than mD. Look at Step 2: $nA : mB = nC : mD$. We know that nA is greater than mB, since $nA - mB = 1$ (Step 1). Hence, by the proportion, it must also be true that nC is greater than mD.

2. From this Theorem, it is evidence that numbers which are prime to each other are the least numbers in their ratio, measuring all others that are in the same ratio with themselves. For example, 5 and 6 are prime to each other, and all other numbers in their ratio are equimultiples of them, such as 10 and 12.

THEOREM 8: *If* N *is the greatest common factor of* A × N *and* B × N*, then* A *and* B *are prime to each other.*

For example, 3 is the greatest common factor of 6 and 15, i.e. of 2×3 and 5×3. And, as the Theorem states, 2 and 5 have no common factor but 1. Now let's prove it generally:

Given: Two numbers $A \times N$ and $B \times N$, and N is their greatest common factor.
Prove: A and B are prime to each other.

If possible, suppose A and B have a common factor other than 1. Then say it is M, and

$$A = R \times M$$

and $B = S \times M$.

Thus $A \times N = R \times M \times N$ (substituting $R \times M$ for A)
and $B \times N = S \times M \times N$ (substituting $S \times M$ for B)

So $M \times N$ is now a common factor of $A \times N$ and $B \times N$. But since M is supposedly greater than 1, it follows that $M \times N$ is greater than N. Therefore $M \times N$ is a common factor of $A \times N$ and $B \times N$ *and it is greater than* N. Which is absurd, since it is given that N is the *greatest* common factor of $A \times N$ and $B \times N$. Therefore our original assumption, namely that A and B have a common factor other than 1, is impossible. Therefore A and B have no common factor other than 1, and so they are prime to each other.

Q.E.D.

THEOREM 9: *If four numbers are proportional, the greatest common factor of the first pair measures each the same number of times that the greatest common factor of the second pair measures each.*

Given: $A \times N : B \times N = C \times M : D \times M$,
N is the greatest common factor of $A \times N$ and $B \times N$
M is the greatest common factor of $C \times M$ and $D \times M$

Prove: $A = C$
$B = D$

[1] First of all, we know that $A : B = C : D$, just by removing the identical multipliers in the given proportion (cf. Ch.5, Thm.13).

[2] We also know that A is prime to B, since N is the greatest common measure of $A \times N$ and $B \times N$ (Given, and Thm.8).

[3] Therefore C and D are equimultiples of A and B (looking at Steps 1 and 2, and applying Thm.7).

[4] Likewise, C is prime to D, since M is the greatest common measure of $C \times M$ and $D \times M$ (Given, and Thm. 8).

[5] Therefore A and B are equimultiples of C and D (applying Thm.7 to Steps 1 and 4).

[6] Thus A and B are equimultiples of C and D (Step 3), and yet C and D are also equimultiples of A and B (Step 5). Which is impossible, except when $A = C$ and $B = D$.
Therefore $A = C$ and $B = D$.

Q.E.D.

THEOREM 9 Remarks:

Verify the Theorem in some numerical examples. Find any proportional numbers, and divide out the greatest common factor of the first two numbers and again the greatest common factor of the last two numbers, and see what you have left.

THEOREM 10: *A prime number is prime to any number that is not its multiple.*

Given: P is a prime number.
N is any number which is not a multiple of P.

Prove: N and P have no common factor but 1.

P has no factors but 1 and P (by the definition of a prime number).
N does not have P as a factor (given).
Therefore 1 is the only common factor of N and P.
Therefore N and P are prime to each other.

Q.E.D.

THEOREM 10 Remarks:

What about 1? Is that a "prime number"? If so, it is an unusual one, since every other number is a multiple of it. Shall we say, then, that 1 is not prime to any number, since it measures them all? But then again, 1 and 3 *are* prime to each other in the sense that nothing except 1 measures both of them. Still, 1 measures 3, and that kind of thing does not happen with other numbers that are prime to each other. Take any other pair of numbers that are prime to each other, such as 3 and 5, and we see that neither one can measure the other; if 3 measured 5, then they would have another common factor besides 1, namely 3. Hence, in the case of numbers other than 1, if two numbers are prime to each other, then neither one measures the other.

THEOREM 11: *If a number is prime to two numbers, it is also prime to their product.*

For example, 3 is prime to 4 and also prime to 5. Hence 3 is also prime to 4×5. Now let's prove it generally:

Given: P is prime to A and also to B.
Prove: P is prime to $A \times B$.

[1] To see it, let F be any common factor of P and $A \times B$. Suppose, then that

$$P = F \times N$$

and

$$A \times B = F \times M.$$

[2] Making a proportion out of these two equalities, we get

$$A \times B : F \times M = P : F \times N$$

[3] According to Ch.5, Theorem 13, we can divide the first two by the same number, and maintain the proportion. So divide the first two equal numbers by B:

$$A : (F \times M) \div B = P : F \times N$$

Now let's alternate that proportion:

$$A : P = (F \times M) \div B : F \times N$$

Again, we can divide both numbers in the second ratio by F and maintain the proportion:

$$A : P = M \div B : N$$

[4] And since A and P are given as prime to each other, it follows that N is a multiple of P (Thm.7).

But $P = F \times N$ (Step 1)

So P is also a multiple of N.

But the only way P and N can be multiples of each other is if $P = N$, and $F = 1$.

[5] Since F was a randomly taken common factor of P and $A \times B$, and F must equal 1, therefore 1 is the only factor common to P and $A \times B$. Therefore P is prime to $A \times B$.

Q.E.D.

THEOREM 11 Remarks

1. For example, 3 is prime to 4 and also to 5. So 3 is also prime to 4×5.

2. It also follows that if P is prime to A and to B and to C, that P must also be prime to the product of all three. For since P is prime to A and also to B, therefore, by the above Theorem, it is also prime to $A \times B$. Since P is prime to $A \times B$ and also to C, therefore, by the above Theorem, it is also prime to $A \times B \times C$.

Obviously, this works for any number of numbers to which P is prime.

3. Does it follow from the Theorem that P does not measure $A \times B$? Not necessarily. Suppose $P = 1$. If P, A, B are 1, 3, 5, then 1 is prime to 3×5 because it has no common factor with it but 1, which is itself. But it also measures 3×5. However, if P is anything besides 1, then it will not measure $A \times B$, since it would then have another factor in common with $A \times B$, namely P itself.

THEOREM 12: *If a prime number measures a product, it also measures at least one of its factors.*

Given: The number $A \times B \times C$, a product which is divisible by prime number P.
Prove: P is a factor of at least one of the numbers A, B, C.

If P were *not* a factor of *any* of the numbers A, B, C, what would follow? Since none of those numbers would be a multipl3 of P, then by Theorem 10 it would follow that P is prime to each of them. Since P would be prime to each of the numbers A, B, C, then by Theorem 11, it would follow that P is prime to their product $A \times B \times C$. But P is *not* prime to that product (since it is given that it *measures* that product, and so P and $A \times B \times C$ have at least P as a common factor). Therefore P cannot be prime to A and to B and to C. Therefore P must measure at least one of them.

Q.E.D.

THEOREM 12 Remarks:

From this Theorem we can draw the following corollary: *If a prime number measures a square number, then it also measures its side*. For suppose that the prime number P measures the square number $S \times S$. By the present Theorem, P must therefore measure at least one of the factors of $S \times S$, i.e. it must measure S, the side of the square number. For example, if a prime number measures 6×6, it must also measure 6. 1, 2, and 3 are the only prime numbers that measure 36, and all of them also measure 6.

THEOREM 13: *Any number is expressible as a product of prime numbers in only one way.*

For example, $24 = 2 \cdot 2 \cdot 2 \cdot 3$. And 24 has no other prime factors. Now let's show that every number is expressible in such a way.

Given: N, any number you like.
Prove: N is expressible as the product of certain prime numbers, and it cannot be expressed as any other product of prime numbers.

Obviously, if N is prime, we can express it as $1 \times N$, where 1 and N are both prime numbers. And it cannot be expressed as the product of any other numbers at all, since, being prime, N has no other factors.

What if N is a composite number? Then it is the product of numbers other than itself and 1, e.g. $N = A \times B$. And if A and B are both composite, then we can express N as a product of *their* factors, i.e. $N = (C \times D) \times (E \times F)$.

But since the factors get ever smaller as we do this, we cannot continue forever, but all composite factors will eventually reduce to factors which have no further factors, i.e. to prime factors. Thus N itself will be expressible as the product of prime numbers, say, like this:

$$N = P_1P_2P_3P_4$$

Of course the order of the terms makes no difference, and some of these prime numbers might be equal (as in $75 = 5 \times 5 \times 3$). But the question before us is this: is it possible to express N as the product of a *different* collection of prime numbers? For example:

$$N = R_1R_2R_3 \ldots$$

Even without comitting ourselves to how many primes are in this "new" expression for N, we can prove that these primes must in fact be the same as those in the product $P_1P_2P_3P_4$. For since N is divisible by P_1, therefore $(R_1R_2R_3 \ldots)$ is too. But since P_1 is prime, therefore it measures one of the primes in $(R_1R_2R_3 \ldots)$, according to Theorem 12. But since $(R_1R_2R_3 \ldots)$ are all primes, each is measured only by itself and 1, and so the prime P_1 must be identical to one of the primes in $(R_1R_2R_3 \ldots)$. Say, then, that $P_1 = R_1$. We can prove, likewise, that $P_2 = R_2$, etc. Therefore all the primes in $(R_1R_2R_3 \ldots)$ end up being none other than the primes $P_1P_2P_3P_4$. Therefore N is expressible as a product of certain prime numbers, and *only* those prime numbers.

We may say, then, that the "prime factorization" of any number is like its DNA code, its fingerprint.

Q.E.D.

THEOREM 13 Remarks:

1. Since every composite number is expressible as a unique product of primes, but no prime number is expressible as a product of composites, there is a kind of priority of prime numbers over composite numbers. That is why they are called *primes*, from *primus* for "first."

2. Exercise: find the prime factorization of the following numbers …
   12, 120, 85, 496, 2808.

THEOREM 14: *All the factors of any number are: the primes in its prime factorization, all their products, and 1.*

Take any number N, and say its prime factorization is $A \times B \times C$.
Suppose F is some factor of N besides 1 and N itself. I say that F is either A, or B, or C, or some product formed out of these numbers (i.e. either $A \times B$ or $A \times C$ or $B \times C$ or $A \times B \times C$).

Proof: Since F is a factor of N, then the prime factorization of F contains only primes which are also in the prime factorization of N (otherwise N could be expressed as a product of primes in more than one way, contrary to Theorem 13). But A, B, and C are the primes in the prime factorization of N. Therefore F contains only A, or B, or C (or some combination of them) in its prime factorization.

Since it is obvious that $A \times B \times C$ does in fact have A, B, C (and any product formed out of these three numbers) as factors, and since we have just shown that these are the *only* factors, the Theorem is proved.

Q.E.D.

THEOREM 14 Remarks:

If two or more of the prime factors are the same, then some of the factors of the number N will simply be powers of those prime factors. For example, suppose we take the number 56, whose prime factorization is $2 \times 2 \times 2 \times 7$. What are the factors of 56? According to the present Theorem, they are 1, 2, 2^2, 2^3, 7, 2×7, $2^2 \times 7$, $2^3 \times 7$.

THEOREM 15: *There is always a bigger prime number.*

Given: P is any prime number you like.
Prove: There is another prime number bigger than P.

Suppose someone thought the prime number 5 was the biggest prime number of all, so that no number after 5 was prime. Let's prove him wrong.

Form the number 5! + 1 (i.e. "five factorial plus one"), which is the number

$$(5 \times 4 \times 3 \times 2 \times 1) + 1$$

If this number is prime, then since it is obviously greater than 5, we are done. On the other hand, if it is composite, since every composite number is expressible as the product of primes (Thm.13), so this number will be also. But since 2 goes evenly into $(5 \times 4 \times 3 \times 2 \times 1)$, then it will not go evenly into $(5 \times 4 \times 3 \times 2 \times 1)$ *plus one*, because the unit is not divisible. For the same reason, $(5 \times 4 \times 3 \times 2 \times 1) + 1$ is not evenly divisible by 3, or by 4, or by 5. Therefore whatever prime number *does* measure $(5 \times 4 \times 3 \times 2 \times 1) + 1$ has to be greater than 5. Therefore there is a prime number greater than 5.

By exactly the same argument, we can always prove that there is a prime number greater than any given prime number P, just by forming the number P! + 1. This means that there is an unlimited multitude of prime numbers.

Q.E.D.

THEOREM 15 Remarks:

Note that sometimes P! + 1 is itself a prime greater than P. In other cases it is not prime, but is still measured by a prime greater than P.

For example, if someone says 3 is the greatest prime number, we can refute him by proving that 3! + 1 is either a greater prime, or is measured by one. In this case, 3! + 1 = 7, which is a prime number greater than 3.

But if someone says 5 is the greatest prime number, we can refute him by proving that 5! + 1 is either a greater prime, or is measured by one. In this case, 5! + 1 = 121, which is not prime, but it is measured by 11, which is a prime greater than 5. Also, notice that 11 is not the *next* prime after 5.

Accordingly, the proof shows that there must always be a greater prime, but gives us no sure way of producing it. To this day, there is no known formula which generates all the prime numbers. The only way to find the primes is to list all the numbers as far as you like and cross out all the ones that have factors other than 1; the remaining numbers are the primes. This method is called the "Sieve of Eratosthenes."

THEOREM 16: *Give me any number: I can always find that many consecutive numbers none of which is prime.*

We might wonder how primes are "spaced" as we take the next biggest prime, the next biggest, etc. Are they more or less evenly spaced, or do they become "rarer"? They become much rarer. For example, is it possible to find 99 consecutive numbers none of which is prime? Yes indeed.

Just form the factorial number (99 + 1)!, or 100!

Now	100! + 1	is, of course, divisible by 1, but it *might* be prime.
Still,	100! + 2	must be divisible by 2, since 100! is divisible by 2, and so is 2. Thus 100! + 2 is *not* prime.
Again,	100! + 3	is divisible by 3, and thus *not* prime.
Again,	100! + 4	is divisible by 4, and thus *not* prime.

etc., etc. all the way up to

	100! + 100	which is divisible by 100, and therefore *not* prime.

So there you have 99 consecutive numbers none of which is prime. Likewise, we can construct a *billion* consecutive numbers none of which is prime! Just begin with the number (1,000,000,000 + 1)! + 2.

Q.E.F.

THEOREM 16 Remarks:

Notice that even though the prime numbers "thin out" as we climb upward in the numbers, that does not mean the distances between consecutive primes steadily increases as we go. That is because strange clusters of prime numbers show up – something still not well understood to this day. For example, prime numbers which have only one number between them are called "Twin Primes", such as 5 and 7, or 29 and 31, or 197 and 199, or 821 and 823. As I write these words, it is still not known to the mathematical community whether or not the "Twin Primes" run out, i.e. whether there is a highest pair of twin primes, or if there is always another pair.

There are only 26 prime numbers less than 100, which are:
1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97. How many pairs of twin primes can you find in this list?

THEOREM 17: *How to find the greatest common factor of any two numbers.*

Given: Two numbers A and B, and A is greater than B.
Find: Their greatest common factor (or "G.C.F." for short).

Repeatedly subtract the lesser from the greater till nothing remains (which must happen, since numbers are finite):

$A - B = C$
$C - B = D$
$B - D = E$
etc. etc. ...
$Z - X = X$
$X - X = 0$

Then X is the greatest common factor of A and B!

First of all, that X is a common factor of A and B is clear already from Theorem 6 (go back and look at it if you forget!). But how do we know that it is the *greatest* factor common to A and B?

Take any number N which is a factor of both A and B. I will show you that it *cannot* be greater than X.

Since N measures both A and B, therefore it measures their difference, which is A – B, i.e. C. Since we now see that N measures both B and C, it must also measure their difference, i.e. D. Since we now see that N measures both C and D, it must measure their difference, E, etc. So N must measure every number in the whole process, including X. Therefore any factor common to A and B must also measure X, and therefore must be equal to or less than it. Therefore X is the greatest factor common to A and B.

Q.E.F.

THEOREM 17 Remarks:

1. If A and B are prime to each other, D will equal 1. If they are not prime to eacah other, D will be some number greater than 1. So for any two numbers there is always a G.C.F., even if it is only 1.

2. We can see from this that *If a number measures two numbers, it will also measure their greatest common factor.*

3. Notice that in this Theorem we have found a way to discover the greatest common factor of two numbers *without factoring either number*! Let's try an example. What is the greatest common factor of 36 and 81?

$81 - 36 = 45$

$45 - 36 = 9$

$36 - 9 = 27$

$27 - 9 = 18$

$18 - 9 = 9$

$9 - 9 = 0$

So 9 is their greatest common factor. Now try it with 176 and 132.

4. Obviously, we can also find the G.C.F. of *three* numbers A, B, C simply by finding the G.C.F. of A and B (say it is N), and then the G.C.F. of B and C (say it is M), and then finding the G.C.F. of N and M (say it is G). G then has to be the GCF of the 3 numbers A, B, C. For since the G.C.F. of the 3 numbers measures both A and B, it must also measure *their* G.C.F., namely N. And since it measures both B and C, it must also measure *their* G.C.F., namely M. And since it measures both N and M, it must measure *their* G.C.F., namely G. So the G.C.F. of A, B, C must measure G. And therefore G *is* the greatest common factor of the 3 numbers. *Q.E.F.*

THEOREM 18: *How to find the least common multiple of any two numbers*.

Given: Two numbers, A and B, and A is the greater one.
Find: Their least common multiple (or "L.C.M.", for short).

[1] First, by Theorem 17, find the greatest factor common to both A and B. Say it is N, and that

$A = C \times N$

$B = D \times N$

[2] Therefore

$A : B = C \times N : D \times N$ (By the two equalities above)

thus $A : B = C : D$ (Chapter 5, Thm.13)

Now since N is the greatest factor common to A and B, it follows that C and D are prime to each other (Thm. 8).

[3] Now the product of the extremes is equal to the product of the means, so

$A \times D = B \times C$ (Thm.4)

Notice that $A \times D$ is measured by A (namely D times), but it is also measured by B (namely C times). Therefore $A \times D$ is a common multiple of A and B. I say that it is also the *least*.

[4] For let A and B measure any *other* number, K, and say that

$A \times R = K$

$B \times T = K$

So that K is also a multiple of both A and B. I say that K is greater than $A \times D$.

[5] Now since the factors of equal numbers are reciprocally proportional (Thm.7),

and	$A \times R = B \times T$	(Since they both equal K)
Thus	$A : B = T : R$	(Thm.7)
But	$\underline{A : B = C : D}$	(Step 2)
so	$C : D = T : R$	
But	C and D are prime to each other	(Step 2)
Thus	D measures R	(Thm.7)
Thus	D is less than or equal to R.	
So	$A \times D$ is less than or equal to $A \times R$ (multiplying both by A).	
i.e.	$A \times D$ is less than or equal to K.	

And since, by supposition, K is a common multiple of A and B *other* than $A \times D$, therefore $A \times D$ is less than K.

Thus $A \times D$ is the least common multiple of A and B.

Q.E.F.

THEOREM 18 Remarks:

1. There is no *greatest* common multiple of two numbers, since all multiples of their least common multiple will also be common multiples of them, and we can multiply their least common multiple as many times as we like.

2. Let's try a numerical example. Take 24 and 18. What is the least number that both of them measure? Divide out their greatest common factor, namely 6, and we form the proportion

$$24 : 18 = 4 : 3$$

Multiplying the extremes (or the means), we get 72, which is the least common multiply of 24 and 18. Now try it with 36 and 15.

3. Notice that the product of two numbers is always a common multiple of them, but it is not always their *least* common multiple. For example, $4 \times 6 = 24$, and 24 is indeed a multiple of both 4 and 6, but it is not their least common multiple – rather, it is 12. Other times, however, it is the least, e.g. $5 \times 6 = 30$, and 30 is the least multiple common to 5 and 6. When does that happen? When the two original numbers are prime to each other. Use the procedure for finding the L.C.M. of any pair of numbers that are prime to each other (such as 5 and 6) and you will quickly see why this must be so.

4. Can you see how to find the L.C.M. of any three numbers?

THEOREM 19: *If the base of a triangular number is N, then the triangular number equals* $\frac{N(N+1)}{2}$.

Consider the number $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$.
By definition, that is a triangular number whose base is 10.
Now rearrange the addition by putting together pairs of numbers starting with the two ends (1 and 10), and working your way in toward the middle (2 and 9 are next):

$$(1 + 10) + (2 + 9) + (3 + 8) + (4 + 7) + (5 + 6)$$

Notice that we have here five sums each equal to 11, so the whole sum is $5 \times 11 = 55$. The pairs will all be the same, since we start out with $1 + 10$, and in the next pair we add 1 to the first number (giving us 2), but we *subtract* 1 from the second number (giving us 9), and so on.

Since there are 10 original numbers being added, that is $\frac{10}{2}$ pairs, or 5 pairs, each of which equals $10 + 1$. So the whole sum is $\frac{10(10+1)}{2}$.

What happens if we have an *odd* number of numbers, you ask?

Consider $1 + 2 + 3 + 4 + 5 + 6 + 7 = K$.

This number K equals *half* of $(1 + 2 + 3 + 4 + 5 + 6 + 7) + (1 + 2 + 3 + 4 + 5 + 6 + 7)$.
Now rearrange, as before, in pairs of numbers, and now K is *half* of

$$(1 + 7) + (2 + 6) + (3 + 5) + (4 + 4) + (5 + 3) + (6 + 2) + (7 + 1)$$

each of these pairs being equal to $(7 + 1)$. And how many pairs are there? Since we simply repeated our original numbers twice, there are 7 pairs, i.e. as many pairs as original numbers. The entire sum, then, is equal to the number of these pairs times the value of each pair, i.e. $7(7 + 1)$. But K is only *half* of that, and so it is equal to

$$\frac{7(7+1)}{2}$$

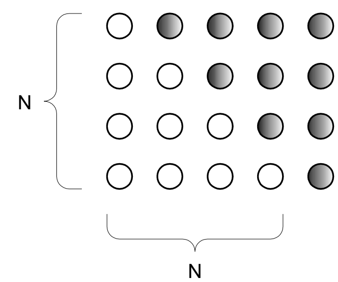
So the formula $\frac{N(N+1)}{2}$ works equally well with odd numbers.

Q.E.D.

THEOREM 19 Remarks:

1. Carl Friedrich Gauss, the great German mathematician, discovered this formula for triangular numbers all by himself at the tender age of 6. His teacher assigned his class this problem: add all the numbers from 1 to 100. The teacher, expecting the students to be busy for a solid hour, was surprised when little Gauss approached him with the correct answer right away: 5050. Gauss saw that 1 + 2 + 3 + … + 98 + 99 + 100 is equal to 50 pairs each equal to 101. So he simply multiplied 50 times 101 and got his answer.

2. It is possible to give a "visual proof" for this Theorem. Let a triangular number, having a base equal to N, be represented by filled circles placed in the form of a right triangle. If we place an identical triangular number of empty circles against it, we complete a rectangle with sides of N and N + 1. The whole rectangle contains N (N + 1) circles. Accordingly, since the original triangular number is only half this rectangle, it contains N (N + 1) ÷ 2 circles. Hence the triangular number of base N is $\frac{N(N+1)}{2}$.



THEOREM 20: *The sum of two consecutive triangular numbers is a square number, and the base of the larger triangular number is the side of the square.*

Given: Two consecutive triangular numbers, namely

$$\frac{N(N+1)}{2} \text{ and } \frac{(N+1)[(N+1)+1]}{2}$$

which, by the last Theorem, are triangular, and which are consecutive because they have consecutive bases N and N + 1.

Prove: The sum of these two numbers is a square number whose side is N + 1.

Our two triangular numbers are half of N(N + 1) and (N + 1)[(N + 1) + 1] respectively. So their sum is half the sum of these two numbers, or

N (N + 1) + (N + 1) [(N + 1) + 1] *divided by two*,

which is N (N + 1) + (N + 1) (N + 2) *divided by two*.

Now, using Theorem 1 of Chapter 5, we can "factor out" the (N + 1) common to both parts of our sum, which is to say that our whole sum is the same as

(N + 1) [N + (N + 2)] *divided by two*

which is (N + 1) (2N + 2) *divided by two*

Again, using Theorem 1 of Chapter 5, we can "factor out" the 2 common to both parts of (2N + 2), which is to say that our whole sum is the same as

$(N + 1)\ (N + 1)2$ *divided by two*

And since our whole sum is reached by multiplying by two but then dividing by two, we can simply say that the sum of our two triangular numbers is

$(N + 1)\ (N + 1)$

which is, by definition, a square number whose side is $N + 1$.

Q.E.D.

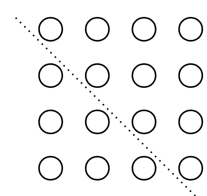
THEOREM 20 Remarks:

1. As a kind of porism, we can say that any square number is equal to the sum of consecutive numbers from 1 up to its square root and back down to 1. For example,

$16 = 1 + 2 + 3 + 4 + 3 + 2 + 1$

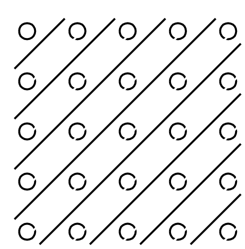
The reason is that $1 + 2 + 3$ is a triangular number
and again $1 + 2 + 3 + 4$ is the next triangular number,
and so, by the Theorem, their sum is equal to 4^2. But their sum is the same as the numbers from 1 up to 4 and back down again to 1.

2. Can a number be both square *and* triangular? Yes. For example, 36 is both 6^2 and also $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$. How many numbers are both square and triangular? An infinity of them, but we won't prove that here.



3. As you have probably guessed, it is possible to give a kind of visual proof for this Theorem. Arrange any square number of dots in a square pattern, and you will see that you can slice it into two consecutive triangular numbers. For example, 4^2 equals the triangular number on base 3 plus the triangular number on base 4.

If you draw lines diagonally through a square number of things arranged in a square pattern, you can see that the longest diagonal – 5 in the diagram – equals the side of the square and is the common base of two identical triangular numbers. You can see why $25 = 1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1$.



THEOREM 21: *Every square number* N^2 *is the sum of* N *consecutive odd numbers beginning with* 1; *and the even number following the last odd number added is double the side of the square.*

Notice	1 is a square, for it equals	1×1,
and	$1 + 3$ is a square, for it equals	2×2,
and	$1 + 3 + 5$ is a square, for it equals	3×3.

Will this always work? Yes, and here's why:

Take any square number N^2, having side N. This number is the sum of two consecutive triangular numbers, say one with base N and one with base N – 1, which is to say that

$$\begin{array}{lll} N^2 = & 1 + 2 + 3 + 4 + \ldots + N & \text{(triangular number with base N)} \\ & \underline{\quad + 1 + 2 + 3 + \ldots + N - 1} & \text{(triangular number with base N – 1)} \end{array}$$

Now add the numbers in pairs, each number in the top row to the one below it in the bottom row, and we have

$$N^2 = 1 + (2 + 1) + (3 + 2) + (4 + 3) + \ldots + (N + N - 1)$$

Since in each case we are adding an even number and an odd one, the parentheses all contain odd sums. And since, moreover, each member of each pair is always *one* more than its corresponding member in the previous pair, the result is that each pair adds up to *two* more than the previous pair. i.e. we are adding *consecutive* odd numbers. i.e.

$$N^2 = 1 + 3 + 5 + 7 + \ldots + 2N - 1$$

And the last number added is $(2N - 1)$, and so the number after it is simply 2N, half of which is N, the side of the square.

Q.E.D.

THEOREM 21 Remarks:

Since 2N is double the number of odd numbers from 1 to N, it follows that N is not only the side of the square, but also the number of consecutive odds which add up to the square. For example, $1 + 3 + 5 = 9$, and the side of this square, namely 3, is also the number of odds added.

THEOREM 22: *The product of two square numbers is a square number (whose side is the product of their sides).*

Given: $A \times A$ and $B \times B$
Prove: $(A \times A)(B \times B)$ is a square

Since the order of multiplication makes no difference, therefore

$(A \times A)(B \times B) = (A \times B)(A \times B)$,

i.e. $(A \times A)(B \times B)$ is a square number whose side is $(A \times B)$.

Q.E.D.

THEOREM 22 Remarks:

For example, $5^2 \times 9^2 = (5 \times 9)^2$
or $25 \times 81 = 45^2 = 2025$.
It is also true that the product of two cubes is a cube whose side is the product of their sides. That is, $(A \times A \times A)(B \times B \times B) = (A \times B)(A \times B)(A \times B)$
For example $2^3 \cdot 5^3 = (2 \cdot 5)^3$
or $8 \cdot 125 = 10^3$

THEOREM 23: If a square times some number makes a square, the number is also a square.

Given: $(A \times A) \times N = (B \times B)$
Prove: N is a square number.

Now either A is prime, or else it is the product of primes (Thm.13). Either way, A can be expressed in terms of nothing but primes, so say

$$A = P_1P_2P_3$$

Since A measures $B \times B$, therefore all these primes measure $B \times B$. But any prime that measures a square number must also measure the side of the square (Thm.12 Remarks). Therefore all these primes measure B. Hence $P_1P_2P_3$ must be found in the prime factorization of B. But that means that each B in $B \times B$ is measured by $P_1P_2P_3$, or more simply, by A. Since each B is divisible by A, it follows that $B \div A$ is a number.

Now $A \times A \times N = B \times B$ (given)
so $N = (B \times B) \div (A \times A)$ (dividing both sides by $A \times A$)
so $N = (B \div A) \times (B \div A)$ (See Ch.6, Thm. 4)

and since $(B \div A)$ is a number, therefore N is a square number. *Q.E.D.*

THEOREM 23 Remarks:

1. For example, consider $5^2 \times N = 30^2$

clearly $N = 30^2 \div 5^2$

or $N = (30 \times 30) \div (5 \times 5)$

so $N = (30 \div 5) \times (30 \div 5) = 6 \times 6$

2. It follows from this Theorem that *No square number is double any other square number*. For if so, i.e. if $Q \times 2 = R$ where Q and R are both square numbers, it would follow by the Theorem that 2 is a square number, which it certainly isn't.

Thus, in general, it follows that *No square number is a non-square multiple of any other square number*. For example, no square number is 3 times another, or 5 times another, or 6 times another, etc.

THEOREM 24: *No prime number is to any other prime number as a square number is to a square number.*

Take any two distinct prime numbers N and M. I say that N and M do not have the same ratio as any two square numbers.

[1] If possible, assume that

$$N : M = A^2 : B^2$$

where A^2 and B^2 are the *least* square numbers having the ratio of N to M.

[2] Since N and M are distinct primes, they have no common factor but 1, and hence are prime to each other. Therefore, according to the proportion above, N must measure A^2 (Thm.7). But since N is prime, it must also measure A, since any prime number that measures a square number also measures its side (Thm.12, Remarks). Likewise M measures B.

[3] So let $A = K \cdot N$ and $B = L \cdot M$

Thus $A^2 = K^2 \cdot N^2$ and $B^2 = L^2 \cdot M^2$

[4] Then $A^2 : B^2 = K^2 \cdot N^2 : L^2 \cdot M^2$

But N and M have the same ratio as A^2 and B^2 (Step 1), so

$$N : M = K^2 \cdot N^2 : L^2 \cdot M^2$$

Now dividing both antecedents by N, and both consequents by M, we have

$$1 : 1 = K^2 \cdot N : L^2 \cdot M$$

From which it obviously follows that

$$K^2 \cdot N = L^2 \cdot M$$

Since the factors of equal numbers are reciprocally proportional (Thm.4), thus

$$N : M = L^2 : K^2$$

[5] By Step 3 it is plain that L^2 and K^2 are less than A^2 and B^2. But by Step 4, we see that L^2 and K^2 have the same ratio as N and M, whereas A^2 and B^2 are supposed to be the least squares in that ratio. Which is impossible. Therefore N and M cannot have the same ratio as two square numbers.

Q.E.D.

THEOREM 24 Remarks:

Citing Step 3, Step 5 asserts that $K^2 < A^2$ and $L^2 < B^2$. How is this clear from Step 3?
Because $A^2 = K^2 \cdot N^2$ and $B^2 = L^2 \cdot M^2$.
But what if $N = 1$? 1 is a prime number, after all. And in such a case
$A^2 = K^2$.
All right, but then since N and M are *distinct* primes, M is not 1, but some number greater than 1. So since $B^2 = L^2 \cdot M^2$ it will still be true that
$B^2 > L^2$.
So the Theorem will hold regardless.

From this Theorem, then, it follows that no two square numbers have the ratio of 1 to 2, or 1 to 3, or 2 to 3, or 3 to 5, etc.

THEOREM 25: *Two numbers whose factors are proportional have a mean proportional between them.*

Consider $A \cdot B$ and $C \cdot D$
where $A : B = C : D$.
I say that $A \cdot B$ and $C \cdot D$ have a mean proportional number between them.

For $A : B = C : D$ (given)
so $A : C = B : D$ (alternating the proportion)
thus $A{\cdot}B : B{\cdot}C = B : D$ (multiplying the 1st two terms by B)
and $A{\cdot}B : B{\cdot}C = B{\cdot}C : C{\cdot}D$ (multiplying the last two terms by C)
i.e. $B \cdot C$ is a mean proportional between $A \cdot B$ and $C \cdot D$.

Q.E.D.

THEOREM 25 Remarks:

1. Since $B \cdot C$ is equal to $A \cdot D$ (since we are given that $A : B = C : D$), it also follows that $A \cdot D$ is a mean proportional between the two given numbers. Although it is the same mean proportional, it is expressed in terms of different factors.

2. Any two square numbers have one mean proportional between them. For $N \times N$ and $M \times M$ obviously have proportional factors: $N : N = M : M$. Thus the mean proportional between them is the product of their sides, $N \times M$. Confirm this with some examples.

3. Also, any two cube numbers have *two* mean proportionals between them. Consider $A \cdot A \cdot A$ and $B \cdot B \cdot B$. The 2 means between them are $A \cdot A \cdot B$ and $A \cdot B \cdot B$. Why?
Because $A \cdot A : A \cdot B = A \cdot A : A \cdot B$ obviously, since these ratios have identical terms. Now we maintain the proportion if we multiply the first two terms by A and the last two terms by B, giving us
(1) $A \cdot A \cdot A : A \cdot A \cdot B = A \cdot A \cdot B : A \cdot B \cdot B$
Again $A \cdot B : B \cdot B = A \cdot B : B \cdot B$ obviously, since these ratios have identical terms. Multiplying the first two terms by A and the last two terms by B, we have
(2) $A \cdot A \cdot B : A \cdot B \cdot B = A \cdot B \cdot B : B \cdot B \cdot B$
Putting together Proportion (1) with Proportion (2), we see that $A \cdot A \cdot B$ and $A \cdot B \cdot B$ are mean proportionals between the two original cube numbers.

THEOREM 26: *Two numbers with a mean proportional between them have proportional factors.*

Given: $X : N = N : Z$
Prove: X and Z have proportional factors

[1] Find the G.C.F. of X and N, and call it G (Thm.17). Since G is a factor of both X and N, therefore each is equal to some number times G, say

$$A \cdot G = X$$
$$B \cdot G = N$$

[2] And since G is the greatest factor common to X and N, it follows that A and B are prime to each other (Thm.8). Now, from the two equalities in Step 1, it is clear that

	$A \cdot G : B \cdot G = X : N$	
thus	$A : B = X : N$	(dividing the first terms by G)
but	$\underline{X : N = N : Z}$	(given)
thus	$A : B = N : Z$	(each is the same as the ratio X : N)

[3] But A and B are prime to each other, and therefore N and Z are equimultiples of A and B (Thm.7). Say N and Z are each F times A and B, i.e.

$$A \cdot F = N$$
$$B \cdot F = Z$$

[4] Thus $A \cdot F = B \cdot G$ (each is equal to N; Steps 1 and 3)
so $A : G = B : F$ (means and extremes are proportional)

But these are the factors of X and Z, since

$A \cdot G = X$ (Step 1)

and $B \cdot F = Z$ (Step 3)

So that X and Z in fact have proportional factors.

Q.E.D.

THEOREM 26 Remarks:

It follows from this Theorem that two numbers whose factors are *not* proportional have *no* mean proportional between them. For if they *did* have a mean proportional between them, their factors *would* be proportional (by this Theorem). Find some numerical examples to confirm this and other numerical examples to confirm the Theorem.

THEOREM 27: *The product of two numbers with proportional factors is square.*

For example, consider 12 and 48, whose factors are proportional, since $3 \times 4 = 12$, and $6 \times 8 = 48$, and since $3 : 4 = 6 : 8$. Hence the product of 12 and 48 is square, and indeed $3 \times 4 \times 6 \times 8 = 576 = 24^2$. Now let's prove it generally ...

Given: $A \times B$ and $C \times D$
$A : B = C : D$

Prove: $A \times B \times C \times D$ is a square number

[1] First, $A : B = C : D$ (given)

[2] Thus $A \times D = B \times C$ (product of means = product of extremes)

[3] Therefore $(A \times D)\ (B \times C)$ is a square number, since by Step 2 these two factors are equal. That is, $A \times B \times C \times D$ is a square number (since the order of multiplication makes no difference).

Q.E.D.

THEOREM 28: *If one number measures the square of another, then there is a third number proportional to the two numbers; if not, not.*

Given: A measures B^2

Prove: There is a third proportional number N such that
$A : B = B : N$

Since A measures B^2 some definite number of times, let that number be N, so that

$$A \cdot N = B \cdot B$$

Thus $A : B = B : N$ (Thm.4)

So N is a third proportional to A and B. *Q.E.D.*

Given: A does *not* measure B^2

Prove: There is no third proportional number N such that
$A : B = B : N$

If possible, suppose that

$$A : B = B : N$$

then $A \cdot N = B \cdot B$ (Thm.4)

i.e. A *does* measure B^2, which is absurd, since we are given that A does *not* measure B^2. So there is no number N such that $A : B = B : N$, unless A measures B^2.

Q.E.D.

THEOREM 28 Remarks:

1. For example, do 5 and 6 have a third proportional number N, such that $5 : 6 = 6 : N$? No, since 5 does not measure 6×6 or 36. On the other hand, 4 and 6 have a third proportional number N, since 4 does measure 6×6, namely 9 times. So $4 : 6 = 6 : 9$.

2. If two numbers (neither of which is 1) are prime to each other, then there is no third proportional to them. Say A and B are prime to each other. Therefore A is prime to B^2, since A is given as prime to B and therefore it must also be prime to the product of two B's (Thm.11). Therefore A does not measure B^2. So there is no third proportional number to A and B (by the present Theorem). Therefore if two numbers are prime to each other, they have no 3rd proportional.

Of course, 1 and 3 are prime to each other, and yet they have a third proportional, namely 9. But that is because 1 is not only prime to every number, but measures every number (see Thm.11, Remark 3). So 1 must measure the square of *every* number, and therefore for 1 and any number X there will always be a third proportional.

THEOREM 29: *If one number measures the product of two others, then there is a fourth number proportional to them; if not, not.*

Given: A measures $B \times C$

Prove: There is a fourth proportional number to A, B, C
i.e. there is some number N such that $A : B = C : N$

A measures $B \times C$ some number of times (given). Say A measures it N times.
Hence $A \cdot N = B \cdot C$
thus $A : B = C : N$ (Thm.4)
i.e. A, B, C have a fourth proportional.

Q.E.D.

Given: A does not measure $B \times C$

Prove: There is no fourth proportional number to A, B, C

If possible, say $A : B = C : N$
thus $A \cdot N = B \cdot C$ (Thm.4)
i.e. A measures $B \times C$ exactly N times, which is absurd, since it is given that A does not measure $B \times C$. So it is impossible to find a fourth proportional number to A, B, C.

Q.E.D.

THEOREM 30: *If numbers are in a continuous proportion that begins from* 1, *then all terms after the second term are consecutive powers of it.*

Consider the numbers 1, 3, 9, 27, 81. They are in a continuous proportion beginning from 1, since $1 : 3 = 3 : 9 = 9 : 27 = 27 : 81$.
Also $9 = 3^2$
and $27 = 3^3$
and $81 = 3^4$
That is, all the terms after 3 are consecutive powers of 3. Every continuous proportion beginning from 1 must be this way. Here's why:

Given: A, B, C, D, E etc. in continuous proportion beginning from 1,
i.e. $1 : A = A : B = B : C = C : D = D : E$ etc.

Prove: $B = A^2$
$C = A^3$
$D = A^4$ etc.

[1] $1 : A = A : B$ (given)
so $B = A^2$ (products of means and extremes)

[2] But $A : B = B : C$ (given)
so $A \cdot C = B \cdot B$ (products of means and extremes)
or $A \cdot C = A \cdot A \cdot A \cdot A$ ($B = A \cdot A$, Step 1)
so $C = A \cdot A \cdot A$ (dividing both sides by A)
i.e. $C = A^3$

and so on.

Q.E.D.

THEOREM 31: *How to find the sum of* N *numbers that are in a continuous proportion beginning from* 1.

Recalling the last theorem, numbers in continuous proportion beginning from 1 look like this: $1, A, A^2, A^3, A^4$ etc.
If we want the sum of these up to and including A^4, just take the difference $(A^5 - 1)$ and divide it by $(A - 1)$. For example

$$1 + 3 + 3^2 + 3^3 + 3^4 = (3^5 - 1) \div (3 - 1)$$

Why does that work? For brevity,
let $S = 1 + A + A^2 + A^3 + A^4$
now $S\,(A - 1) = S \cdot A - S$ (Ch.5, Thm.1)
or $S\,(A - 1) = A\,(1 + A + A^2 + A^3 + A^4) - 1 - A - A^2 - A^3 - A^4$
so $S\,(A - 1) = A + A^2 + A^3 + A^4 + A^5 - 1 - A - A^2 - A^3 - A^4$
or $S\,(A - 1) = A^5 - 1$
hence $S = (A^5 - 1) \div (A - 1)$

Q.E.F.

THEOREM 31 Remarks:

1. Although the proof sums terms only up to A^4, you can see why $1 + A + A^2 + A^3 + A^4 + A^5 = (A^6 - 1) \div (A - 1)$. In general, the sum of continuously proportional numbers from 1 to A^M is $(A^{M+1} - 1) \div (A - 1)$.

2. In particular, then, $1 + 2 + 2^2 + 2^3 + ... + 2^X = (2^{X+1} - 1)$.

THEOREM 32: *If* $(2^N - 1)$ *is a prime number* P, *then the product* $T = (2^N - 1)\ (2^{N-1})$ *is a perfect number.*

[1] To see this, recall that a perfect number is a number equal to the sum of all its factors other than itself. Now what are all the factors of the product T? Since $(2^N - 1)$ is prime P, and since (2^{N-1}) is a power of 2 made of nothing but a bunch of twos multiplied together (and 2 is prime), therefore *all* the factors of $T = (2^N - 1)\ (2^{N-1})$ are:

{a} 1, plus all the powers of 2 up to 2^{N-1},

and {b} the products of P with 1 and with all those different powers of 2.

This we know from Theorem 14.

[2] As for the sum of {a}, namely of 1 and all the powers of 2 up to 2^{N-1}, this is equal to $2^{(N-1)+1} - 1$ (Thm.31, Remark 2).

So the sum of {a} = $(2^N - 1)$.

[3] As for the sum of {b}, namely the products of P with 1 and with all the powers of 2, this is equal to the product of P with the sum of 1 plus all those powers of 2 (Ch.5, Thm.1),

so the sum of {b} = $P\ (1 + 2 + 2^2 + 2^3 + ... + 2^{N-1})$

i.e. the sum of {b} = P [sum of {a}]

that is, sum {b} = $P\ (2^N - 1)$

[4] Since *all* the factors of T add up to sum {a} + sum {b}, hence all the factors of T add up to

add up to	$(2^N - 1) + P\ (2^N - 1)$	(Steps 2 and 3)
or	$(2^N - 1)\ (1 + P)$	(factoring out $(2^N - 1)$; Ch.5, Thm.1)
or	$(2^N - 1)\ (1 + 2^N - 1)$	(writing out the expression for P)
so	$(2^N - 1)\ (2^N)$ is the sum total of all the factors of T.	

[5] So what is the sum of T's factors that are less than itself? Nothing but the sum of all its factors minus itself, i.e.

$$(2^N - 1)\ (2^N)\ -\ T$$

Replacing T with its whole expression, this is

$$(2^N - 1)\ (2^N)\ -\ (2^N - 1)\ (2^{N-1})$$

But 2^N is *double* 2^{N-1}, since every power of two is double the previous power of two. Hence $2^N = (2^{N-1} + 2^{N-1})$. Substituting this expression for 2^N, all the factors of T less than itself add up to

$$(2^N - 1)\ (2^{N-1} + 2^{N-1})\ -\ (2^N - 1)\ (2^{N-1})$$

But the product on the left equals the sum of $(2^N - 1)$ times each term in the parentheses (Ch.5, Thm.1). So we have

$$(2^N - 1)\ (2^{N-1})\ +\ (2^N - 1)\ (2^{N-1})\ -\ (2^N - 1)\ (2^{N-1})$$

But this simply leaves

$$(2^N - 1)\ (2^{N-1})$$

which is T !

[6] So the sum of all T's factors less than itself is equal to T itself. Therefore T is a perfect number.

Q.E.D.

THEOREM 32 Remarks:

1. For example, $2^2 - 1$ is prime, since it is 3. Therefore $(2^2 - 1)(2^{2-1})$ must be perfect. Working it out, this product is $(4 - 1)(2^1)$, or simply $(3)(2)$, which is 6. And indeed all the factors of 6 which are less than 6 itself are 1, 2, 3. And these add up to 6.

2. Perfect numbers are relatively rare. The first 10 of them are:

N = 2	6
N = 3	28
N = 5	496
N = 7	8128
N = 13	33550336
N = 17	8589869056
N = 19	137438691328
N = 31	2305843008139952128
N = 61	2658455991569831744654692615953842176
N = 89	191561942608236107294793378084303638130997321548169216

Note that all the values for N are prime. The next several perfect numbers are generated by plugging in the following values for N in the perfect number formula: 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701. The last few of these, when spelled out in all their digits, take up several sheets of paper. In 1981, the largest known prime was of the form $2^N - 1$, namely $2^{44497} - 1$, and so 44497 is also a value for N generating a perfect number.

3. There are some interesting facts about perfect numbers. First, it is unknown to this day whether or not any *odd* perfect numbers exist. All the perfect numbers of the type described in this Theorem are obviously even. It is known, however, that all perfect numbers that are even are of the form described in this Theorem.

4. If you add the digits in a perfect number, and do the same with the resulting number, and so on until you can go no further, the result is always 1. The only exception is with the first perfect number, 6. But the next is 28, and its digits add up to 10, whose digits add up to 1. The next perfect number is 496, whose digits add to 19, whose digits add to 10, whose digits add to 1. The next perfect number is 8128, whose digits add to 19, whose digits add to 10, whose digits add to 1, etc.

5. It has been proved that there is an infinity of perfect numbers, but fewer than 50 are known today.

6. Every number of the form $(2^N - 1)(2^{N-1})$, including those that are perfect (namely when $2^N - 1$ is prime), is a triangular number whose base is $(2^N - 1)$. So the prime number which is distinctive of each even perfect number is also its triangular base.

7. No even perfect number is square. That is obvious, since its prime factorization is a bunch of twos and one odd prime. There is no way to divide up those factors into two equal factors.

8. No even perfect number measures any other. That is clear because if they are different, their unique primes are different. But then if one measures the other, all its factors will have to measure the other, also. And so its unique prime will measure the other, and so the other will have that prime in its prime factorization, too, which is impossible. For each even perfect number has only one odd prime in its factorization, namely $2^N - 1$.

9. All even perfect numbers end their digits with either 6 or 8.

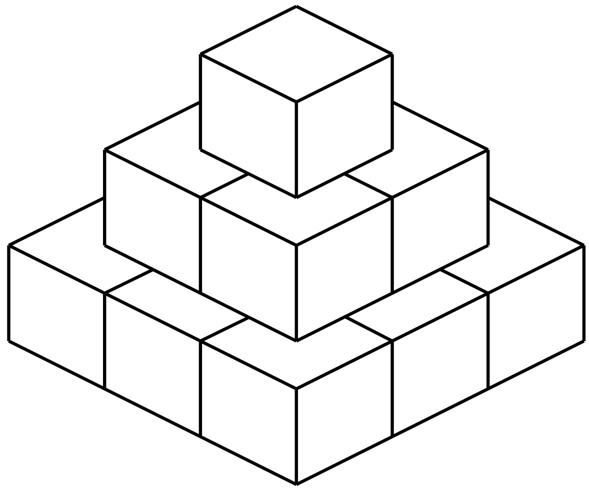
10. All even perfect numbers (except 6) have the same remainder when you divide them by 6, namely a remainder of 4.

"HOOK": THE ONLY SQUARE SQUARE-PYRAMIDAL NUMBER.

If you add consecutive numbers starting from 1, the sum is called a "triangular number." But if you add consecutive *square* numbers starting from 1, the sum is called a "square pyramidal number," because the units can be arranged in the form of a "square pyramid".

Question: Is any square pyramidal number also a square number? Yes! The number $4900 = 70^2 = 1^2 + 2^2 + 3^2 + ... + 24^2$.

What is more amazing, 4900 is the only square pyramidal number which is also a square number (other than the trivial example of 1).



"HOOK": THE ONLY TRIANGULAR SQUARE-PYRAMIDAL NUMBERS.

Are any square pyramidal numbers triangular? Yes, namely these:

1
55
91
208335

And those are the only ones!

"HOOK": NUMBERS EXPRESSIBLE AS SUMS OF CUBES IN TWO WAYS.

When G. H. Hardy pulled up in front of Ramanujan's flat for the first time, in a cab numbered 1729, Hardy remarked it was a singularly uninteresting number, to which Ramanujan famously replied "Not at all, it is a very interesting number, since it is the first to be expressible as the sum of two cubes in two ways," meaning

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

Apparently, this identity was found in Ramanujan's notebooks dated earlier than this meeting, so he did not simply come up with it on the spot.

It is perhaps unsatisfying that 1 is involved, since 1 is in a way a "cube number" only in a degraded sense. So here are a couple other examples which don't involve 1:

$$6507811154 = 31^3 + 1867^3 = 397^3 + 1861^3$$

$$6058655748 = 61^3 + 1823^3 = 1049^3 + 1699^3$$

"HOOK": FERMAT'S LAST THEOREM.

Is it ever the case that the sum of two square numbers is itself a square number? Yes. For example:

$$3^2 + 4^2 = 5^2$$

This is plainly connected to the Pythagorean Theorem.

But what about cube numbers? Is it ever the case that the sum of two cube numbers is itself a cube?

No! With the trivial exception of 0 (which, for the purposes of this book, is not considered a "number" anyway), it is never the case that

$$a^3 + b^3 = c^3$$

where a, b, c are all numbers, i.e. positive integers.

In fact, it is never the case that

$$a^n + b^n = c^n$$

where a, b, c, n are all integers, and $n > 2$.

This more general statement is Fermat's famous last theorem, proved only in the 20th century.

Chapter Eight

Irrational Magnitudes

DEFINITIONS

1. A MAGNITUDE is a quantity which is divisible forever.

For example, a line is a magnitude because it can be divided into parts, its parts can be divided into parts, and so on without ever coming to an end. But a number (as defined in Ch.7, definitions 2 & 3) is not a magnitude, because its divisibility ends with its units, which are indivisible.

2. Two magnitudes are COMMENSURABLE if they have a common measure, but INCOMMENSURABLE if they do not.

(The sense of "measure" in this definition is the one defined in Ch.5, Def.1.)

For magnitudes to be commensurable or incommensurable they must be comparable in terms of greater than, less than, or equal to – e.g. two lines, or two areas, not a line and an area.

3. Designating any straight line as our unit length, straight lines commensurable with it are called RATIONAL LINES in reference to it; those incommensurable with it are called IRRATIONAL LINES. And any areas commensurable with the square on the unit line (i.e. the "unit square") are called RATIONAL AREAS in reference to it; those incommensurable with the unit square are called IRRATIONAL AREAS.

4. A FRACTION of a magnitude is any measure of it taken any number of times.

For example, "five eighths" of a straight line is a fraction of it, since that means taking one eighth of it give times, and it is written

$$\frac{5}{8}$$

And "eight fifths" of a straight line is a fraction of it, since that means taking one fifth of it eight times, and it is written

$$\frac{8}{5}$$

Though eight fifths is larger than the original line, it is still called a fraction (although sometimes it is called an "improper" fraction).

PRINCIPLES

1. Any two commensurable magnitudes have to each other the same ratio as some pair of numbers, and any two magnitudes having the same ratio as some pair of numbers are commensurable with each other.

For example, if a straight line measures another straight line 5 times, and yet another straight line 8 times, then the two measured lines have the ratio 5 : 8. And if two straight lines have the ratio 5 : 8, then they are commensurable.

2. No two incommensurable magnitudes have to each other the same ratio that any two numbers have, and magnitudes not having the same ratio that any two numbers have are incommensurable with each other.

3. Any two multiples of the same magnitude are commensurable with each other, and have the same ratio as the multiplying numbers.

For example, 5A *and* 3A *are commensurable, having* A *as a common measure, and they have the same ratio as* 5 *and* 3.

THEOREMS

THEOREM 1: *Any fraction of a magnitude is commensurable with it.*

Given: Any magnitude M
Any fraction of it, F

Prove: F is commensurable with M

Since F is a fraction of M, it is equal to some measure of M taken some number of times. Say it is equal to 5 times the seventh part of M.

Then	a seventh of M measures F	(5 times)
and	a seventh of M measures M	(7 times)
so	F and M have a common measure	(namely a seventh of M)
thus	F is commensurable with M	

Q.E.D.

THEOREM 1 Remarks:

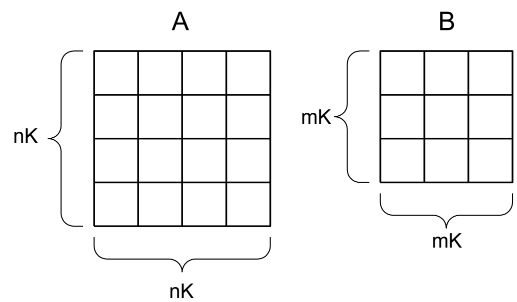
1. It does not matter whether F is greater or smaller than M. The proof still works in exactly the same way.

2. Any fraction of the unit length is rational. For a fraction of the unit length will have to be commensurable with it (by this Theorem), and any length commensurable with the unit length is rational (Def.3).

3. Any fraction of the unit square is rational. For a fraction of the unit square will have to be commensurable with it (by this Theorem), and any area commensurable with the unit square is rational (Def.3).

4. No irrational line length is expressible as a fraction of the unit line, for then it would be rational (Remark 2 above). So if we call a certain line length "1," then it is impossible to designate an irrational line (i.e. one that is incommensurable with the line we have chosen to call 1) as a fraction, e.g. as "two thirds" or as "one-hundred-and-twenty-three thousandths."

THEOREM 2: *Squares with commensurable sides have the same ratio as two square numbers.*



Given: Square A and square B, which have commensurable sides.

Prove: Square A has to square B the same ratio as a square number to a square number.

Since the side of A is commensurable with the side of B, let K, their common measure, go n times into the side of A, and m times into the side of B.

Now divide the sides of A into n equal parts, and divide the sides of B into m equal parts, each part therefore being equal to K, and complete the "grid" in each square.

The area of square A is now divided into n rows each containing n of the equal squares, each with a side of K. So the area of A equals $n \times n$ such squares. Likewise, the area of B equals $m \times m$ of those same squares, each with a side of K.

Thus A : B = $n \times n$ squares : $m \times m$ squares

that is A : B = $n \times n$: $m \times m$ (Principle 3)

And so squares A and B have the same ratio as a pair of square numbers.

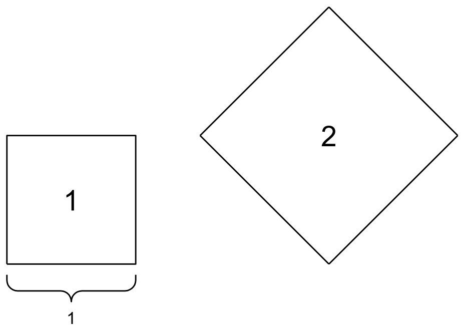
Q.E.D.

THEOREM 2 Remarks:

1. The proof reveals that if the sides of two squares are as the numbers n and m, then the squares themselves are as the numbers n^2 and m^2.

2. It follows that squares with areas that do *not* have the ratio of any two square numbers have incommensurable sides. For if their sides were commensurable, then their areas would have the same ratio as a pair of square numbers.

THEOREM 3: *Any square whose area is a non-square number of times the unit square has an irrational side.*



Let your unit line length be set out, and make a square on it, which is thus the unit square (with 1 square unit of area). Now take any square whose area is some number of times this unit square, but choose a *non-square* number, say 2.

Since 2 is a non-square number, I say that the side of the new square (having 2 times the area of the unit square) is irrational.

For these two squares have the ratio of 1 : 2 (given).
But no two square numbers have the ratio of 1 : 2.

For, if possible, say $1 : 2 = N^2 : M^2$

Then, multiplying the means and extremes in this numerical proportion,

we get $M^2 = 2N^2$ (Ch.7, Thm.4)

And so it seems that 2 times a square number (namely N^2) makes a square number (namely M^2). From this fact, it follows that 2 is itself a square number, since only a square number times a square number produces a square number (Ch.7, Thm.23). But 2 is *not* a square number, so our initial assumption that two square numbers can have the ratio of 1 : 2 must have been false.

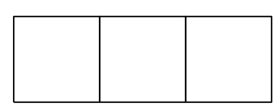
So no two square numbers have the ratio of 1 : 2.

Therefore our two squares, which *do* have the ratio of 1 : 2, do not have the ratio of any two square numbers, either. Therefore our two squares have incommensurable sides (Thm.2, Remarks). Thus the side of the square with 2 times the area of the unit square, being incommensurable with the unit length, is irrational.

And the same is true for the side of a square with 3 square units of area, or 5, or 6, or 7, or *any* non-square number of square units of area. They all have irrational sides.

Q.E.D.

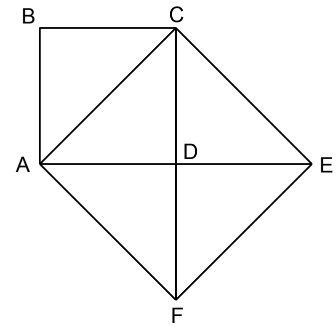
THEOREM 3 Remarks:



1. Incidentally, how would you make a square with 2 times the area of the unit square? How do you make a square that is *double* the area of a given square? How about *triple*? How about 4 times? Can you see how to make a square that has *any* number of times the area of a given square? Start by taking the given square the required number of times, and putting them together into one rectangle. Now make a square equal to that rectangle.

2. Even if we do not use the unit square, but just any old square S, the proof works the same way to show that the side of another square that is a non-square number of times the area of S is incommensurable with the side of the first square.

THEOREM 4: *The diagonal of a square is incommensurable with its side.*



Take any square ABCD. I say that side AB is incommensurable with diagonal AC.

Draw the square on AC, namely ACEF. If you extend CD and AD, they pass through F and E, dividing square ACEF into 4 triangles identical to triangle ACD. But square ABCD is divided into 2 triangles identical to triangle ACD.

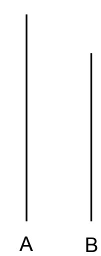
Therefore ☐ABCD : ☐ACEF = 2 : 4

or ☐ABCD : ☐ACEF = 1 : 2

So these two squares do not have the ratio of a square number to a square number, since *no* square numbers have the ratio of 1 : 2 (as we saw in Thm.3). And therefore the sides of these two squares are incommensurable with each other (by Thm.2, Remarks). Thus AB is incommensurable with AC.

Q.E.D.

THEOREM 4 Remarks:

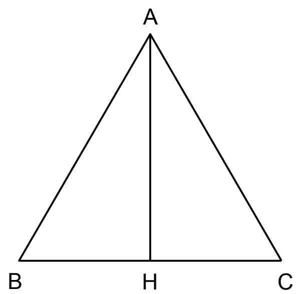


This Theorem, even more clearly than the last one, shows that some magnitudes are incommensurable. We can easily make a square – now just draw the diagonal, and it will be incommensurable with the side of the square. In the absence of proof, it might seem impossible for two comparable magnitudes to be incommensurable. Given any pair of straight lines, A and B, can't we find a tiny straight line that fits exactly into each of them some number of times? After all, there is no limit to how small straight lines can get – there *must* be one small enough to measure both A and B exactly ... right?

Wrong! Some magnitudes are incommensurable. We have just shown that *nothing* measures both the side of a square and its diagonal.

Before reading this Chapter, you might have thought that all ratios can be expressed numerically, as a ratio between two numbers. After all, there is an infinity of numbers, and an infinity of numerical ratios. So given any two straight lines, such as A and B, musn't their ratio be expressible as a ratio between two numbers? Not if A and B are incommensurable, as the side of a square and its diagonal are. Not all ratios are numerical ratios, and all the numerical ratios are not all the ratios.

THEOREM 5: *The height of an equilateral triangle is incommensurable with its side*.



Take any equilateral triangle ABC, drop AH perpendicular to BC. Thus AH is the height.

I say side AB is incommensurable with height AH.

Proof: Since BH is half the length of the side of the triangle, thus AB = 2BH. Hence the square on AB is *four* times the square on BH, i.e.

	☐AB : ☐BH = 4 : 1	
so	☐AB : ☐AB – ☐BH = 4 : 4 – 1	(Ch.5, Thm.16, Remark 3)
i.e.	☐AB : ☐AH = 4 : 3	(☐AB - ☐BH = ☐AH; Pythagorean)

Now 4 and 3 don't have the same ratio as any two square numbers. For if possible, suppose that

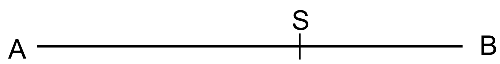
	$4 : 3 = M^2 : N^2$	
thus	$4N^2 = 3M^2$	(Ch.7, Thm.4)

And since $4N^2$ is a square number (since a square number times square number yields a square number), thus $3M^2$, its equal, is also a square number. But since only a square number times a square number yields a square number (Ch.7, Thm.23), thus 3 must be a square number. But that's impossible. Therefore, too, it is impossible that 4 and 3 should have the same ratio as any two square numbers.

But since the ratio of 4 : 3 is not the ratio of any two square numbers, therefore the ratio of ☐AB : ☐AH is likewise not the ratio of any two square numbers. Therefore (by Thm.2 Remarks) it follows that their sides are incommensurable, i.e. AB is incommensurable with AH.

Q.E.D.

THEOREM 6: *The parts of a magnitude cut into mean and extreme ratio are incommensurable.*



Take any straight line AB and cut it in mean and extreme ratio at S.
Hence BA : AS = AS : SB.

I say that AS and SB are incommensurable with each other.

If possible, suppose AS and SB are commensurable, and therefore have a numerical ratio. Say AS : SB = n : m, where n and m are the least numbers in that ratio, and so are prime to each other (Ch.7, Thm.7, Remark 2).

[1]	Since	AS : SB = n : m	(assumed)
	thus	AS : AS + SB = n : n + m	(Ch.5, Thm.15, Remarks)
	or	AS : AB = n : n + m	(AS + SB is AB)
	but	$\underline{\text{AS : AB = SB : AS}}$	(inverse of given)
	so	SB : AS = n : n + m	(each ratio is the same as AS : AB)
	but	$\underline{\text{SB : AS = m : n}}$	(inverse of the assumed proportion)
	so	m : n = n : n + m	(each ratio is the same as SB : AS)

[2] Since m and n are the least numbers in their ratio, hence they measure the numbers in the same ratio with them (Ch.7, Thm.7). So the antecedent measures the antecedent in the proportion above, i.e. m measures n. But since m and n are prime to each other, they have no common measure but 1. So m = 1.

[3]	So	1 : n = n : n + 1	(replacing m with 1 in Step 1 proportion)
	hence	$n \times n = n + 1$	(product of means = product of extremes)

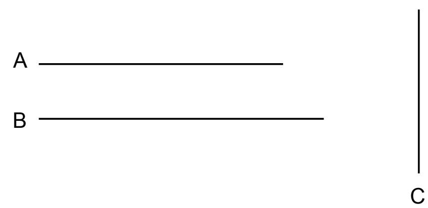
But this is impossible for any number n.

Consider: $1 \times 1 < 1 + 1$
but $2 \times 2 > 2 + 1$
and $3 \times 3 > 3 + 1$
and $4 \times 4 > 4 + 1$
and $5 \times 5 > 5 + 1$
etc.

[4] Since it is impossible for n to be a number, it is impossible to express mean and extreme ratio as the ratio between two numbers, n and m. Hence AS is incommensurable with SB.

Q.E.D.

THEOREM 7: *Two magnitudes each commensurable with the same magnitude are also commensurable with each other.*



Given: A and C are commensurable
B and C are commensurable

Prove: A and B are commensurable

Since A and C are commensurable, they have a common measure, say W, that goes into each of them some number of times. Say

$A = 5W$
$C = 7W$

Since B and C are commensurable, they have a common measure, say Z, that goes into each of them some number of times. Say

$B = 3Z$
$C = 4Z$

[1] Now $A : 5W = B : 3Z$ ($A = 5W$, and $B = 3Z$)

[2] thus $7 \cdot 4 \cdot A : 7 \cdot 4 \cdot 5W = 7 \cdot 4 \cdot B : 7 \cdot 4 \cdot 3Z$

All we have done in this step is multiply every term in the proportion of Step 1 by 7×4, i.e. by 28. Doing this does not destroy the proportion. A few inconsequential rearrangements of the multipliers gives us

[3] $28A : 4 \cdot 5 \cdot 7W = 28B : 7 \cdot 3 \cdot 4Z$

And since $7W = C$ (as we set out in the beginning), and $4Z = C$, we can replace those two terms in the proportion with C:

so $28A : 4 \cdot 5 \cdot C = 28B : 7 \cdot 3 \cdot C$
or $28A : 20C = 28B : 21C$
thus $28A : 28B = 20C : 21C$ (alternating the proportion)
i.e. $A : B = 20 : 21$

So A and B have to each other the ratio that a number has to a number, and so they are commensurable.

Q.E.D.

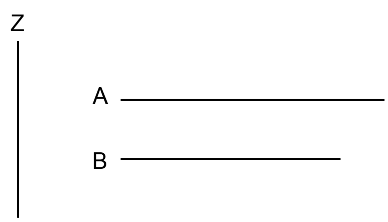
THEOREM 7 Remarks:

1. In particular, *if one magnitude is commensurable with another, then it is also commensurable with any measure of it*. That is, if M is commensurable with 35Q, then M is also commensurable with Q. For

	M is commensurable with 35Q	(given)
	Q is commensurable with 35 Q	(since Q measures both)
thus	M and Q must be commensurable	(both being commensurable with 35Q)

2. We have seen many relationships which are *transitive*, i.e. which are such that if two things have that relationship to a third thing, they also have it to each other. This is true about the relationships of (a) equality, (b) congruence, (c) parallelism, (d) sameness of ratio, (e) similarity, and now (f) commensurability.

THEOREM 8: *Any magnitude commensurable with one of two incommensurable magnitudes must be incommensurable with the other one*.



Given: A and B are incommensurable with each other
Z is commensurable with A

Prove: Z is incommensurable with B

If Z *were* commensurable with B,

then	A and B would both be commensurable with Z	(it is given that A is),
and so	A and B would be commensurable with each other	(by Thm.7),
but	A and B are *not* commensurable with each other	(given),

and therefore neither can Z be commensurable with B.

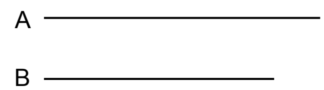
Q.E.D.

THEOREM 8 Remarks:

Now it is also clear that if a straight line is commensurable with an irrational line, it is irrational too.

THEOREM 9: *The sum of two incommensurable magnitudes is incommensurable with each of them – and the difference of two incommensurable magnitudes is incommensurable with each of them.*

Given: A is incommensurable with B



Prove: A + B is incommensurable with A
A + B is incommensurable with B

If A were commensurable with A + B, then they have some common measure M going into each some number of times, say

$$\begin{array}{lr}
 & A + B = 5M \\
\text{and} & \underline{A = 3M} \\
\text{thus} & B = 2M \qquad \text{(subtracting equals from equals)}
\end{array}$$

And so M also measures B, not just A. Thus A will be commensurable with B. But it is given that it is not. Therefore neither can A be commensurable with A + B.
Likewise B cannot be commensurable with A + B.

Q.E.D.

Similarly with the differences, if A is incommensurable with B, then
A – B is incommensurable with A
A – B is incommensurable with B

For suppose, if possible, that A and A – B were commensurable, and so each is measured by a common measure, M, some definite number of times, say

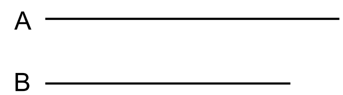
$$\begin{array}{lr}
 & A = 12M \\
 & \underline{A - B = 8M} \\
\text{thus} & B = 4M \qquad \text{(subtracting equals from equals)}
\end{array}$$

And so again A and B would have to have M as a common measure, which is contrary to what is given about them. Therefore A must be incommensurable with A – B.
Likewise B is incommensurable with A – B.

Q.E.D.

THEOREM 10: *Either of two commensurable magnitudes is commensurable with their sum, and also with their difference.*

Given: A and B are commensurable.



Prove: A is commensurable with A + B
B is commensurable with A + B
A is commensurable with A – B
B is commensurable with A – B

Since A and B are commensurable, they have a common measure, M,which goes into each some number of times, say

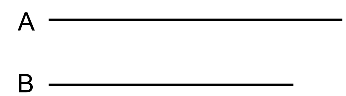
A = 7M
B = 5M

Thus A + B = 12M, and so their sum is also measured by M, the common measure of A and B, and so A + B is commensurable with A and with B.

And A – B = 2M, and so their difference is also measured by M, and thus is commensurable with A and also with B.

Q.E.D.

THEOREM 11: *How to make an infinity of magnitudes, every one of which is incommensurable with all the rest.*



Given: Two incommensurable magnitudes to start with, A and B, such as the side and diagonal of a square.

Find: A way to generate from them an infinity of magnitudes every one of which is incommensurable with every other.

If we make the following magnitudes:

A + 1B
A + 2B
A + 3B … etc.

I say that every one of these is incommensurable with all the rest. Consider, for example, A + 5B and A + 12B. I say they are incommensurable.

For, if possible, suppose that

	A + 5B is commensurable with A + 12B	
then	A + 5B is commensurable with 7B	(i.e. with their difference; Thm.10)
then	A + 5B is commensurable with B	(i.e. with a measure of 7B; Thm.7)
then	A + 5B is commensurable with 5B	(i.e. with a multiple of B)

Now, since 5B is commensurable with A + 5B, it follows that 5B is also commensurable with their *difference*, namely with A (by Thm.10).

thus A is commensurable with 5B
so A is commensurable with B

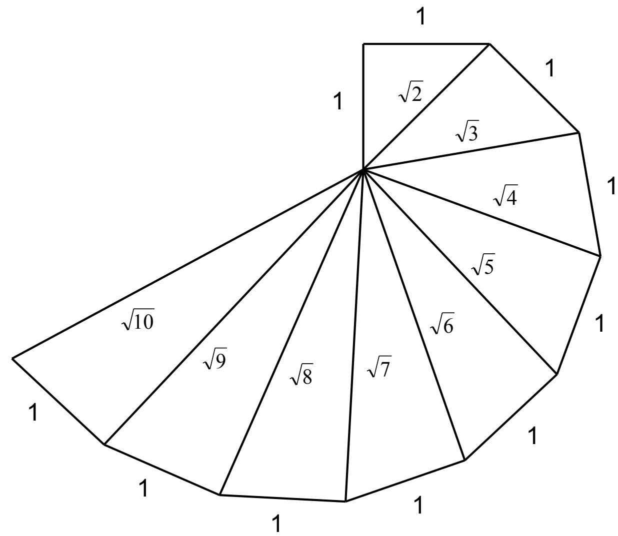
since B is a measure of 5B, and anything commensurable with 5B must also be commensurable with anything that measures it (see Thm. 7, Remark).

But it is impossible for A to be commensurable with B, since it is given that they are not commensurable. Therefore our initial supposition, namely that A + 5B and A + 12B are commensurable, is impossible. Therefore A + 5B and A + 12B are incommensurable.

Likewise any two magnitudes in our infinite list are incommensurable with each other.

Q.E.D.

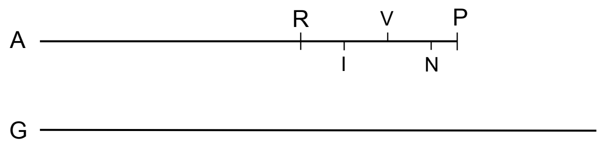
THEOREM 11 Remarks:



Here is another way to set out an infinity of lines all incommensurable with each other. Set out the unit line 1, make another line equal to it at right angles and join the endpoints. Since this hypotenuse is in fact the diagonal of the square on the unit line, the square on that hypotenuse equals two times the unit square. Call this line *side of square two*. Now draw another unit line at right angles to the side of square two, and join its endpoint to the other end of the side of square two. By the Pythagorean Theorem, the square on this new hypotenuse equals the square on 1 (i.e. 1 square unit) plus the square on the side of square two (i.e. 2 square units), and so it is equal to 3 square units of area. Call this line *side of square three*. Repeat the process and you will get the hypotenuses which are the sides of squares 4, 5, 6 etc., having 4 square units of area, 5 square units of area, and so on.

Looking at the hypotenuses, the squares on any two of them are as the numbers under the radical signs, which are simply all the numbers (if the process is carried on indefinitely). Now attend only to the hypotenuses with prime numbers. The squares on any two of these will be as those prime numbers – but no two prime numbers have the same ratio as two square numbers (Ch.7, Thm.24). So the squares on any two prime-numbered hypotenuses will *not* be as a square number to a square number, and so those hypotenuses themselves will be incommensurable (Thm.2, Remark 2). But we can make as many such hypotenuses as there are prime numbers, and there is an infinity of those (Ch.7, Thm.15). Therefore we can make an infinity of straight lines, each one of which is incommensurable with all the others.

THEOREM 12: *How to make an infinity of straight lines commensurable with a given line, but falling between two given lengths.*



Suppose I give you a straight line G, and also two other lengths AR and AP differing by however much or little you please. Can you find a way to make an infinite series of different lengths falling *between* AR and AP, which are all commensurable with G? Absolutely.

[1] First notice that however small RP may be, some multiple of it will exceed G.
Suppose $\quad 57\ RP > G$
Now divide G into more than 57 equal parts, say 5700 equal parts (Ch.6, Thm.8), and call each one F. So $G = 5700\ F$.
So now $\quad 57\ RP > 5700\ F$
hence $\quad RP > 100\ F$
So if we now multiply F until it first exceeds AR, we will have to add almost another 100 F's before we reach P. In other words, there are plenty of multiples of F between AR and AP.

[2] So let AI and AN both be multiples of F falling between AR and AP in length. Bisect IN at V.

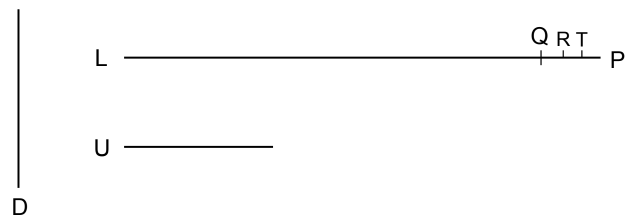
[3] Since AI and AN are both commensurable with G (all three are multiples of F),

hence	AI is sommensurable with AN	(Thm.7)
so	AI is commensurable with IN	(their difference; Thm.10)
but	IV is commensurable with IN	(IV is half IN)
so	AI is commensurable with IV	(Thm.7)
thus	AI is commensurable with AV	(their sum; Thm.10)
but	AI is commensurable with G	(both are multiples of F)
so	AV is commensurable with G	

[4] So AV lies between AR and AP in length, and it is commensurable with G, the given line. If we now bisect IV at Q, AQ will likewise lie between AR and AP and be commensurable with G by similar reasoning, and their is no limit to how many lines we can make like this.

Q.E.F.

THEOREM 13: *Between any two irrational lengths there is an infinity of rational lengths; and between any two rational lengths there is an infinity of irrational lengths, each incommensurable with all the others.*



[1] Set out the unit length U, and any two irrational lengths LQ and LP, with as small a difference between them as you like. By Thm.12, we can construct an infinity of straight lines falling between LQ and LP in length, but all commensurable with U, and hence rational. So the first part of the Theorem is proved.

[2] Take any two rational lengths LR and LT (falling between LQ and LP if you wish), with as small a difference between them as you like. Set out any irrational line D (such as the diagonal of the square on U).

Consider the length D + U. Since D is incommensurable with U, hence their sum is also incommensurable with U and hence irrational (Thm.9). But by Thm.12, we can make an infinity of lines between LR and LT in length, all commensurable with D + U, and thus all irrational. Choose any one of these and call it X_1.

[3] Now consider the length D + 2U. It must likewise be irrational, since D is incommensurable with U and hence with all its multiples such as 2U, and so 2U is incommensurable with their sum D + 2U (Thm.9). Therefore U, being commensurable with 2U, is also incommensurable with D + 2U (Thm.8). Therefore D + 2U is irrational.

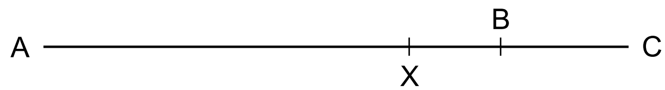
Also, D + 2U is incommensurable with D + U (Thm.11). But again, using Thm.12, we can make an infinity of lines between LR and LT in length, all commensurable with D + 2U, all of them therefore irrational and incommensurable with D + U. Choose any of these, and call it X_2.

Because X_1 is commensurable with D + U, but X_2 is commensurable with D + 2U, it follows that X_1 and X_2 are incommensurable with each other.

[4] We can continue with D + 3U, and make X_3 between LR and LT, another irrational length, again incommensurable with both X_1 and X_2. And so on forever.

Q.E.D.

THEOREM 14: *If a straight line is divided into two parts incommensurable with each other, it is impossible to divide it into two other parts commensurable with the original two parts*.



Given: AC is divided at B, and AB is incommensurable with BC

Prove: It is impossible to divide AC at another point X so that AX is commensurable with AB *and* XC is commensurable with BC.

For let it be given that AX is commensurable with AB.
Then I say that XC must be *incommensurable* with BC.

For   AX is commensurable with AB   (given)

and so the difference between them, XB, must be commensurable with each of them,

so   XB is commensurable with AB   (Thm.10)

Now since AB and BC are incommensurable (given), and XB is commensurable with one of them, namely AB, it follows that

XB is *incommensurable* with BC   (Thm.8)

and so the sum of these two must be incommensurable with each of them (Thm.9),

i.e.   XC is *incommensurable* with BC.

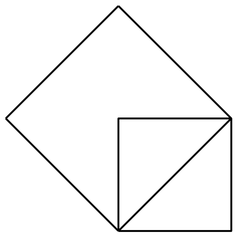
And so it is impossible to divide the line AC at any point other than B so as to get two parts that are each commensurable with AB and BC respectively.

Q.E.D.

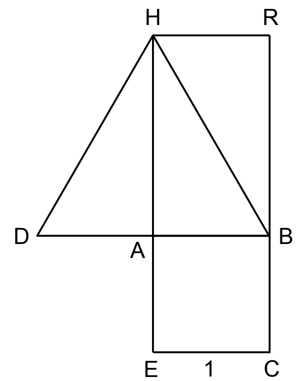
THEOREM 14 Remarks:

Obviously we could make X the same distance from A that point B is from C, and thus AX would be commensurable with BC (being equal to it), and XC with AB (being equal to it). But then we would not have really divided the line into two new *kinds* of parts.

THEOREM 15: *It is possible to contain a rational area with irrational lines, and also to contain an irrational area with rational lines*.



Begin with the square on the unit line, the unit square. Form the square on its diagonal. Since this diagonal is incommensurable with the unit side (Thm.4), it is irrational. So the square on the diagonal of the unit square is contained by four irrational lines. And yet its area is rational, being double (and so commensurable with) the area of the unit square. So it is possible to contain a rational area with irrational lines.



Begin again with the square on the unit line, and extend one of its sides, AB, to D, so that DA = AB (and so DB is called 2 in reference to the unit length). Now make equilateral triangle DHB on DB. Thus AH is both the height of △DHB, and the extension of one side of the unit square (namely EA). Complete the rectangle AHRB.

Now, ☐EABC : rect. AHRB = EA : AH
(Ch.6, Thm.1)

But AH is the height of the equilateral triangle, and so it is incommensurable with its side, DB (Thm.5). Thus height AH is also incommensurable with AB, half the side. Thus AH is also incommensurable with EA, which is equal to AB. So EA is incommensurable with AH, and thus EA : AH *is not a numerical ratio*.

Thus ☐EABC : rect AHRB *is not a numerical ratio*.

But rectangle AHRB is equal to the equilateral triangle DHB. Therefore

☐EABC : △DHB *is not a numerical ratio*,

which is to say that these two areas are incommensurable with each other. But since square EABC is the unit square, hence △DHB is an irrational area. And yet it is contained by rational sides, each of its sides being double (and thus commensurable with) the unit length. So it is possible to contain an irrational area with rational lines.

Q.E.D.

Chapter Nine

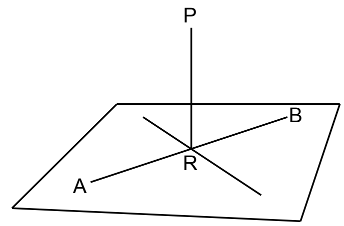
Basic Solid Geometry

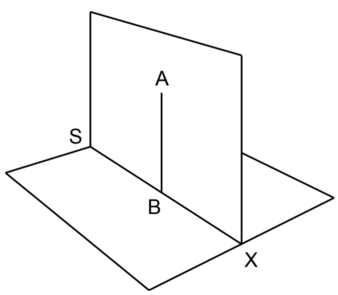
DEFINITIONS

1. A SOLID is whatever has length, width, and depth.

2. You have a STRAIGHT LINE PERPENDICULAR TO A PLANE if it is perpendicular to all the straight lines it stands on in that plane.

For example, PR *stands on* AB, *and is perpendicular to it. And if* PR *is perpendicular to all such lines in the plane passing through* R, *then* PR *is perpendicular to the plane*.

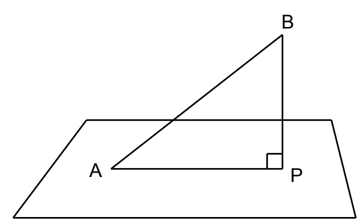


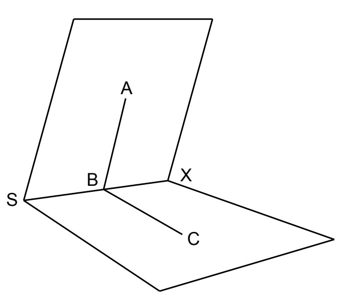


3. You have PERPENDICULAR PLANES if every straight line in one of them that is perpendicular to their intersection is also perpendicular to the other plane.

For example, let two planes intersect along SX. *If every straight line* AB *drawn perpendicular to* SX *in one of the planes is also perpendicular to the other plane, then the two planes are perpendicular to each other.*

4. Consider a straight line AB that passes through some plane at point A, and is not perpendicular to the plane, but leans over somewhat. How much does it lean? If we choose any point B along it, and BP falls perpendicular to the plane, then the INCLINATION OF THE STRAIGHT LINE TO THE PLANE is angle BAP.

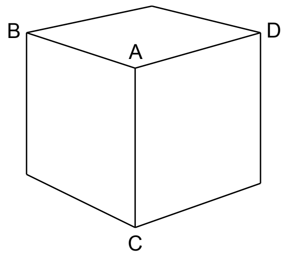




5. The INCLINATION OF A PLANE TO A PLANE is the angle between two straight lines, one drawn in each plane, and both drawn perpendicular to the line of intersection and from the same point on it.

For example, angle ABC *is formed by two perpendiculars to* SX, *each of them drawn in one of the two planes. So angle* ABC *is the inclination of the planes*.

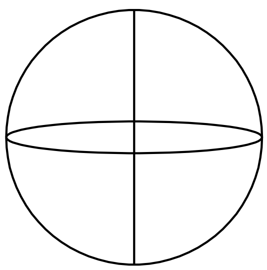
6. PARALLEL PLANES are those which never meet, no matter how far they are extended.



7. A PLANAR SOLID ANGLE is formed by three or more planes meeting at a point.

For example, in a cube, one angle of it is formed by three right-angled faces, namely BAC, BAD, CAD.

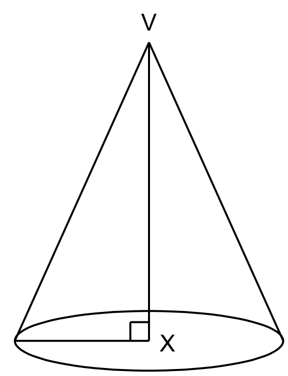
8. A SOLID FIGURE is a figure contained by one or more surfaces.

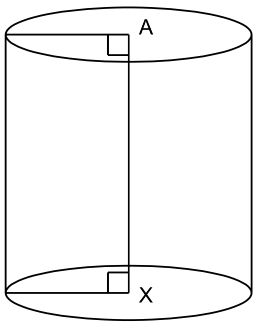


9. A SPHERE is a solid figure contained by one surface which is at all points equidistant from one point within called the CENTER. If a semicircle is rotated all the way around on its diameter once, the solid figure it describes is a sphere. The center and diameter of the semicircle are also the center and diameter of the sphere. Any straight line drawn through the center of a sphere and stopping at the surface of the sphere in each direction is a DIAMETER of the sphere.

10. A RIGHT CONE is a solid figure described by rotating a right triangle all the way around once about one of the sides forming the right angle. The side about which the triangle was rotated is called the AXIS of the cone, and the circle described by the other side of the right angle is the BASE of the cone. The point at which the axis meets the hypotenuse of the original triangle is the VERTEX of the cone.

Note: this kind of cone is called a right *cone because its axis is at right angles to its base. There are other kinds of cones called* oblique *cones, but since these will not come up in this book, the simple term* cone *will always refer to a right cone*.





11. A RIGHT CYLINDER is a solid figure described by rotating a rectangle all the way around once about one of its sides. The side about which the rectangle was rotated is called the AXIS of the cylinder, and the two circles described by the two sides of the rectangle adjacent to the axis are the BASES of the cylinder.

Note: this kind of cylinder is called a right *cylinder because its axis is at right angles to its bases. There are other kinds of cylinders called* oblique *cylinders, but since these will not come up in this book, the simple term* cylinder *will always refer to a right cylinder*.

12. SIMILAR CONES or SIMILAR CYLINDERS are those in which the axes and the diameters of the bases are proportional.

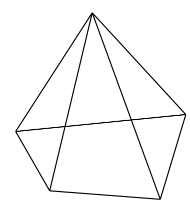
13. A POLYHEDRON is a solid figure contained by four or more rectilineal plane figures. *Note: the plural for* polyhedron *is often written* polyhedra. *I prefer to say* polyhedrons.

14. SIMILAR POLYHEDRONS are those whose faces are similar, each to each, and similarly arranged.

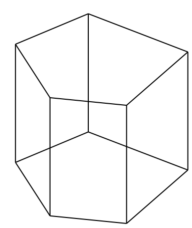
By "similarly arranged" I mean that if any two faces in one solid meet each other, then the two correspondingly similar faces in the other solid also meet each other, forming an edge; also, if a solid angle in one solid is convex, then the corresponding solid angle in the other solid is also convex, but if concave, then concave.

SIMILAR AND EQUAL POLYHEDRONS are similar polyhedrons whose corresponding faces are equal in size. These can also be called *congruent* polyhedrons.

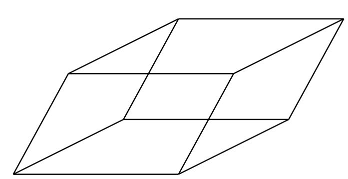
15. A PYRAMID is a polyhedron contained by a plane and three or more triangles drawn down to it from one point. The portion of the plane bounding the pyramid is called its BASE, whereas the point is called its VERTEX.



16. A PRISM is a polyhedron contained by two congruent and parallel polygons similarly oriented, and all the parallelograms joining their corresponding sides. The two identical and parallel polygons are the BASES of the prism.



17. A PARALLELEPIPED is a prism whose bases are parallelograms.



BASIC PRINCIPLES OF SOLID GEOMETRY

1. If any two points of a straight line lie in a plane, the whole straight line lies in that plane.

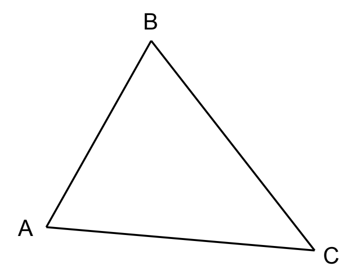
2. If two planes intersect, their intersection is a straight line, and they have no other points in common.

3. Any plane can be extended as far as we please in any of its directions.

4. Any plane can be rotated about any straight line that lies within it.

THEOREMS

THEOREM 1: *Any three points not lying in a straight line lie only in one plane, and every triangle lies only in one plane.*



Consider any three points A, B, C which do not lie in one straight line. It is impossible for all three of these points to lie in more than one plane.

If possible, suppose A, B, C all lie in two distinct planes: plane Q and also plane Z.

Now, since A and B both lie in plane Q, therefore straight line AB lies in plane Q (Princ. 1). And since A and B both lie in plane Z, therefore straight line AB lies in plane Z (Princ. 1). Therefore plane Q and plane Z have line AB in common, i.e. they intersect along that straight line. But then they have no other points in common, beyond those lying in a straight line with AB (Princ. 2). Therefore point C, not lying in line with AB (given), is *not* common to planes Q and Z. And thus it is not possible for A, B, C to lie all in plane Q, and also all in plane Z.

Again, the whole triangle ABC lies only in one plane. For any plane containing all of triangle ABC must also contain its three vertices A, B, and C. But we have just showed that there is only one such plane. Therefore the whole of any triangle lies only in one plane.

Q.E.D.

THEOREM 1 Remarks:

1. One point can have many straight lines passing through it, but any two points lie only in one straight line. Similarly, two points can have many planes passing through them, but any three points (if they are not in a straight line) lie only in one plane.

It is obvious that any two points can in fact have a straight line passing through them. Is it also obvious that any three points can have a plane passing through them? Yes. Say the points are A, B, C. Join AB, and pass any plane through AB. Now rotate the plane around AB like a hinge until it hits C, and the result is a plane containing points A, B, and C.

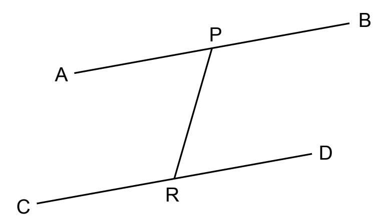
So any three points do lie in one plane. But a fourth point might not lie in the same plane.

2. Obviously, if you have 3 points in a straight line, there is an infinity of planes that contain those 3 points. Pass any plane through the straight line containing the 3 points, and this plane will contain all 3 points.

3. If it were not obvious enough by itself, it is now obvious that *One and only one plane can be drawn through a given straight line and a given point not on that straight line*. For example, only one plane goes through straight line AB and point C – otherwise, more than one plane would contain the three points A, B, C.

4. If it were not obvious enough by itself, it is now obvious that *One and only one plane can be drawn through a given pair of intersecting straight lines*. For example, only one plane goes through the straight lines AB and BC – otherwise, more than one plane would contain the three points A, B, C.

THEOREM 2: *One and only one plane passes through any pair of parallel straight lines, and any straight line joining any two points on the parallels also lies in that plane.*



Given: AB and CD, a pair of parallel straight lines, with P and R being random points on each of them.

Prove: One and only one plane passes through both AB and CD, and PR lies in that plane.

Put a pencil down on the table, and imagine it indicating a straight line going on forever in both directions, say North and South. Now hold a pen over the pencil, but pointing East and West. These two straight lines will never intersect each other, and yet we do not call them "parallel." Why? Because they are not in the same plane. It is especially interesting that even in the same plane two straight lines can be so oriented that they will never meet – there is in fact only *one* orientation you can give a straight line to make it parallel to another. And thus "parallel" means not only "never meeting," but also "in one plane." Therefore the first part of the theorem, namely that any two straight lines that are parallel must lie in the same plane, is really self-evident. It is part of what "parallel" means.

It is also clear that the parallels AB and CD lie *only* in one plane – it is not possible for more than one plane to contain them both. For supposing it were so, then two distinct planes would contain points A, B, C, even though these do not lie in a straight line with each other, which is impossible (Thm.1). Thus it is impossible for more than one plane to contain a given pair of parallel straight lines.

And since points P and R both lie in the plane containing the parallels AB and CD, therefore the line PR lies in that plane, too (Princ. 1).

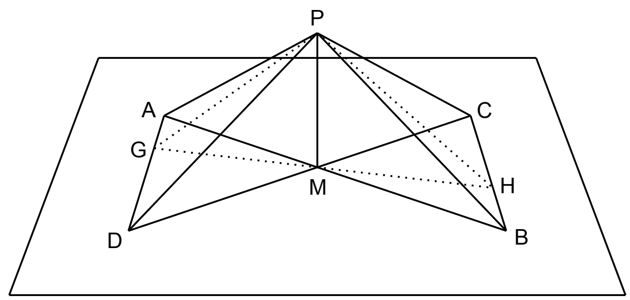
Q.E.D.

THEOREM 2 Remarks:

A pair of lines that never meet, but are *not* in the same plane as each other, are called *skew* lines.

THEOREM 3: *If a straight line is perpendicular to two intersecting straight lines at the point where they meet, then it is perpendicular to the plane in which they lie.*

Suppose AB and CD meet each other at M, and PM is perpendicular to both AB and CD. Then I say PM is perpendicular to the plane passing through AB and CD.



Cut off MA = MD = MB = MC.
Join AD, BC, AP, DP, BP, CP.
In the plane of AB and CD, draw GMH through M at random, cutting AD and BC.
Join GP, HP.

I say that PM is at right angles to the line GMH drawn through M randomly in the plane.

[1] Since ∠AMD and ∠BMC are vertical and are contained by equal lines, hence △MAD ≅ △MBC, so ∠MAD = ∠MBC.

[2]	Now	∠MAG = ∠MBH	(being the same as ∠MAD and ∠MBC)
	but	∠AMG = ∠BMH	(being vertical)
	and	<u>AM = MB</u>	(we made them so)
	so	△MAG ≅ △MBH	(Angle Side Angle)
[3]	Again	PA = PB	(△MAP ≅ △MBP by S-A-S)
	and	PD = PC	(△MDP ≅ △MCP by S-A-S)
	and	<u>AD = BC</u>	(△MAD ≅ △MBC; Step 1)
	so	△PAD ≅ △PBC	(Side Side Side)
	so	∠PAD = ∠PBC	
[4]	Again	∠PAG = ∠PBC	(being the same as ∠PAD and ∠PBC)
	and	PA = PB	(△MAP ≅ △MBP)
	and	<u>AG = BH</u>	(△MAG ≅ △MBH; Step 2)

	so	$\triangle PAG \cong \triangle PBH$	(Side Angle Side)
[5]	Now	$PG = PH$	($\triangle PAG \cong \triangle PBH$; Step 4)
	and	$MG = MH$	($\triangle MAG \cong \triangle MBH$; Step 2)
	and	<u>PM is common</u>	(to $\triangle PMG$ and $\triangle PMH$)
	so	$\triangle PMG \cong \triangle PMH$	(Side Side Side)
	so	$\angle PMG = \angle PMH$	

But these equal angles are adjacent. Hence PM is at right angles to GMH.

[6] Since PM is thus at right angles to any straight line drawn through M in the plane of AB and CD, therefore PM is perpendicular to that plane.

Q.E.D.

THEOREM 3 Remarks:

1. If GH is drawn through M so that it does not cut AD and BC, then it will cut AC and DB, and we use them for the proof instead.

2. A kind of converse to this Theorem is: *All perpendiculars to one point on a straight line lie in one plane*. All the perpendiculars to PM drawn from M lie in the plane of AB and CD.

3. Prove that the line drawn perpendicular to a plane from a point above it is the shortest straight line that can be drawn from that point to the plane.

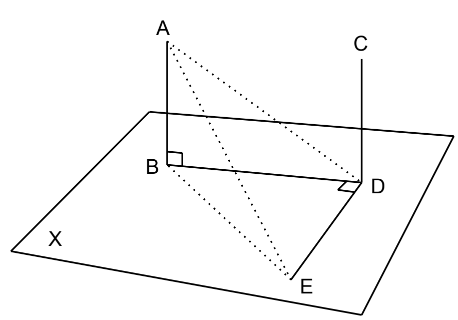
4. Prove that only one straight line can be drawn from a given point perpendicular to a given plane.

THEOREM 4: *If one of two parallels is perpendicular to a plane, so is the other.*

Given: AB is parallel to CD, and AB is perpendicular to plane X.

Prove: CD is also perpendicular to plane X.

Join BD.
Draw DE (in plane X) perpendicular to BD, and cut off DE equal to AB.



Join BE, AE, AD.

[1] Now, $\triangle ABD \cong \triangle EDB$ (SAS)

[2] Thus AD = BE (see Step 1)
but DE = AB (we made them equal)
and AE is common (to $\triangle ABE$ and $\triangle EDA$
so $\triangle ABE \cong \triangle EDA$ (SSS)

[3] Now $\angle ABE = \angle EDA$ (by Step 2)
but $\angle ABE$ is right (since AB is given perpendicular to plane X)
so $\angle EDA$ is right

[4] So ED is perpendicular to DA
but ED is perpendicular to DB (by construction)
so ED is perpendicular to the plane through DA and DB (Thm.3), i.e. the plane containing points A, B, D.

[5] Now, there is only one plane containing points A, B, D (Thm.1), but the plane containing parallels AB and CD (Thm.2) contains points A, B, D, and therefore the plane containing points A, B, D is the same as the plane containing parallels AB and CD. Thus ED is perpendicular to the plane of the parallels, i.e. to the plane containing triangle BDC. Therefore $\angle EDC$ is a right angle (see Def.2).

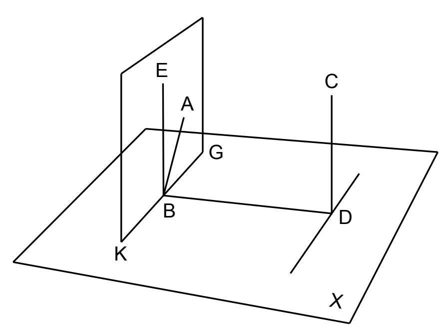
[6] So CD is perpendicular to DE (Step 5)
and CD is perpendicular to BD ($\angle ABD$ is right, and CD is parallel to AB)
so CD is perpendicular to two straight lines intersecting in plane X, and thus CD is perpendicular to plane X (Thm.3).

Q.E.D.

THEOREM 5: *If two straight lines are perpendicular to the same plane, they are parallel.*

Given: AB is perpendicular to plane X.
CD is perpendicular to plane X.

Prove: AB is parallel to CD.



Suppose, if possible, that AB is *not* parallel to CD. Then since B, D, C are all in one plane, draw BE in this plane parallel to CD. Therefore BE is perpendicular to plane X (Thm.4).

Now, A, B, E are all in one plane. Let the intersection of their plane with plane X be called GK.

Since BE is perpendicular to plane X, therefore ∠GBE is right.

Since AB is perpendicular to plane X, therefore ∠GBA is right.

Thus ∠GBE = ∠GBA, i.e the whole is equal to the part, which is impossible. Thus our initial assumption was impossible – AB in fact *is* parallel to CD.

Q.E.D.

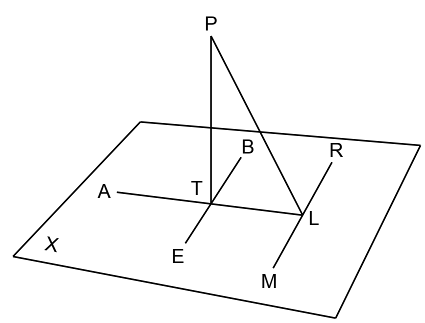
THEOREM 5 Remarks:

From this it is clear that *You can't have two straight lines perpendicular to the same point on a plane* (except, of course, on opposite sides of the plane, i.e. one above it and one below it).

THEOREM 6: *How to drop a straight line perpendicular to a plane from a given point above it.*

Suppose P is the point above our plane X. Choose any straight line RM in plane X. Thus P, R, M are in one and only one plane – drop PL perpendicular to RM in that new plane (as we learned to do in Ch. 1).

Now draw LA perpendicular to RLM in plane X. Thus P, L, A are in one and only one plane. Drop PT perpendicular to LA in that new plane.



I say that PT is perpendicular to plane X.

In plane X, draw BTE parallel to RLM.

Now RLM is perpendicular to plane PLT, since it is perpendicular to both PL and LT by construction (Thm.3). Thus BTE, parallel to RLM, is also perpendicular to plane PLT (Thm.4).

Thus BT is perpendicular to all lines through T in plane PLT (Def.2).

So BT is perpendicular to PT.

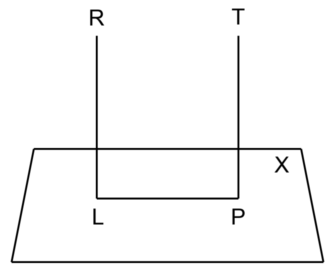
But LT is perpendicular to PT, by construction.

So PT is perpendicular to BT *and* LT, which both lie in plane X.

Therefore PT is perpendicular to plane X (Thm.3).

Q.E.F.

THEOREM 7: *How to set up a straight line perpendicular to a plane from a given point on it.*



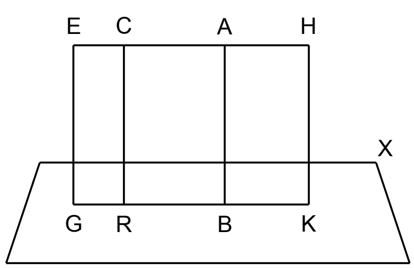
Given: Point P in plane X.
Make: A straight line perpendicular to plane X at P.

Choose any point R at random above plane X, and drop RL perpendicular to plane X (Thm.6).
In the plane of R, L, P draw PT parallel to RL.

Now	RL is perpendicular to plane X	(by construction)
and	<u>PT is parallel to RL</u>	(by construction)
so	PT is perpendicular to plane X	(Thm.4)

Q.E.F.

THEOREM 8: *Any plane containing a straight line perpendicular to another plane is itself perpendicular to that plane.*



Given: AB is perpendicular to plane X,
EHKG is a containing through AB

Prove: Plane EHKG is perpendicular to plane X.

Choose any random point R on GK, the intersection of plane EHKG and plane X.
Draw RC perpendicular to GK in plane EHKG.
We already know that AB is also perpendicular to GK, since AB is perpendicular to all straight lines through B in plane X.

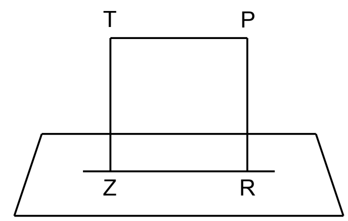
Since	AB is perpendicular to plane X	(given)
and	<u>RC is parallel to AB</u>	(RC and AB, in one plane, are $\perp$ to GK)
thus	RC is perpendicular to plane X	(Thm.4)

For the same reasons, *any* straight line (in plane EHKG) drawn perpendicular to GK will be perpendicular to plane X. Therefore plane EHKG is perpendicular to plane X (Def. 3).

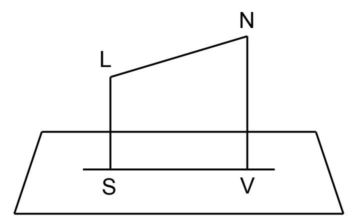
Q.E.D.

THEOREM 8 Remarks:

From this it is clear how to Drop a plane perpendicular to a given plane from a given straight line above the given plane, and how to Set up a plane perpendicular to a given plane upon a given straight line in the given plane.



Given a plane and a straight line in it, to set up a plane on that line perpendicular to the given plane: (1) pick any 2 points R and Z on the given line, (2) set up ZT and RP perpendicular to the given plane (Thm.7), (3) since ZT and RP are perpendicular to the same plane, therefore they are parallel (Thm.5), and thus are in one and only one plane together (Thm.2), (4) since the plane containing them passes through lines that are at right angles to the given plane, therefore their plane is at right angles to the given plane (Thm.8).



Given a plane and a straight line above it, to construct the plane which contains that line and is perpendicular to the base plane: (1) pick any 2 points L and N on the given line, (2) drop LS and NV perpendicular to the given plane (Thm.6). The rest of the proof is the same as above.

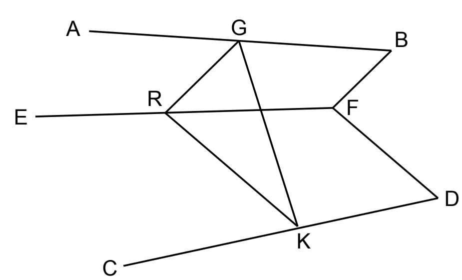
THEOREM 9: *If three straight lines are not all in one plane, and yet one of them is parallel to the other two, then the other two are also parallel to each other.*

Given: AB, CD, EF are three lines not all in one plane.

AB is parallel to EF.

CD is parallel to EF.

Prove: AB is parallel to CD.



[1] Choose R at random on EF.
Draw RG perpendicular to EF in the plane of parallels AB and EF.
Draw RK perpendicular to EF in the plane of parallels CD and EF.

[2] Now R, G, K are all in one plane (Thm.1).
And since ER is perpendicular to both KR and RG in the plane of R, G, K, therefore ER is perpendicular to the plane of KRG (Thm.3).

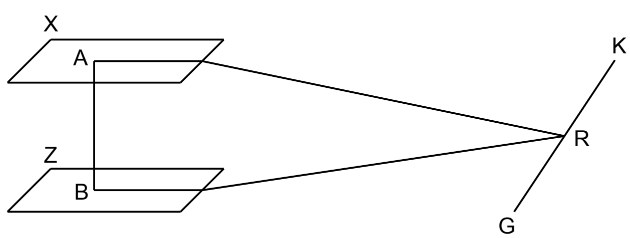
[3] Now AG is parallel to ER (given)
and ER is perpendicular to plane K, R, G (Step 2)
so AG is perpendicular to plane K, R, G (Thm.4)

[4] But CK is parallel to ER (given)
and ER is perpendicular to plane K, R, G (Step 2)
so CK is perpendicular to plane K, R, G (Thm.4)

[5] Since AG and CK are both perpendicular to the same plane, namely the plane of points K, R, G, therefore AG and CK are parallel to each other (Thm.5).

Q.E.D.

THEOREM 10: *If one straight line is perpendicular to two planes, the planes are parallel.*



Given: AB is perpendicular to plane X and to plane Z.

Prove: Plane X is parallel to plane Z.

If possible, suppose planes X and Z are *not* parallel, but eventually meet each other – let KG be the line of their intersection. Pick point R at random on KG.

Join AR.
Join BR.

[1] Now, A and R are both in plane X, and so line AR is in plane X.
And B and R are both in plane Z, and so line BR is in plane Z.

[2] Since BA is perpendicular to plane X (given), therefore any straight line in plane X passing through A is at right angles to BA. But AR is in plane X (Step 1), and it passes through point A. Therefore AR is at right angles to BA.

Thus ∠BAR is right.

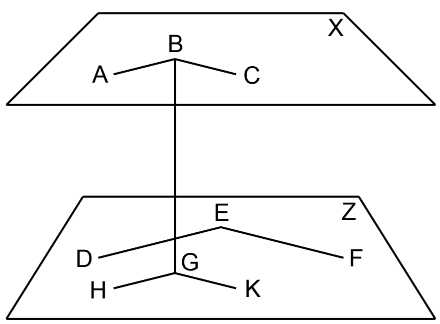
[3] Since AB is perpendicular to plane Z (given), therefore any straight line in plane Z passing through B is at right angles to AB. But BR is in plane Z (Step 1), and it passes through point B. Therefore BR is at right angles to AB.

Thus ∠ABR is right.

[4] Thus ABR is a triangle two of whose angles are right angles – which is impossible. Therefore our initial assumption was impossible, namely that planes X and Z should meet. Therefore planes X and Z never meet – and so they are parallel.

Q.E.D.

THEOREM 11: *If two intersecting lines in one plane are parallel to two intersecting lines in another plane, the two planes are parallel.*



Given: AB and BC intersect in plane X,
DE and EF intersect in plane Z,
AB is parallel to DE,
BC is parallel to EF.

Prove: Plane X is parallel to Plane Z

[1] Drop BG perpendicular to plane Z (Thm.6).
In plane Z, draw GH parallel to ED, and GK parallel to EF.

[2] Since BG is perpendicular to plane Z,
thus ∠BGH is right
and ∠BGK is right

[3] But GH is parallel to DE (we made it so)
and AB is parallel to DE (given)
so AB is parallel to GH (Thm.9)

Thus ∠ABG is right (since ∠BGH is right; Step 1)

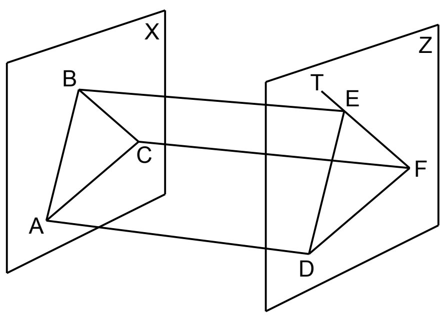
[4] Again GK is parallel to EF (we made it so)
and BC is parallel to EF (given)
so BC is parallel to GK (Thm.9)

Thus ∠CBG is right (since ∠BGK is right; Step 1)

[5] Therefore BG is at right angles to both AB and BC (Steps 3 and 4), which are two lines intersecting in plane X. Therefore BG is at right angles to plane X (Thm.3). But BG is at right angles to plane Z (we dropped BG at right angles to plane Z; Step 1). Therefore planes X and Z have a common perpendicular, namely BG, and thus these two planes are parallel to each other (Thm.10).

Q.E.D.

THEOREM 12: *A pair of intersecting lines parallel to another pair of intersecting lines in another plane will contain the same angle (or supplementary angles).*



Given: AB and BC intersect in plane X,
DE and EF intersect in plane Z,
AB is parallel to DE,
BC is parallel to EF.

Prove: ∠ABC = ∠DEF.

Cut off AB = DE, and cut off BC = EF.
Join AC, DF, AD, BE, CF.

[1] AB and DE are parallel (given), and so they are in one plane.
But we have just cut off AB and DE equal to each other.
Therefore the lines joining their endpoints are also parallel and equal (Ch.1).
i.e. AD and BE are parallel and equal to each other.

[2] BC and EF are parallel (given), and so they are in one plane.
But we have just cut off BC and EF equal to each other.
Therefore the lines joining their endpoints are also parallel and equal (Ch.1).
i.e. BE and CF are parallel and equal to each other.

[3]	Since	AD is parallel and equal to BE	(Step 1)
	and	<u>CF is parallel and equal to BE</u>	(Step 2)
	thus	AD is parallel and equal to CF	(Thm.9)

And so the lines joining their endpoints are also parallel and equal (Ch.1), i.e. AC is parallel and equal to DF.

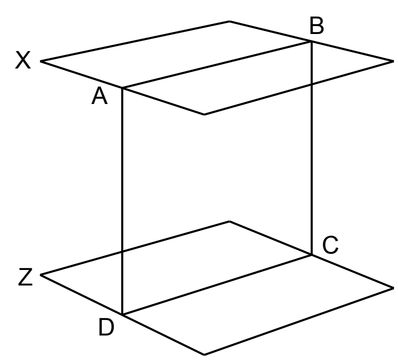
[4]	Now	AB = DE	(we cut them off equal)
	and	BC = EF	(we cut them off equal)
	and	<u>AC = DF</u>	(Step 3)
	thus	ΔABC ≅ ΔDEF	(Side-Side-Side)
	so	∠ABC = ∠DEF	

Q.E.D.

THEOREM 12 Remarks:

If we extend FE to T, then TE is parallel to BC, too, but ∠TED will not be equal to ∠ABC (unless ∠TED and ∠FED are both right angles). Still, ∠TED is supplementary to ∠DEF, and therefore also supplementary to ∠ABC.

THEOREM 13: *If one plane intersects two parallel planes, the two lines of intersection are parallel.*



Given: Plane X is parallel to plane Z, each is cut by plane ABCD, namely at AB and CD.

Prove: AB is parallel to CD.

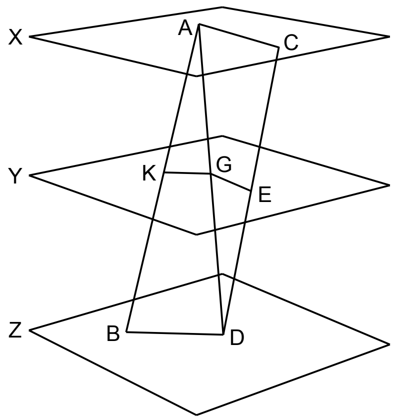
Since planes X and Z never meet in any direction, a line contained in one can never meet a line contained in the other. Therefore AB can never meet CD.

But since AB and CD are both in the one plane ABCD (given), therefore they are non-meeting straight lines *in the same plane*, and therefore they are parallel to each other.

So AB is parallel to CD.

Q.E.D.

THEOREM 14: *Straight lines cut by parallel planes are cut in the same ratios*.



Given: AB and CD are cut by three parallel planes X, Y, Z, cutting them off at A, K, B and C, E, D.

Prove: AK : KB = CE : ED.

[1] Join AD, AC, DB, GE, GK.

[2] Since A and C are both in plane X, thus AC is in plane X.
Since A and C are both in the plane of A, C, D, thus AC is in the plane of A, C, D.
Therefore AC is the line of intersection of plane X and plane A, C, D.

[3] Likewise EG is the intersection of plane Y and plane A, C, D.
GK is the intersection of plane Y and plane A, B, D.
DB is the intersection of plane Z and plane A, B, D.

[4] Thus AC is parallel to EG, being intersections of plane ACD with the parallel planes X and Y. (Thm.13)

and GK is parallel to DB, being intersections of plane ABD with the parallel planes Y and Z. (Thm.13)

[5] And so, since GK is parallel to DB (Step 4) in ΔABD, thus

AK : KB = AG : GD

and since AC is parallel to EG (Step 4) in ΔACD, thus

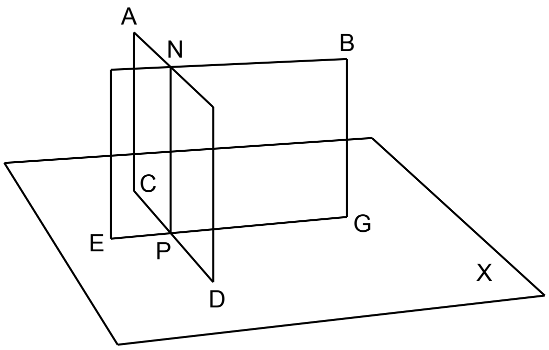
CE : ED = AG : GD

and since in these two proportions two ratios are the same as a third, it follows that they are the same as each other, i.e.

AK : KB = CE : ED

Q.E.D.

THEOREM 15: *The intersection of two planes each perpendicular to a third plane is a straight line perpendicular to the third plane.*



Given: Planes A and B, both perpendicular to plane X, and intersecting each other along PN, P being in plane X.

Prove: PN is perpendicular to plane X.

[1] Since plane A is perpendicular to plane X, and CD is their intersection, therefore every line drawn in plane A perpendicular to CD is also perpendicular to plane X (Def. 3). Therefore the straight line drawn from P (in plane A), perpendicular to CD, is perpendicular to plane X.

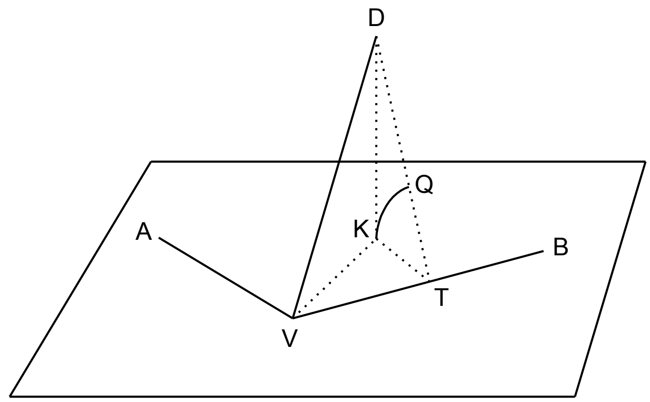
[2] Likewise since plane B is perpendicular to plane X, and EG is their intersection, therefore every line drawn in plane B perpendicular to EG is also perpendicular to plane X (Def. 3). Therefore the straight line drawn from P (in plane B), perpendicular to EG, is perpendicular to plane X.

[3] Therefore there is a perpendicular to plane X standing on point P *that lies in plane* A (Step 1), and again there is a perpendicular to plane X standing on point P *that lies in plane* B (Step 2). But there is only one perpendicular to plane X standing on point P (Thm.5 Remark). Therefore the line perpendicular to plane X, standing on point P, must be a line common to planes A and B. But the only line common to them is their line of intersection (Princ. 2), namely NP. Therefore the line perpendicular to plane X, standing on point P, is NP.

So PN is perpendicular to plane X.

Q.E.D.

THEOREM 16: *In a solid angle formed by three rectilineal angles, any two of those angles together are greater than the third.*



Let V be the vertex of a solid angle made up of the three rectilineal angles AVD, DVB, and AVB. I say that any two of these together are greater than the third.

[1] Drop DK perpendicular to the plane of AVB (Thm.6). In plane AVB, draw KT perpendicular to VB. Join DT.

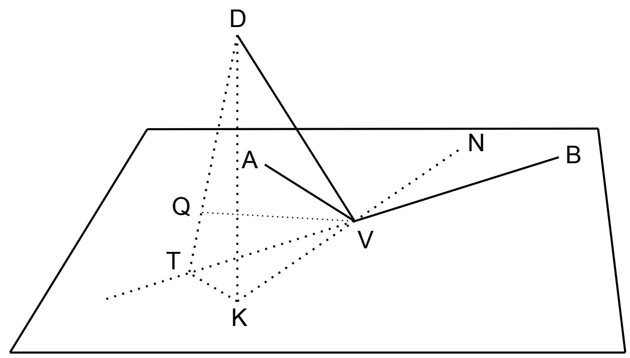
[2]	Now since	DK is perpendicular to AVB,
	thus	plane DKT is perpendicular to plane AVB (Thm.8).
	So	any line in plane AVB that is perpendicular to KT (which is the intersection of planes DKT and AVB) must be perpendicular to plane DKT (Def.3).
	But	VT is perpendicular to KT (Step 1).
	Hence	VT is perpendicular to plane DKT.
[3]	Since	VT is perpendicular to plane DKT (Step 2),
	thus	VT is perpendicular to every line through T in plane DKT (Def.2).
	So	VT is perpendicular to DT, i.e. $\angle DTV$ is right.
[4]	Now since DK is perpendicular to plane AVB, hence $\angle DKT$ is right.	
	Thus	DT > TK (since DT is hypotenuse in right $\triangle DTK$)
	So cut off	TQ = TK.
	Now	$\angle VTK = \angle VTQ$ (both are right; $\angle VTQ$ is $\angle DTV$)
	and	<u>VT is common</u> (to triangles VTK and VTQ)
	so	$\triangle VTK \cong \triangle VTQ$
	so	$\angle QVT = \angle KVT$
[5]	Now	$\angle DVT > \angle QVT$ (the whole is greater than the part)
	so	$\angle DVT > \angle KVT$ ($\angle KVT = \angle QVT$, Step 4)
[6]	So	$\angle DVB > \angle KVB$ (Step 5)
	Similarly	$\underline{\angle DVA > \angle KVA}$
	hence	$\angle DVB + \angle DVA > \angle KVB + \angle KVA$ (adding)
	or	$\angle DVB + \angle DVA > \angle AVB$

So these two given angles are greater than the third. Since there was nothing special about the two angles we chose among the given three, it follows the same way that any two of them will be greater than the third.

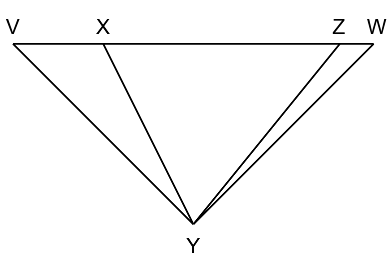
Q.E.D.

THEOREM 16 Remarks:

1. A solid angle contained by 3 plane angles is called a *trihedral angle*.

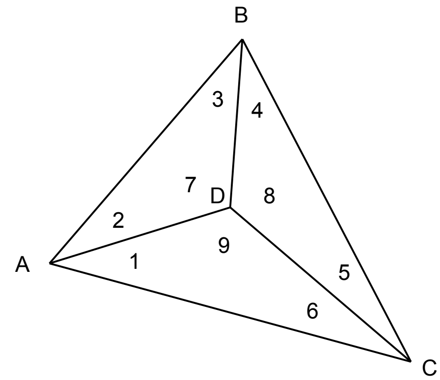


2. What if K lands outside angle AVB? Then the proof is identical up to Step 4, where we said $\angle DVT > \angle KVT$. Now extend KV through $\angle AVB$.
Thus $\angle KVT = \angle NVB$ (vertical)
so $\angle DVT > \angle NVB$.
And since $\angle DTV$ is right, hence $\angle DVT$ is acute (in $\triangle DTV$), and so its supplementary angle, $\angle DVB$, is obtuse.
Hence $\angle DVB > \angle DVT$
so $\angle DVB > \angle NVB$ (since $\angle DVT > \angle NVB$ above)
and $\angle DVA > \angle NVA$ by the same reasoning. And the remainder of the proof is the same as in the Theorem.



3. To illustrate why this Theorem is true, draw any angle XYZ on a piece of paper, and on each side of it draw angles VYX and ZYW which together add up to an angle *less than* angle XYZ. Cut out $\triangle VYW$, and fold along XY and YZ. Do triangles VYX and ZYW form a solid angle with triangle XYZ? Do they meet above the plane of $\triangle XYZ$? What happens if $\angle VYX + \angle ZYW = \angle XYZ$?

THEOREM 17: *Any solid angle is contained by plane angles adding up to less than four right angles*.



Let's start once more with a "trihedral" angle, an angle formed by three plane angles, namely 7, 8, 9, all coming up to a point D. (You must imagine that point D is above the plane of this page.) I say that 7 + 8 + 9 is less than four right angles.

Choose A, B, C at random along the legs of the solid angle, and join AB, BC, CA, thus forming solid angles again at A and at B and at C. Looking at the diagram, then, you must remember that you are looking down like a bird at the peak of a solid pyramid – so ABC is the base of the pyramid, but angles 1 through 9 all lie in planes that rise up toward you from that base.

[1] Because A is a trihedral angle, thus $1 + 2 > \angle CAB$ (Thm.16)

[2] Because B is a trihedral angle, thus $3 + 4 > \angle ABC$ (Thm.16)

[3] Because C is a trihedral angle, thus $5 + 6 > \angle BCA$ (Thm.16)

[4] Adding together all these inequalities, keeping the greater things on one side,

$$1 + 2 + 3 + 4 + 5 + 6 > \angle CAB + \angle ABC + \angle BCA$$

but $\underline{\angle CAB + \angle ABC + \angle BCA = \text{two rights}}$ (triangle ABC)

so $1 + 2 + 3 + 4 + 5 + 6 > \text{two rights}$

[5] Now angles 1 through 9, added together, equal all the angles in three triangles, and so all together they add up to three times the angle-sum of a triangle, i.e. three times two rights, i.e. six rights. So

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = \text{six rights}$$

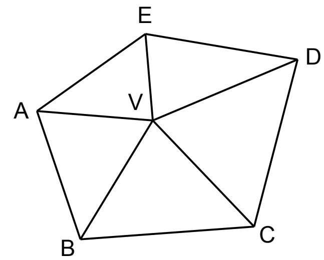
[6] Thus, if we subtract *more* than two rights from these nine angles, *less* than four rights will remain. But 1 + 2 + 3 + 4 + 5 + 6 is *more* than two rights (Step 4). Therefore, when subtracted from the nine angles, *less* than four rights remain, i.e.

$$7 + 8 + 9 < \text{four rights.}$$

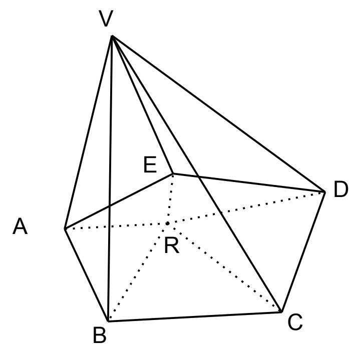
So the three plane angles forming a trihedral angle must add up to less than four right angles.

Q.E.D.

This Theorem is not limited to solid angles made of three plane angles. Take any solid angle with vertex V formed out of n plane angles. Pass a plane through the legs of the angle, forming a polygon base and a pyramid with vertex V. The polygon base will thus have n sides, and if we pick a random point R inside it, we can divide it into n triangles.



Now the angle-sum of the polygon base equals the angles of all those n triangles minus the angles around R, i.e. minus 360°. So the angles of the polygon $= (n \times 180° - 360°)$.



Since every vertex of the polygon base is also the vertex of a trihedral angle in the pyramid, hence very angle of the polygon must be less than the two angles above it which form the angles at the foot of the pyramid. For example, $\angle ABC < \angle ABV + \angle CBV$ (Thm.16). So all $2n$ angles about the foot of the pyramid add up to more than the n angles of the polygon, i.e. more than $(n \times 180° - 360°)$. So let those angles at the foot of the pyramid add up to $(n \times 180° - 360° + Z°)$.

Now the n plane angles forming the solid angle at V equal the angles in the n triangular faces of the pyramid *minus* their $2n$ angles at the foot of the pyramid. So the n angles forming solid angle V add up to

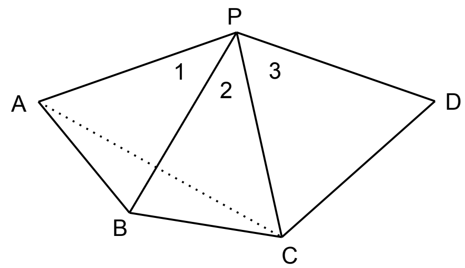
$$(n \times 180°) - (n \times 180° - 360° + Z°)$$

or

$$360° - Z°$$

So the n plane angles forming solid angle V add up to less than four rights.

THEOREM 18: *If among three angles in a plane any two are greater than the third, and they are made the peak angles of three isosceles triangles of the same leg-length, then likewise for the bases of these triangles, any two together will be greater than the third.*



Given: Three isosceles triangles whose legs are all equal, i.e. PA = PB = PC = PD, and whose peak angles (1, 2, 3) are such that any two are greater than the third.

Prove: Any two bases of these triangles will be greater than the third.

For example, I say that AB + BC > CD.

[1] Join AC.

[2]	Since	AP = CP = DP	(given)
	but	$\underline{\angle APC > \angle CPD}$	(given)
	thus	AC > CD	(Ch.1, Thm. 16 Question 1)
[3]	Now	AB + BC > AC	(triangle ABC)
	and	$\underline{AC > CD}$	(Step 2)
	so	AB + BC > CD	

Since there was nothing special about AB and BC, the same proof works just as well to show that BC + CD > AB, and again that AB + CD > BC. To show that AB + CD > BC, just rearrange the triangles so that angles 1 and 3 are next to each other, and 2 is on the outside.

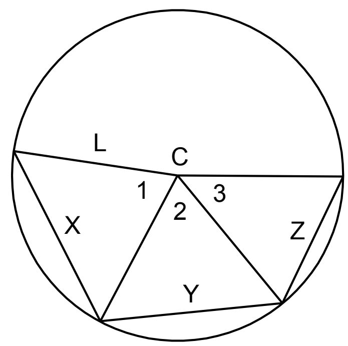
So whenever three isosceles triangles of the same leg-length are formed with three peak angles any two of which are greater than the third, likewise for their bases any two of them together will be greater than the third. *Q.E.D.*

THEOREM Remarks:

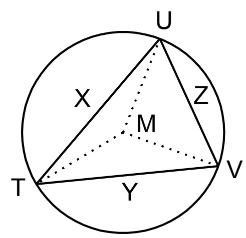
A quick corollary follows from this Theorem: we can make a triangle out of lengths AB, BC, CD, since any two of them are greater than the third. Thus we conclude: *When three isosceles triangles of the same leg-length are formed with three peak angles any two of which are greater than the third, then it will be possible to make a triangle out of the lengths of their bases.* For short, call such a triangle a "base triangle."

Obviously, this Theorem is simply a matter of plane geometry, but we will need it for the upcoming Theorem 20, here in solid geometry, where we shall construct a solid angle.

THEOREM 19: *If the peak angles of three isosceles triangles with a common leg-length* L *add up to less than four right angles, then* L *is greater than the radius of the circle circumscribing their "base triangle."*



Again, this is a matter of plane geometry, but it is crucial for the solid geometry in the next theorem. Start with three isosceles triangles of leg-length L, with peak angles 1, 2, 3 adding up to less than 360°, and bases X, Y, Z. Since they have the same leg length, L, if we place their equal sides together and give them a common vertex, C, the circle of center C and radius L will pass through the endpoints of bases X, Y, Z. Since 1 + 2 + 3 is less than 360°, hence the chords X, Y, Z do not cut off the circle's entire circumference.

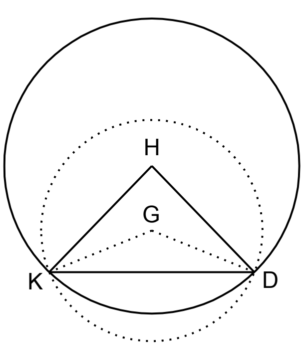


Now if the angles 1, 2, 3 are such that any two are together greater than the third, we can make a triangle out of X, Y, Z (Thm.18). So suppose this condition is met, and make △TUV with sides equal to X, Y, Z. Circumscribe a circle about △TUV (Ch.4). Call its center M.

Obviously the chords X, Y, Z together cut off the entire circumference of circle M. But these same chords together cut off only a portion of the circumference of circle C. Therefore circle C is greater than circle M, and so L (the radius of circle C) is greater than the radius of circle M.

Q.E.D.

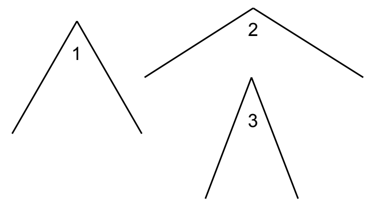
THEOREM 19 Remarks:



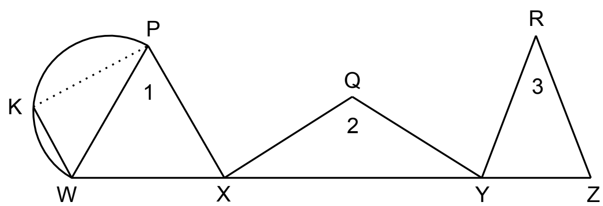
The proof takes it as evident that if the same chord length cuts off a greater portion of the circumference in one circle than it does in another, the other circle is greater than the one. For example, if KD cuts off an arc in circle G corresponding to ∠KGD, and an arc in circle H corresponding to ∠KHD, and ∠KGD > ∠KHD, then circle H is larger than circle G. To see it, compare isosceles triangles KGD and KHD. Since ∠KGD is greater than ∠KHD, the base angles of isosceles △KGD must be less than those of △KHD, and so KG and DG must meet *inside* △KHD. Hence the legs of △KGD are less than those of △KHD. So GK < HK, which means circle G is smaller than circle H.

THEOREM 20: *How to make a solid angle out of three plane angles. Thus it is required that they add up to less than four right angles, and that any two of them are greater than the third.*

Let our three given plane angles be 1, 2, 3. By Theorem's 16 and 17 we know that it is impossible to make a solid angle out of them unless they meet the conditions that any two of them are greater than the third, and they add up to less than four right angles. So let them meet these conditions.



To make a solid angle out of them,

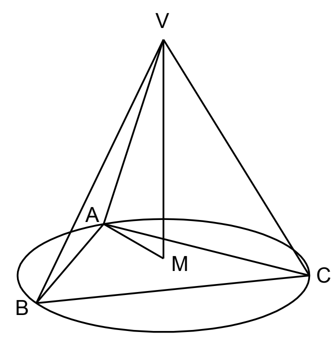


[1] Cut off any length PW along the leg of angle 1, and make three isosceles triangles PWX, QXY, RYZ, all having leg-length PW.

[2] Thus a triangle can be made out of their bases (Thm.18). So make triangle ABC with

AB = WX
and BC = XY
and CA = YZ.

Draw a circle around triangle ABC, find center M, and join MA.



[3] Draw a semicircle on PW. Setting your compass to length MA, make a circle (not shown) around center W, and where it cuts the semicircle call K. Thus WK = MA. This can be done because MA is less than diameter WP (by Thm.19).

[4] Join PK. Thus ∠PKW is right (Ch.3).
Set up MV perpendicular to the plane of the circle (Thm.7), making MV = PK.

[5] Now	MV = KP	(we made it so; Step 4)
and	MA = KW	(we made it so; Step 3)
and	∠VMA = ∠PKW	(both are right; Step 4)
so	△VMA ≅ △PKW	(Side-Angle-Side)
thus	VA = PW	

Likewise VC and VB are also each equal to PW, the common leg-length of our original isosceles triangles.

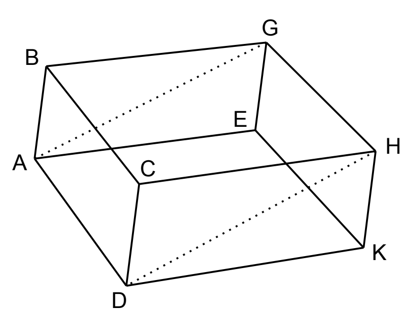
[6] But AB, BC, CA are equal to the bases of our isosceles triangles WX, XY, YZ (Step 2). So the three triangles standing on AB, BC, CA from point V are congruent to the three isosceles triangles (SSS), and hence the three peak angles forming solid angle V are equal to the given angles 1, 2, 3.

Q.E.F.

THEOREM 20 Remarks:

This Theorem is the converse of Theorems 16 and 17. In 16 and 17 we learned that any trihedral angle must be made of plane angles which add up to less than four rights and any two of which add up to more than the third one. But we were left wondering: are there *more* conditions required for three plane angles to be able to form a solid angle, or are those two conditions sufficient? Also, we might wonder this: the three angles must be less than four right angles – but do they in fact have to be less than *three* right angles, too? Or is it enough for them to be less than *four* right angles? This Theorem answers all those questions: as soon as the three plane angles are such that they are less than four right angles (by whatever amount you like), and such that any two of them are greater than the third, we can make them into a solid angle. Those conditions are not only necessary, but sufficient.

THEOREM 21: *If a solid is contained by three pairs of parallel planes, the opposite faces are congruent parallelograms (i.e. the solid is a parallelepiped).*



Suppose solid AH is contained by three pairs of parallel planes, namely BE and CK, and BH and AK, and BD and GK. I say that each pair of opposite faces, such as ABCD and EGHK, are identical parallelograms.

[1] Since AB and CD are the intersections of plane AC with the parallel planes BE and CK, therefore AB is parallel to CD (Thm.13).

[2] Since BC and AD are the intersections of plane AC with the parallel planes BH and AK, therefore BC is parallel to AD (Thm.13).

[3] Since AB is parallel to CD (Step 1)
and BC is parallel to AD (Step 2)
thus ABCD is a parallelogram.

Likewise the remaining 5 faces are parallelograms.

[4] Join AG, DH.

[5] Since AD is parallel to EK (because ADKE is a parallelogram)
and GH is parallel to EK (because GHKE is a parallelogram)
thus AD is parallel to GH (Thm.9)

[6] And thus A, D, G, H are all in one plane (Thm.2). And their plane intersects the parallel planes BE and CK at AG and DH, and therefore AG is parallel to DH (Thm.13). But AD was just proved parallel to GH (Step 5), and therefore AGHD is a parallelogram.

[7] So AG = DH (opp. sides in parallelogram AGHD)
and AB = DC (opp. sides in parallelogram ABCD)
and BG = CH (opp. sides in parallelogram BGHC)
so $\triangle ABG \cong \triangle DCH$ (Side-Side-Side)

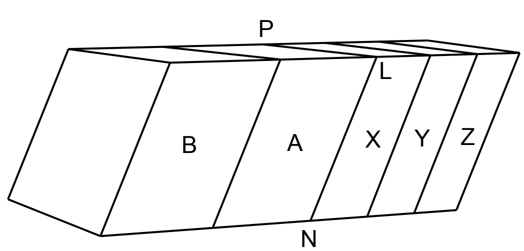
[8] But ABGE is just two of $\triangle ABG$, and DCHK is just two of $\triangle DCH$, similarly arranged. Therefore

$$ABGE \cong DCHK.$$

Likewise the other opposite parallelograms containing the solid are congruent to each other.

Therefore if a solid is contained by 3 pairs of parallel planes, then its six faces are three pairs of congruent parallelograms, and such a solid is called a *parallelepiped*.

THEOREM 22: *If a parallelepiped is cut by a plane parallel to one of its pairs of opposite faces, the two resulting parts have to each other the same ratio as the bases on which they stand.*



Given: Parallelepiped A + X, cut by a plane at PLN parallel to one pair of its opposite faces, thus dividing it into two parallelepipeds, namely A and X.

Prove: volume of A : volume of X = area of base of A : area of base of X

[1] Place a solid B, identical to A, right next to it, and a solid Y, identical to X, right next to it. And thus multiply solids A and X however many times you like. Say you double A, and triple X.

[2] Because of the identical shape and size of solids A and B, it is clear that the *base* of the whole solid A + B is double the *base* of A.

Likewise the base of the whole solid X + Y + Z is triple the base of solid X.

[3] Now, because they lie inside the same parallels and have identical angles, if the *solid* A + B is equal in volume to the *solid* X + Y + Z, this can only be because they stand on equal bases, i.e. the *base* of A + B must be equal to the *base* of X + Y + Z.

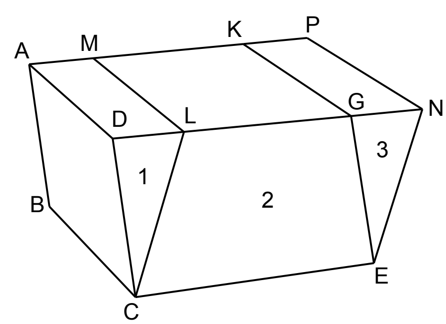
But if the solid A + B is *bigger* than solid X + Y + Z, then A + B must stand on a bigger base than X + Y + Z does. And if the solid A + B is *smaller* than solid X + Y + Z, then A + B must stand on a smaller base than X + Y + Z does.

[4] Therefore, whatever multiple we take of solid A (and therefore of its base), and whatever multiple we take of solid X (and therefore of its base), the multiple solids must compare the same way as the corresponding multiple bases.

[5] Therefore   solid A : solid B = base of A : base of B (Ch.5, Def.8)

Q.E.D.

THEOREM 23: *Parallelepipeds standing on the same base and having the same height are equal (i.e. they have the same volume).*



Given: Parallelepipeds AE and ME, both standing on base BCE and having their tops in the same plane.

Prove: AE and ME have the same volume.

First, suppose solids AE and ME not only have their tops in the same plane, but also that some other pair of their faces lie in the same plane, say CG and CN lie in the same plane – and therefore also the parallel faces BK and BP lie in the same plane on the opposite side.

But CDA and EGK do *not* coincide with CLM and ENP (if they did, the two solids would coincide entirely).

I say that solids AE and ME have the same volume.

[1]	For since CDGE and CLNE are both parallelograms, therefore		
		DG = LN	(each is equal to CE)
	so	DL = GN	(subtracting part LG from both sides)
	but	DC = GE	(in parallelogram CDGE)
	and	CL = EN	(in parallelogram CLNE)
	so	$\triangle$DCL ≅ $\triangle$GEN	(Side-Side-Side)

[2]	Now	AD is parallel to BC	(ABCD is a parallelogram)
	and	ML is parallel to BC	(BCLM is a parallelogram)
	so	AD is parallel to ML	(Thm.9)
	thus	ADLM is a parallelogram.	

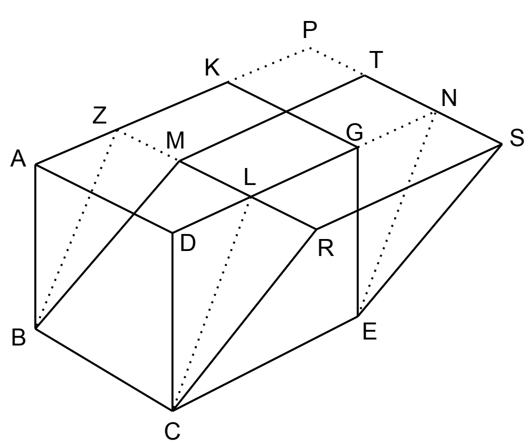
Clearly KGNP is also a parallelogram, and it is congruent to ADLM.
And, because they are opposite faces in the parallelepipeds,

AC and KE are congruent parallelograms

and MC and PE are congruent parallelograms

[3] Clearly, then, the two triangles and three parallelograms containing prism 1 are congruent with and arranged similarly to the two triangles and three parallelograms containing prism 3 (Steps 1 and 2). And thus they can be made to coincide and therefore have equal volumes.

[4]	So	prism 1 = prism 3	(Step 3)
	so	solid 1 + 2 = solid 2 + 3	(adding solid 2 to each side)
	i.e.	solid AE is equal to solid ME.	



Next, suppose that solids AE and ME have only their tops and bottoms in the same planes, and the front face of ME, namely CRSE, does *not* lie in the same plane as CDGE, the front face of solid AE.

AE and ME are still going to be equal in volume.

Let MRST be the top face of solid ME, in the same plane as ADGK, the top face of solid AE.

[1] Extend RM to Z on AK, and ST to P on the extension of AK.
Extend LG to N.
Join ZB, LC, NE, and P to X, the back corner of base BCE (which, to avoid cluttering up the diagram, I have not drawn).

[2] Now ZLNP is a part of the top plane, and the top plane is parallel to base BCEX (given). Thus
plane ZLNP is parallel to plane BCEX.

[3] And CLNE is a part of the face plane CDGE, which is parallel to the back plane BAKX (in solid AE). But BZPX is a part of the back plane. Thus
plane CLNE is parallel to plane BZPX.

[4] And ZLCB is a part of the side plane MRCB, which is parallel to the opposite side plane TSEX (in solid ME). But PNEX is a part of that opposite side plane. Thus
plane ZLCB is parallel to plane PNEX.

[5] Therefore the solid contained by planes
ZLNP and BCEX
and CLNE and BZPX
and ZLCB and PNEX
is contained by 3 pairs of parallel planes (Steps 2 – 3).
Therefore that solid, namely ZE, is a parallelepiped (Thm.21), and it stands on base BCEX and under the same height as the two given solids)

[6] Since solid ZE has its face CLNE in the same plane as CDGE, the face of solid AE, therefore solid ZE = solid AE, by the first part of this Theorem.

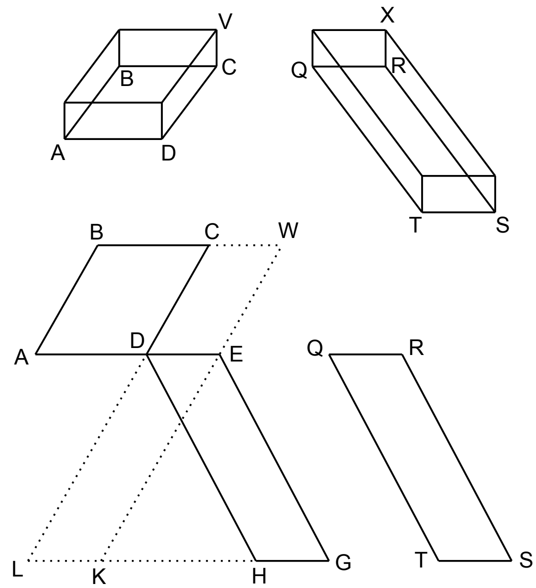
[7] Again, since solid ZE has its face ZLCB in the same plane as BMRC, the face of solid ME, therefore solid ZE = solid ME, by the first part of this Theorem.

[8] Therefore solid AE = solid ME (each being equal to solid ZE; Steps 6 and 7).

Therefore, no matter what, when two parallelepipeds have the same base and stand under the same height, they have the same volume.

Q.E.D.

THEOREM 24: *Parallelepipeds which are of the same height and on bases of equal area are equal.*



Conceive two parallelepipeds, AV and TX, with the same height and with bases ABCD and QRST having the same area. I say the solids have the same volume.

[1] Let's take the simplest case first: let the sides of these solids all be perpendicular to their bases – thus CV and RX are perpendicular to the bases and CV = RX (because the heights are the same). Because the walls of these solids are thus all standing at right angles to the bases, we can imagine the solids like two buildings, and just look at their "floor plans," namely their bases ABCD and QRST.

Now, to prove that AV = TX …

[2] We place a solid identical to TX in line with AD, that is, letting DEGH (identical to base QRST) be its base, we place DE in a straight line with AD. Complete parallelogram CDEW in the base plane, and build a "building" on it with the same height again as the solids on ABCD and DEGH.

[3] Extend CD to where it meets GH extended, namely at L, and complete parallelogram EDLK in the base plane, and build another "building" on top of it with the same height once more.

[4] Now, there is an undrawn rectangle standing straight up on DE (coming up at you out of the page) which is a *wall* for the building on DEKL; but it is also a wall for the building on DEGH. Since there is no absolute up and down in geometry, this wall can also be thought of as a base of each of these two solid buildings, and both are under the same height, i.e. both their tops lie in the plane standing on LKHG. Therefore they are equal in volume (Thm.23).

So The solid on DEKL = the solid on DEGH

[5] Notice that the buildings on DEKL and CDEW together make up one big parallelepipedal building, since they are in line with each other. Therefore, by Thm.22,

building on DEKL : building on CDEW = area of DEKL : area of CDEW,

[6] But, looking just at the parallelograms in the base plane,

	DEKL = DEGH	(both stand on DE, and are in the same parallels)
but	DEGH = QRST	(we made DEGH identical to QRST)
and	QRST = ABCD	(given)
so	DEKL = ABCD	

[7] So, substituting ABCD for DEKL in the proportion from Step 5, we have:

building on DEKL : building on CDEW = area of ABCD : area of CDEW,
But also by Theorem 22, we have

building on ABCD : building on CDEW = area of ABCD : area of CDEW

Since we have two ratios the same as a third ratio, they are the same as each other, i.e.

blding on DEKL : blding on CDEW = blding on ABCD : blding on CDEW.

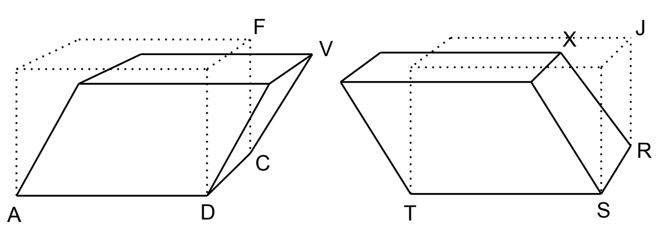
Notice in this proportion the buildings on DEKL and ABCD both have the same ratio to the building on CDEW. From this, it follows that they are equal. Thus

building on DEKL = building on CDEW.

[8]	Now	solid on DEKL = solid on ABCD	(Step 7)
	but	solid on DEKL = solid on DEGH	(Step 4)
	so	solid on ABCD = solid on DEGH	
	but	solid on QRST = solid on DEGH	(we made it thus in Step 2)
	so	solid on ABCD = solid on QRST	

Therefore the solid AV is equal in volume to the solid TX.

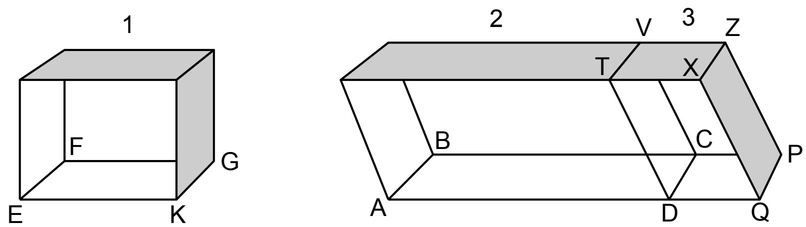
[9] Now what if the solids on ABCD and QRST, although having their tops and bottoms in the same planes, yet have their walls tilted in different ways? Will they still be equal? Yes.



Just build the solids on those same bases whose walls *are* perpendicular to the bases, having their tops also in the same top-plane as the "tilty" solids. Then, by Theorem 23, each upright solid is equal to the tilty solid whose base it shares. But, by the proof we just gave, the two upright solids are equal to each other – since they stand on equal bases and between the same parallel planes. Therefore the tilty solids are equal, too.

Q.E.D.

THEOREM 25: *Parallelepipeds of the same height are to each other as their bases*.



Given: Parallelepipeds 1 and 2 of the same height, standing on bases EFGK and ABCD.

Prove: Solid 1 has to solid 2 the same ratio that base EFGK has to base ABCD.

[1] Extend base ABCD so that parallelogram DCPQ, while having the same angles as parallelogram ABCD, nonetheless has the same *area* as EFGK.

[2] Complete the parallelepipedal solid on DCPQ by extending the planes of solid 2, and by capping it off with plane QXZP parallel to plane DTVC. Thus we have solid 3, and solids 2 and 3 together form one big parallelepiped.

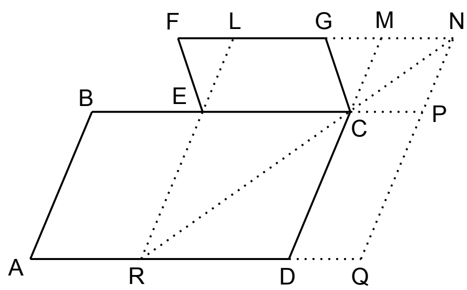
[3] Now solid 3 : solid 2 = DCPQ : ABCD (Thm.22)

[4] But solid 3 = solid 1,
since they stand between the same parallel planes, and have bases of equal area (Thm.24). Substituting solid 1 for solid 3 in the proportion from Step 3, then, we have:
solid 1 : solid 2 = DCPQ : ABCD

[5] But DCPQ = EFGK, by Step 1. Substituting EFGK for DCPQ in the proportion, we now have
solid 1 : solid 2 = EFGK : ABCD,
which is what we sought to prove.

Q.E.D.

THEOREM 25 Remarks:



You might be wondering how we accomplish Step 1. How do we extend the base ABCD with a parallelogram DCPQ that is equiangular with ABCD, but equal in area to EFGK?

Since that all takes place in the base plane, it is a matter of simple plane geometry, and Chapter 1 gives us all we need:

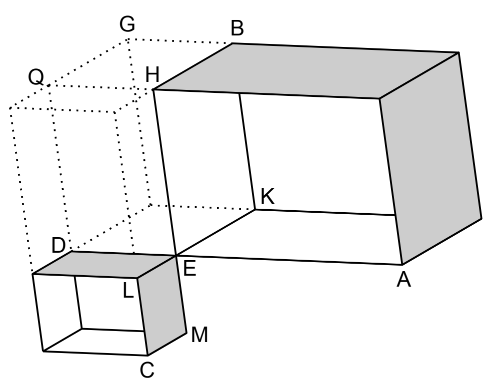
[1] Place EFGK on BC so that K is on point C.

[2] Draw LER parallel to AB and CD. Join RC. Extend RC and FG until they meet at N. Extend DC to M. Complete parallelogram DMNQ. Extend BC to P.

[3] Parallelogram DCPQ is clearly equiangular with parallelogram ABCD. But it is equal to parallelogram EFGC in area,

since	DCPQ = ECML	(complements in parallelogram RLNQ)
and	<u>EFGC = ECML</u>	(in the same parallels and on the same base)
so	DCPQ = EFGC	

THEOREM 26: *Similar parallelepipedal solids are to one another in the triplicate ratio of their corresponding sides.*



Given: Similar parallelepipeds AB and CD, with sides AE and ED being a pair of corresponding sides.

Prove: Solid AB : solid CD is the ratio triplicate of AE : ED.

[1] Place solids AB and CD so that they have a common corner at E, and the corresponding sides AE and ED lie in a straight line. Thus the corresponding sides LE and EK will also line up (since ∠LED = ∠KEA in the similar solids).

[2] In angles HED and HEK complete parallelepiped EG.
In angles HED and HEL complete parallelepiped LQ.

[3] Because of the similarity of the solids, AE, KE and HE are proportional to ED, EL, and EM. Hence

AE : ED = KE : EL = HE : EM

[4] Now, because parallelograms under the same height are to one another as their bases (Ch.6, Thm.1), it follows that:

AE : ED = AK : KD
and KE : EL = KD : DL
and HE : EM = HD : DM.

Because of Step 3, the first in each of these pairs of ratios are all the same ratio. Therefore the second in each of these pairs of ratios are also all the same,
i.e. AK : KD = KD : DL = HD : DM.

[5] But since parallelepipeds under the same height are to each other as their bases (Thm.25), it follows further that:

AK : KD = solid AB : solid EG
and KD : DL = solid EG : solid LQ
and HD : DM = solid LQ : solid CD

Because of Step 4, the first in each of these pairs of ratios are all the same ratio. Therefore the second in each of these pairs of ratios are also all the same,
i.e. solid AB : solid EG = solid EG : solid LQ = solid LQ : solid CD

[6] Since that proportion is continuous, and contains four terms, therefore the first has to the last the triplicate ratio of the first to the second, i.e.

solid AB : solid CD is the triplicate ratio of solid AB : solid EG.

[7] But, as we saw above in Steps 5 and 4,

solid AB : solid EG = AK : KD = AE : ED.
Therefore solid AB : solid CD is the triplicate ratio of AE : ED.

So similar parallelepipeds have to each other the triplicate ratio of their corresponding sides.

Q.E.D.

THEOREM 26 Remarks:

The most important instance of this, of course, is with cubes. All cubes are similar parallelepipeds, and so it follows that they are to each other in the ratio triplicate of their corresponding sides.

For example, suppose you had a pair of cubes, and the side or edge of one was *double* the side or edge of the other, i.e. their sides were in the ratio of 1 : 2. Then what is the ratio of their volumes? It will be 1 : 8, since

1 : 2 = 2 : 4 = 4 : 8,

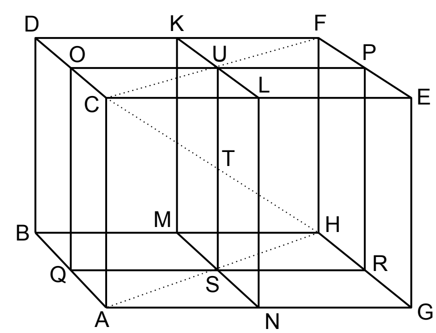
and thus 1 : 8 is the ratio triplicate of 1 : 2.

This Theorem should make you wonder about the ratios of other kinds of similar solids, such as curved ones. Do spheres have to each other the triplicate ratio of their diameters?

THEOREM 27: *If the sides of opposite faces in a parallelepiped are bisected by two planes, then the intersection of these two planes bisects (and is bisected by) the diagonal of the solid.*

Given: Parallelepiped BE, with diagonal CH. Planes QOPR and MKLN bisect the edges at Q, O, M, K, R, P, N, L. SU is the intersection of these two cutting planes.

Prove: SU and CH bisect each other.



[1] Join CU, UF.

[2] It is easily seen that OULC and UPEL are parallelograms.

Thus	OU = CL	
and	UP = LE.	
but	<u>CL = LE</u>	(given)
thus	OU = UP	
but	OC = PF	(being halves of the equal sides DC and EF)
and	<u>$\angle$UOC = $\angle$UPF</u>	(each is equal to $\angle$PEL)
so	$\triangle$UOC $\cong$ $\triangle$UPF	(Side-Angle-Side)
thus	$\angle$OUC = $\angle$PUF.	

[3] But OUP is a straight line, and therefore CUF is also a straight line, since the vertical angles OUC and PUF are equal.

Likewise ASH is a straight line.

And since AC and FH are equal and parallel lines, ACFH is a parallelogram.

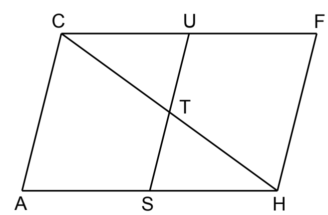
[4] Thus SU lies in the plane of parallelogram ACFH, since it joins points U and S which lie on its opposite sides. Thus CH and SU must meet, say at T.

[5]	Now	CU = UF	(since $\triangle$UOC ≅ $\triangle$UPF; Step 2)
	and	AS = SH	(since similarly $\triangle$SQA ≅ $\triangle$SRH)

Thus SU joins the midpoints of the opposite sides in parallelogram ACFH. Therefore SU bisects the diameter of ACFH, namely CH, and also is bisected by it.

Q.E.D.

THEOREM 27 Remarks:



1. If it is not perfectly clear why the line joining the midpoints of a parallelogram's opposite sides must bisect and be bisected by the diagonal, consider the following. Let ACFH be a parallelogram, and let CU = UF, and AS = SH.

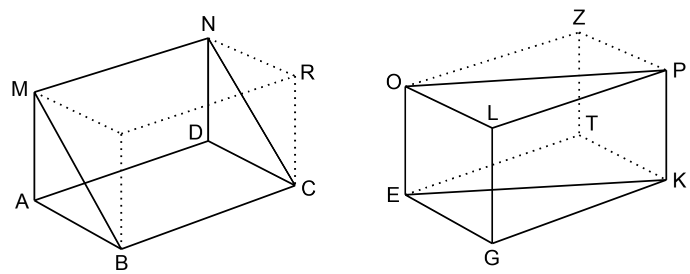
Now CU = SH		being halves of the opposite sides of a parallelogram,
and	∠HCF = ∠CHS	since CF is parallel to HA
and	<u>∠CUS = ∠HSU</u>	since CU is parallel to SH
so	$\triangle$CUT ≅ $\triangle$HST	(Angle-Side-Angle)
so	UT = TS	
and	CT = TH	*Q.E.D.*

2. Obviously, this Theorem is true about cubes in particular – if the sides of a cube are bisected by two planes, the intersection of those planes will bisect the diagonal of the cube, and be bisected by it.

3. In parallelepipeds other than cubes, the four diagonals can be unequal to each other. But that doesn't make any difference to this Theorem – take any diagonal you like, the proof did not require that we choose a special one.

THEOREM 28: *If a triangular prism lies on one of its parallelogram sides, and in this position has the same height as another triangular prism lying on its triangular base, and if the parallelogram is double the triangle, then the prisms will have the same volume.*

Imagine a prism with triangular bases ABM and DCN, lying on one of its parallelogrammic sides ABCD, and another prism with triangular bases EGK and OLP, lying on EGK, which has half the area of ABCD.



Now if we further suppose that ABCD and EGK lie in the same plane, and also that OLP and MN lie in the same plane, then I say that the prisms will have the same volume.

[1] Complete the parallelepiped AR contained by the angles ADC, ADN, NDC.
Complete parallelogram EGKT, and
Complete the parallelepiped GZ contained by the angles GKT, GKP, PKT.

[2] Since ABCD is double triangle EGK in area, and EGKT is also double triangle EGK in area, therefore ABCD = EGKT.

[3] But that means that solids AR and GZ stand on equal bases. And yet they also have the same height, since it is given that the height of the prisms is the same, and we made the parallelepipeds to have that same height. Therefore AR and GZ have the same volume (Thm. 24).

[5] Since AR and GZ are the same volume, therefore also their halves have the same volume. But the triangular prisms are obviously their halves. Therefore the two prisms are equal in volume, too.

Q.E.D.

THEOREM 28 Remarks:

We assumed in this Theorem that each prism is obviously half the volume of the parallelepiped of which it is a part. Why is that obvious?

Consider the prism contained by triangles OLP and EGK. It makes up a parallelepiped by being combined with another prism, the one contained by triangles OZP and ETK. Now EGKT and OLPZ are parallelograms, and so are OZTE and all the other faces of the parallelepiped.

Thus $\triangle$OLP ≅ $\triangle$OZP

and $\triangle$EGK ≅ $\triangle$ETK

and LPKG ≅ OZTE

and OLGE ≅ ZPKT

and, of course, OPKE is a common face for both prisms.

So the two prisms are contained by an equal number of congruent and similarly arranged faces. Therefore they are congruent and contain equal volumes.

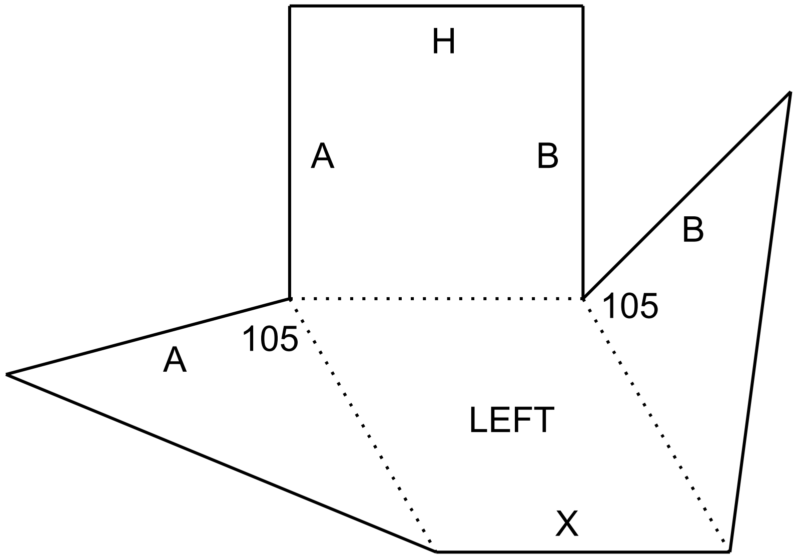
Does that mean that these prisms can *coincide*? Not necessarily.

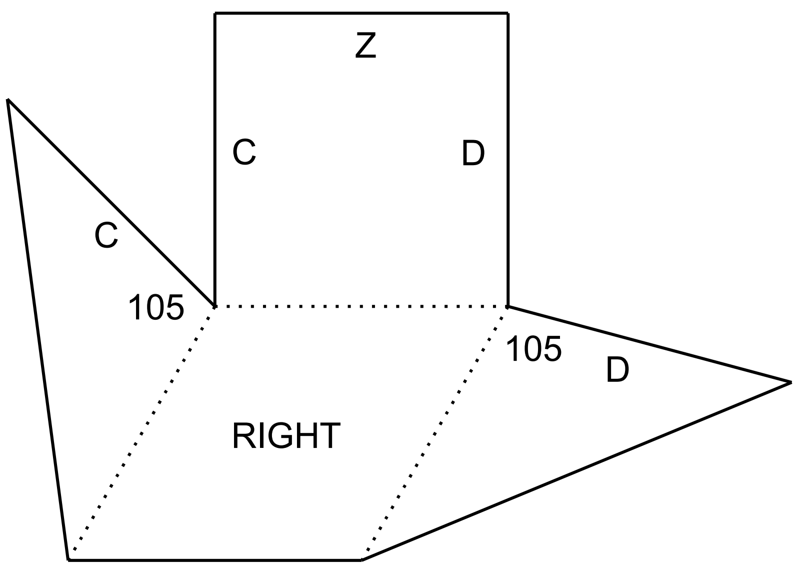
Consider your right hand and your left hand. Even if they were perfectly symmetrical, and of a ghostly quality so that they could pass through each other, they would not be able to coincide with each other and form one self-same hand. A right hand simply can't be a left hand!

Now, can the two prism halves of a parallelepiped be like that? Can they be perfect mirror images of each other, and yet not be able to coincide? Yes. It is almost impossible to represent this in a two-dimensional diagram in a clear and convincing way, so the best thing to do is to *make* a pair of such prisms. It is best not to use paper, since that is too flimsy – you need something more rigid like cardstock or a manila folder. Transfer the diagrams below onto a piece of manila: each consists of a square, a rhombus with angles of 60° and 120° (it is made of two equilateral triangles), and two isosceles triangles with peak angles of 105° (i.e. 60° + 45°) placed at the bottom corners of the square. The legs of the isosceles triangles are equal to the sides of the square.

After you have transferred the diagrams, cut out the two figures along the *solid* lines. Next, with all the labeling face up on the table, fold up the triangles and square along all the *dotted* lines. Bring together the edges marked with the same letters, such as "A", and tape them together. When you are done, you will have two triangular prisms, each with one open face. If you place the square faces down on the table and turn H and Z toward you, you will see that the prisms are symmetrical, but, like a right hand and a left hand, cannot be made to coincide. Their corresponding faces can be made to coincide one at a time, but not all of them simultaneously. If you pick them up in your hands, and place edges X and Z together, and in that position bring together the two open faces of the prisms, you will be holding a parallelepiped.

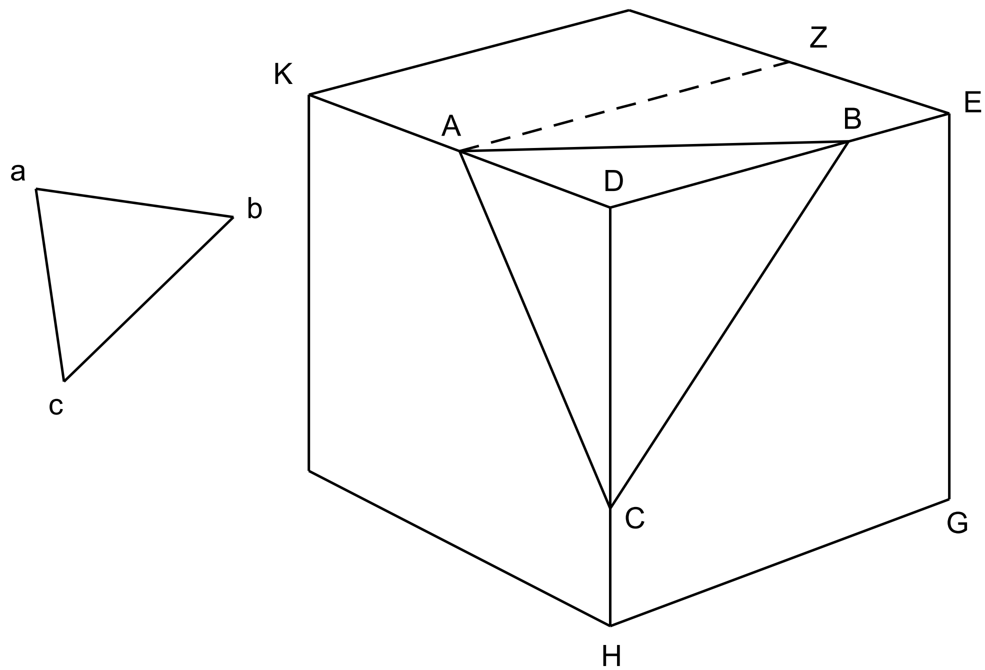
What makes the equality of these two prisms obvious, then, is not that they could be made to coincide. Rather, like your two hands, it is their perfect symmetry – one is a perfect mirror image of the other.


H
A
B
B
105
105
A
LEFT
X


Z
C
D
C
105
105
D
RIGHT

“HOOK”: TRIANGULAR SECTIONS OF A CUBE.

If you are given a cube and a triangle abc, will it be possible to slice the cube with a plane so that there will be formed a triangular facet which is similar to abc? Not if abc is right or obtuse. But if abc is acute, it can always be done.



Chapter Ten

Volumes and Areas

DEFINITIONS

1. A SERIES is a multitude of quantities produced in an order determined by some repeated process. It is an INFINITE SERIES if the process producing it can always be repeated to produce the next quantity.

For example, the numbers 2 4 8 16 *constitute a series because they are quantities produced in this order by the repeated process of doubling. They are part of an infinite series because there is no end to how many times we can repeat the process of doubling, and so we can always say which number comes next in the series*.

2. An infinite series is said to APPROACH a quantity that is not a member of it if, every time we are given an assigned amount, it is possible to find a member of the series differing from the quantity by *less* than that amount.

For example, take a series of lengths beginning with one foot:

First Term: 1

Second Term: $1 + \frac{1}{2}$

Third Term: $1 + \frac{1}{2} + \frac{1}{4}$

Fourth Term: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ etc.

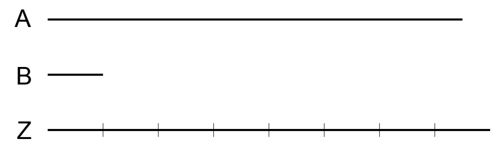
These lengths constitute an infinite series, since we can always form the next new term by adding half of the last thing added in the previous term. Now, two feet, or 2, *is not a member of this series – no matter how far we go in it, every term will always be less than two feet long. But each new member of the series gets us closer to two feet, and in fact the terms come as close to two feet as you want (as we will prove in the Theorems). For example, take one millionth of an inch – there will be a term in the above series that falls short of being two feet long by LESS than a millionth of an inch. And so this series is said to* approach *two feet. It never gets there, but it gets closer than any assigned difference*.

3. A GROWING series is one each new member of which is greater than the one before it. A SHRINKING series is one each new member of which is less than the one before it.

Note: you can also have oscillating series, e.g. where all the odd terms are growing but all the even terms are shrinking. It is not within the scope of this book to exhaust all the kinds of series that are possible.

THEOREMS

THEOREM 1: *If we subtract half from the greater of two unequal magnitudes, and always half again of what remains of it, by this process we will eventually have left a magnitude less than the lesser one set out.*



Recall from Chapter 8 that a *magnitude* is any quantity which is infinitely divisible, such as a line, or a surface, or a solid. Now consider any two unequal magnitudes, A and B, and suppose A is the greater one. What I wish to prove amounts to this:

The process $A - \frac{1}{2}A - \frac{1}{4}A - \frac{1}{8}A$ etc. will eventually leave a remainder that is less than B.

[1] Since B has a ratio to A (A being greater than B), therefore some multiple of B is greater than A. Thus if we double B, and then double its double, and double this again, etc., we will eventually arrive at a magnitude Z that is greater than A.

[2] Suppose, then, that $Z > A$
and $Z = 8B$
8B being a number of B's reached by repeatedly doubling B.

[3] Now $A < Z$ (Step 2)

so $\frac{1}{8}A < \frac{1}{8}Z$ (obviously)

so $\frac{1}{8}A < B$ (B equals one eighth Z by Step 2)

[4] But $\frac{1}{8}A$ is a part of A that remains after repeatedly halving A, and halving what remains, etc. That is,

$$\frac{1}{8}A = A - \frac{1}{2}A - \frac{1}{4}A - \frac{1}{8}A$$

[5] Therefore, by repeatedly halving A, and halving its half, etc., we must eventually arrive at a remainder that is less than B.

Q.E.D.

THEOREM 1 Remarks:

1. In Step 3 we mentioned $\frac{1}{8}$ of A, and also $\frac{1}{8}$ of Z. $\frac{1}{8}$ of Z poses no problem, because that is just B – we *started* with B, and got Z by taking B 8 times. But with A we actually started with A, not with something which, taken 8 times, equals A.

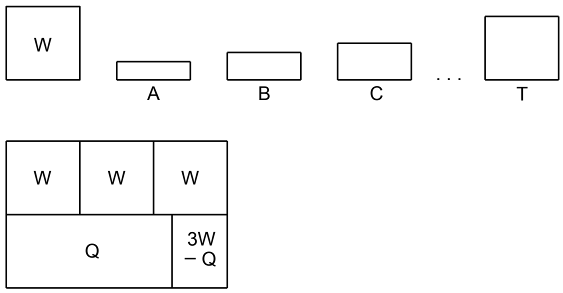
So how do we make $\frac{1}{8}$ of A? Do we know how to find exactly one eighth of any random magnitude? No. But this theorem does not assume we know how to do that – it only says that *if* we can take half and half again as often as we like (and thus leave $\frac{1}{8}$, or $\frac{1}{16}$, or $\frac{1}{32}$, etc.) *then* we will by this process eventually leave a remainder that is less than any given magnitude.

2. Even if we don't know how to take exactly half of any magnitude, as long as its parts always have a ratio to each other we can easily take *more* than half of it. Divide it at random, and take the larger piece (if they happen to be equal, then you *did* divide it in half, after all).

And clearly, if repeatedly taking just *half* of what remains of A gets us to a remainder less than B, then all the more quickly will repeatedly taking *more* than half of what is left get us to a remainder less than B. So suppose we have a growing series of magnitudes Q, R, S, T etc., and another greater magnitude X. Suppose further that each new term in the series takes up more than half of the difference by which X exceeded the last term, i.e. X – R is less than half of X – Q, and X – S is less than half of X – R etc. Then the series approaches X.

3. Can you prove that repeatedly taking *one third* of A, and then *one third* of what remains of A etc., will eventually leave a remainder less than B? Remember, we are not subtracting one third of A each time (we could only do that 3 times, and we would have nothing left): we are subtracting one third of each new remainder, so what we are subtracting is always getting smaller. Do you think this will get us to less than B? How about repeatedly taking one *millionth* of what remains each time? Will that get us down to less than B, regardless of how small B is? Believe it or not, the answer is yes. But I leave it to the reader to find proof – it is a digression from the goals of this book.

THEOREM 2: *If some multiple of magnitude* W *is greater than magnitude* Q, *and a series approaches* W, *then the same multiple of some term in that series is also greater than* Q.



(We are bothering with this only because we need it for the next Theorem. Don't worry, though, it's easy.)

Suppose we are given that $3W > Q$, and we are also given a series of magnitudes A, B, C etc., which approaches W (i.e. there is no limit to how close the terms in the series get to being equal to W, even if none of them actually equal it).

I say that there is some term T in the series A, B, C such that $3T > Q$.

[1] Take any part of the difference between 3W and Q which is less than a third of that difference, and call it d.

Thus $3d < 3W - Q$ (we chose d this way)

[2] Since A, B, C etc. *approaches* W (given), therefore there is always a term in that series differing from W by less than any specified amount. So take any term T in the series which differs from W by less than d.

Thus $W - T < d$

[3] So $3W - 3T < 3d$ (multiplying both sides by 3)

but $\underline{3d < 3W - Q}$ (Step 1)

so $3W - 3T < 3W - Q$

thus $3T > Q$

Q.E.D.

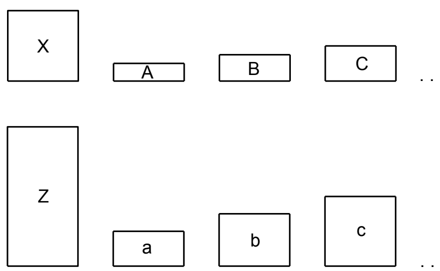
THEOREM 2 Remarks:

1. We chose 3W and then took less than a *third* of $3W - Q$. There is nothing special about 3 and one third. The exact same argument would work for 5W, as long as we also take less than one *fifth* of $5W - Q$, etc.

2. In Step 1 we "take less than a third of 3W – Q." How do we do that? As a matter of fact, as with Theorem 1, we will not be applying this Theorem except to magnitudes that we *can* do this with. So all this Theorem needs to assert (for our purposes) is that whenever we *can* find a way to take less than one third of the difference (or whatever fraction, depending on the multiple of W we start with), there must be a term in series A, B, C such that 3T > Q.

3. Obviously, if A, B, C is a *growing* series, then it is likewise true for every term *after* T, such as V, that 3V > Q. And if A, B, C is a *shrinking* series, then the Theorem would hold for every term *before* T.

THEOREM 3: *If two growing series approach two quantities, and the corresponding terms in the two series always have the same ratio, then the two quantities they are approaching also have that ratio.*



Given: Series A, B, C … etc. is a growing series approaching X,
Series *a, b, c* … etc. is a growing series approaching Z,
and A : *a* = B : *b* = C : *c* etc.

Prove: X : Z = A : *a*

Take any multiple of X and A, say 5 times each: 5X and 5A.
Take any multiple of Z and *a*, say 3 times each: 3Z and 3*a.*

Suppose that 5X > 3Z

[1] Since 5X > 3Z,
therefore in the series approaching X it is possible to take a term T so that
5T > 3Z (Thm.2)

[2] Let *t* be the term in the series approaching Z that corresponds to T.
Thus T : *t* = A : *a* (given)
Now since the series approaching Z is growing toward it (given), therefore every member of it is less than Z, and therefore
t < Z

[3]	Thus	$3t < 3Z$		(multiplying both sides of the inequality by 3)
	but	$\underline{5T > 3Z}$		(Step 1)
	so	$5T > 3t$		
[4]	Now	$T : t = A : a$		(Step 2)
	thus	$5A > 3a$		(since $5T > 3t$, Step 3)
Therefore if		$5X > 3Z$	then	$5A > 3a$.
Likewise if		$NX < MZ$	then	$NA < Ma$,
and if		$QX = RZ$	then	$QA = Ra$.

That is, any random multiples of X and Z must compare the same way as the corresponding multiples of A and *a*. Therefore

$X : Z = A : a$ (Ch.5, Def.8)

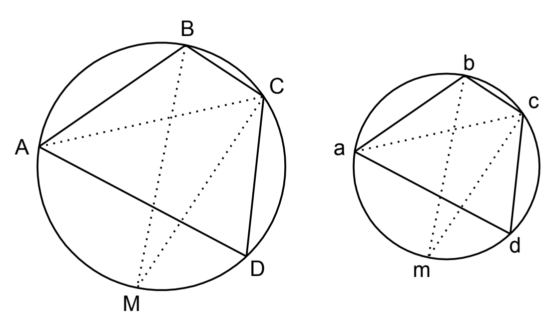
Q.E.D.

THEOREM 3 Remarks:

I chose to speak about *growing* series in this Theorem not because there is something special about them, but because we will be looking only at growing series in this Chapter. The Theorem, in fact, would also apply as well to other kinds of approaching series.

THEOREM 4: *Similar polygons inscribed in circles are to each other as the squares on the diameters*.

Let ABCD and *abcd* be similar polygons, each inscribed in a circle. Let diameters be drawn through corresponding points in the polygons, say points *b* and B, making diameters BM and *bm*.



I say that ABCD : *abcd* = ☐BM : ☐*bm*.

[1] Join AC, *ac*, CM, *cm*.

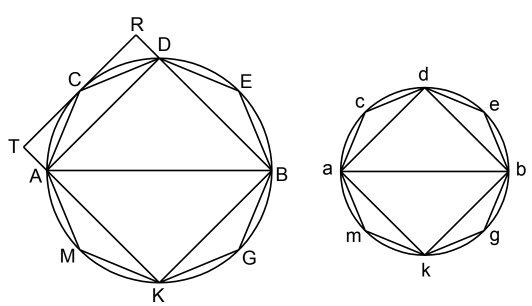
[2] Because similar polygons are made of similar triangles (Ch.6, Thm.14, Remarks),
therefore $\triangle$ABC is similar to $\triangle abc$
so $\angle$BAC = $\angle bac$

[3] But $\angle$BAC = $\angle$BMC (both stand on BC; Ch.3, Thm.20)
and $\underline{\angle bac = \angle bmc}$ (both stand on *bc*; Ch.3, Thm.20)
so $\angle$BMC = $\angle bmc$
But $\underline{\angle MCB = \angle mcb}$ (both are right; Ch.3, Thm.24)
so $\triangle$BCM is similar to $\triangle bcm$,
since they have two angles equal to two angles (Ch.6, Thm.4).

[4] Thus $\triangle$BCM : $\triangle bcm$ = $\square$BC : $\square bc$ (Ch.6, Thm.16)
but $\underline{\text{ABCD} : abcd = \square\text{BC} : \square bc}$ (Ch.6, Thm.17)
so ABCD : *abcd* = $\triangle$BCM : $\triangle bcm$
but $\underline{\square\text{BM} : \square bm = \triangle\text{BCM} : \triangle bcm}$ (Ch.6, Thm.16)
so ABCD : *abcd* = $\square$BM : $\square bm$

Q.E.D.

THEOREM 5: *Circles are to each other as the squares on their diameters.*



Take any two circles. Call their diameters AB and *ab*. I say that the circles are to each other in the same ratio as the squares on their diameters, i.e. as $\square$AB to $\square ab$.

[1] Bisect the semicircumferences in the circle AB at D and K, and inscribe square ADBK. Now bisect the quarter circumferences in circle AB at C, E, G, M, and inscribe the regular octagon ACDEBGKM. Thus we are making a *series* of inscribed regular polygons, each having twice the number of sides as the one before it.

[2] Each new polygon takes up more than half of what was left of the circle by the last polygon. For example, the square ADBK leaves 4 pieces of the circle such as the one contained by straight line AD and arc ACD. But the octagon removes ΔACD from that piece, which is more than half of it – because ΔACD is half of the rectangle ATRD (Ch.1, Thm.33), and since this rectangle is *greater* than the circle's segment ACD, it follows that △ACD is *more* than half segment ACD. So each polygon in the series leaves some amount of the circle left over, but the *next* polygon takes up more than half of that leftover. Therefore the series of polygons is *approaching* the area of the circle (Thm.1, Remark 2).

[3] Likewise the series of regular polygons inscribed in circle *ab*, made by repeatedly bisecting arcs, is approaching the area of circle *ab*. And just as the square in circle AB is similar to the square in circle *ab*, so every polygon in circle AB is similar to the corresponding polygon in *ab*.

[4] But similar polygons inscribed in circles are as the squares on their diameters (Thm.4). Therefore all the corresponding polygons in the two series have the ratio of ☐AB : ☐*ab*. And therefore the two magnitudes *approached* by these two series also have that ratio (Thm.3). That is

circle AB : circle *ab* = ☐AB : ☐*ab*.

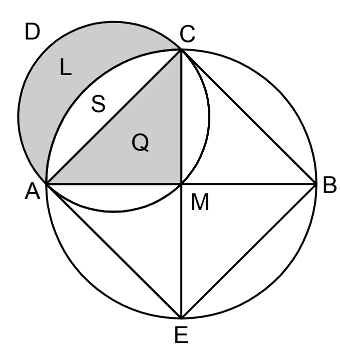
Q.E.D.

THEOREM 5 Remarks:

1. In Step 2 I asserted that rectangle ATRD is more than the segment ACD. How do I know that? I form the rectangle by drawing TCR tangent to point C. See if you can complete the proof by doing the following:

(a) Prove that TR is parallel to AD. (This allows you to complete a rectangle between TR and AD.)

(b) Argue from the fact that TCR is tangent that segment ACD of the circle must be *less* than that rectangle.



2. From this Theorem we can show how to make a square equal to an interesting curvilinear figure called a "lunule," or little moon.

(i) Consider a circle with center M, diameter AMB, and CM drawn perpendicular to AMB. Join AC and circumscribe a circle about triangle ACM.

(ii) Since ∠AMC is right, therefore AC is the diameter of the circle ADCM (Ch.3). So

circle ADCM : circle M = ☐AC : ☐AB (by the present Theorem)

(iii) But ☐AC is the square ACBE, and ☐AB is the square on its diagonal, so

☐AC : ☐AB = 1 : 2

thus circle ADCM : circle M = 1 : 2 (because of Step ii)

(iv) Thus circle M is *double* circle ADCM,

so one quarter circle M = one half circle ADCM

i.e. quadrant MAC = semicircle ADC

or areas S + Q = areas S + L

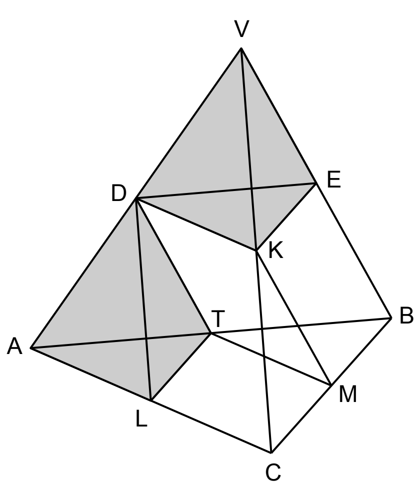
thus area Q = area L (subtracting S from each)

i.e. △AMC = lunule L

3. Since we can make a square equal to any rectilineal figure, we can thus make a square equal to the lunule L. This is called *quadrature*, or *squaring* an area. You can see why the quadrature of this and other kinds of lunules gave the ancient Greeks great hopes of finding a way to make a circle equal to a square. The various attempts at the "quadrature of the circle" over the centuries make up an interesting part of the history of mathematics. The end of the story came in modern mathematics (only about a couple hundred years ago), with a proof that it is actually *impossible* to construct a square equal to a circle using nothing but straight lines and circles (the tools we are using in this geometry book). Worse than that, even if you allow yourself all kinds of curves described by algebraic equations, you *still* cannot make a square equal to a circle!

THEOREM 6: *Any pyramid on a triangular base is divisible into two congruent pyramids (each similar to the whole) and two equal triangular prisms.*

TWO CONGRUENT PYRAMIDS SIMILAR TO THE WHOLE



Conceive a pyramid on triangular base ABC, vertex V.

[1] Bisect its six edges at D, K, E, L, T, M. Join DE, EK, KD, DT, TL, LD.

[2] Since the sides of the pyramid's triangular faces are all cut proportionally by these lines joining the midpoints, therefore these lines are parallel to the edges of the pyramid. For example, DE is parallel to AB.

So $\triangle VDE$ is similar to $\triangle VAB$
but $\underline{\triangle DAT \text{ is similar to } \triangle VAB}$
so $\triangle VDE$ is similar to $\triangle DAT$

And the corresponding sides AD and DV are equal,
so $\triangle VDE \cong \triangle DAT$.

[3] Similarly $\triangle VDK \cong \triangle DAL$.

[4] Now $AL = DK$ (Step 3 congruence)
and $AT = DE$ (Step 2 congruence)
and $\underline{\angle LAT = \angle KDE}$ (since $KD \parallel LA$ and $AT \parallel DE$, Ch.9 Thm.12)
so $\triangle LAT \cong \triangle KDE$

[5] Similarly $\triangle DLT \cong \triangle VKE$

[6] Hence all the triangles containing pyramid VDKE are congruent to those containing pyramid DALT. So these pyramids are congruent. And since they are contained by triangles similar to the triangular facees of the whole pyramid on base ABC, hence they are similar to the whole pyramid.

TWO EQUAL TRIANGULAR PRISMS

[7] Join KM, MT.
Now $DK \parallel LC$ (since $DK \parallel AC$)
but $\underline{DL \parallel KC}$ (since $DL \parallel VC$)
so DKCL is a parallelogram

[8] Likewise LTMC is a parallelogram.

[9] So DK is parallel and equal to LC (opp. sides of DKCL)
and <u>TM is parallel and equal to LC</u> (opp. sides of LTMC)
so DK is parallel and equal to TM
hence DKMT is a parallelogram

[10] Since the opposite sides of parallelograms are equal,
thus $DL = KC$
and $LT = CM$
and $\underline{TD = MK}$
so $\triangle DLT \cong \triangle KCM$

[11] Since $\triangle DLT$ and $\triangle KCM$ are congruent, and their corresponding sides are sides of parallelograms, therefore they contain a prism on LTMC as base.

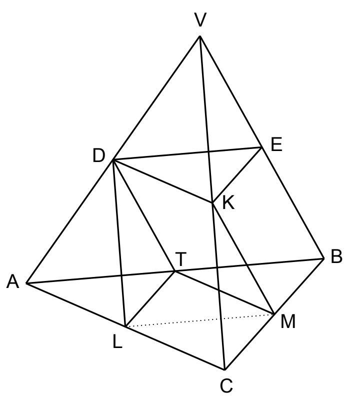
[12] Likewise $\triangle DKE$ and $\triangle TMB$ are congruent, and their corresponding sides are sides of parallelograms, so they contain a prism on $\triangle TMB$ as base.

[13] Now △TMB is half parallelogram LTMC, since they stand on equal bases CM and MB, and in the same parallels LT and CB. Hence the two triangular prisms standing on these bases, being under the same height (since the plane of DKE is parallel to the plane of CLTB), are equal. (Ch.9, Thm.28)

[14] So the whole pyramid is composed of the two congruent pyramids and these two equal triangular prisms

Q.E.D.

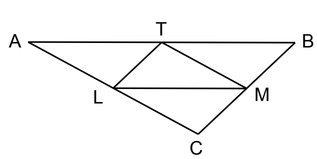
THEOREM 7: *When a pyramid on a triangular base is divided as in the last Theorem, the two equal prisms are more than half the whole.*

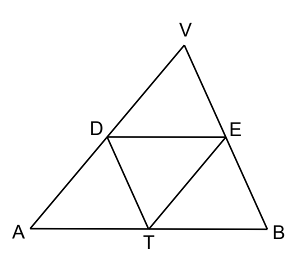


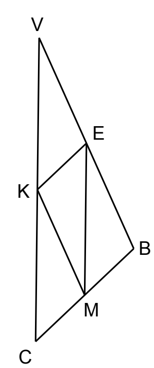
[1] Reproduce the diagram of Theorem 6, and join LM. Can you see that KLCM and ETMB are pyramids congruent to VDKE and DALT? (For clarity I leave some lines undrawn.)

[2] The triangular faces of the whole pyramid had their sides bisected, as at D, E, and T. Joining these midpoints divides each triangular face into four congruent triangles, each similar to the whole face.

[3]	So	$\triangle BMT \cong \triangle ALT$	
	but	$\underline{\triangle ALT \cong \triangle DKE}$	(Thm.6, Step 4)
	so	$\triangle BMT \cong \triangle DKE$	
	again	$\triangle ETB \cong \triangle VDE$	for the same reasons,
	and	$\triangle EMB \cong \triangle VKE$	for the same reasons again.







[4] Since the corresponding sides in these congruent triangles are equal,

hence MT = DK
and TE = DV
and EM = VK
so △TEM ≅ △DVK (SSS)

[5] Hence all the triangles containing pyramid ETMB are congruent to those containing pyramid DALT,

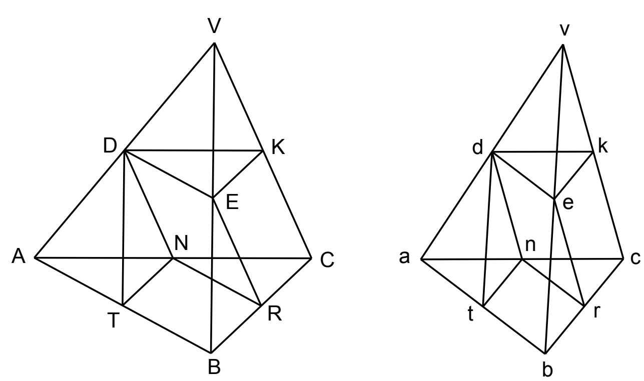
so ETMB ≅ DALT.
Likewise KLCM ≅ DALT.

Now pyramids ETMB and KLCM are only *parts* of the two equal prisms, and so pyramids VDKE and DALT, equal to those pyramids, are less than the two prisms. Since VDKE and DALT, together with those two prisms, exhaust the whole pyramid, hence the two prisms are more than half of it.

Q.E.D.

THEOREM 8: *If two triangular pyramids under the same height are divided each into their two prisms and two pyramids (as in the last Theorem), then the two prisms in one are to the two prisms in the other as the base to the base.*

Suppose you have two pyramids whose triangular bases ABC and *abc* are in the same plane, and also their vertices V and *v* are in a parallel plane. Let them each be divided, as in Theorem 6, into two congruent pyramids each similar to the whole and two equal prisms.



[1] Now the prism on TBRN with top edge DE is half the parallelepiped on that base and with its top in the plane of DEK (Ch.9, Thm.28, Remarks). So too, the prism on *tbrn* is half the parallelepiped on *tbrn*. So

prism TBRN : prism *tbrn* = parallelepiped TBRN : parallelepiped *tbrn*

[2] Since the original pyramids are under the same height, therefore the planes cutting their edges in half are also of the same height, i.e. planes DEK and *dek*. Therefore the parallelepipeds just mentioned are under the same height, and therefore they are to each other as their bases (Ch.9, Thm.25). And therefore, the prisms which are their halves are also as their bases. So

prism TBRN : prism *tbrn* = TBRN : *tbrn*

[3] And likewise the prism on NRC (with top DEK) is to the prism on *nrc* (with top *dek*) as the base NRC is to the base *nrc*. That is,

prism NRC : prism *nrc* = NRC : *nrc*

[4] Now if we join TR, then $\triangle$ABC is divided into four congruent triangles ATN, NTB, BRN, NRC. Likewise if we join *tr*, then Δabc is divided into its four congruent triangles.

So	TBRN : $\triangle$ABC = *tbrn* : $\triangle abc$	(both are the ratio 2:4
thus	TBRN : *tbrn* = $\triangle$ABC : $\triangle abc$	(alternating)
and	NRC : $\triangle$ABC = *nrc* : $\triangle abc$	(both are the ratio 1:4)
thus	NRC : *nrc* = $\triangle$ABC : $\triangle abc$	(alternating)

[5]	Now	prism TBRN : prism *tbrn* = TBRN : *tbrn*	(Step 2)
	But	<u>TBRN : *tbrn* = $\triangle$ABC : $\triangle abc$</u>	(Step 4)
	so	prism TBRN : prism *tbrn* = $\triangle$ABC : $\triangle abc$	
	Also	prism NRC : prism *nrc* = NRC : *nrc*	(Step 3)
	But	<u>NRC : *nrc* = $\triangle$ABC : $\triangle abc$</u>	(Step 4)
	so	prism NRC : prism *nrc* = $\triangle$ABC : $\triangle abc$	
[6]	So	prism TBRN : prism *tbrn* = $\triangle$ABC : $\triangle abc$	(Step 5)
	but	<u>prism NRC : prism *nrc* = $\triangle$ABC : $\triangle abc$</u>	(Step 5)
	so	prisms TBRN + NRC : prisms *tbrn* + *nrc* = $\triangle$ABC : $\triangle abc$	

since the sums of things in the same ratio remain in the same ratio (Ch.5, Theorem 15). And so the two prisms in the one whole pyramid are to the two prisms in the other whole pyramid in the same ratio as the bases of the whole pyramids.

Q.E.D.

THEOREM 8 Remarks:

If we now divide the small pyramids on DEK and *dek* in the same way, the sums of *their* corresponding prism-parts will be as their bases DEK and *dek*. So

2 prisms in pyramid DEK : 2 prisms in pyramid *dek* = DEK : *dek* (this Thm.)

But since △DEK is identical to △NRC, and △*dek* is identical to △*nrc*, therefore

DEK : *dek* = NRC : *nrc*

Therefore, putting together these two proportions,

2 prisms in pyr. DEK : 2 prisms in pyr. *dek* = NRC : *nrc*

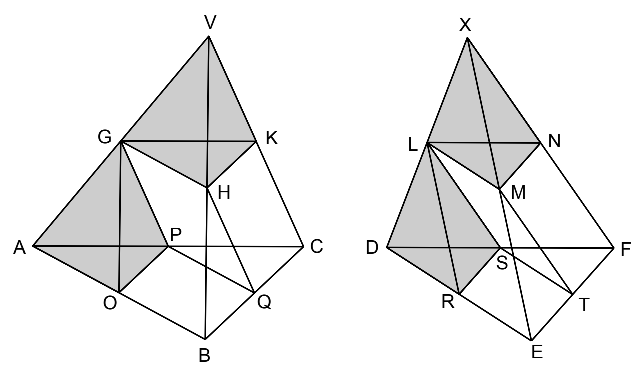
But, as we saw, NRC is one fourth of ABC, and *nrc* is one fourth of *abc*. So

2 prisms in pyr. DEK : 2 prisms in pyr. *dek* = △ABC : △*abc*

So even the little prisms in the leftover little pyramids also have to each other the ratio of the original base ABC to the original base *abc*. Likewise, if we now take the little pyramids into which DVEK and *dvek* are divided, and divide *them*, the equal prisms in them will be as DEK to *dek*, and hence as ABC to *abc*. And therefore *all* the prisms so made in pyramid ABC have to *all* the prisms so made in pyramid *abc* the ratio of △ABC to △*abc*.

THEOREM 9: *Pyramids on triangular bases and under the same height are to each other as their bases.*

Consider a pyramid with vertex V on base ABC, and another pyramid with vertex X on base DEF, and suppose their bases are in the same plane, and their vertices are in a parallel plane. I say that



pyramid V : pyramid X = ABC : DEF.

[1] Divide each pyramid into two congruent pyramids, similar to the whole, and two equal triangular prisms, as before. In each pyramid, the 2 prisms take up more than half the whole (Thm.7). And so if we now divide the *smaller* leftover pyramids in the same way, such as pyramids VGHK and GAOP, *their* prisms will also take up more than half of *them*. And they, too, will have two left over pyramids, and so on. So, continuing in this way we will have a series of magnitudes approaching pyramid V, namely:

1. The 2 prisms in pyramid V
2. The 2 prisms in pyramid V *plus* the 4 prisms in the two leftover pyramids
3. The 2 prisms in pyramid V *plus* the 4 prisms in the two leftover pyramids, *plus* the 8 prisms in the four leftover pyramids etc.

In each step of this process, we take what we had before and add *more than half* of what remained of pyramid V. Thus this series is approaching pyramid V (Thm.1, Remark 2).

So, too, there is a corresponding series of prisms approaching pyramid X.

[2] Now since, at any step in the series, all the prisms in pyramid V have to all the corresponding ones in pyramid X the same ratio as ABC to DEF (Thm.8, Remarks), therefore the magnitudes *approached* by these two series of prisms also have that same ratio (Thm.3). That is

Pyramid V : Pyramid X = ABC : DEF.

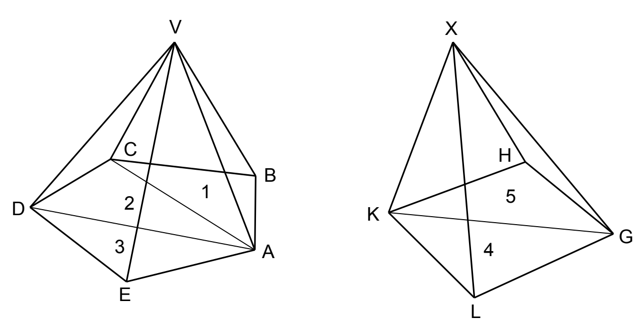
Q.E.D.

THEOREM 9 Remarks:

In particular: *Pyramids under the same height which are on equal triangular bases are equal*. For they must be as their bases (by this Theorem), which are equal.

THEOREM 10: *Pyramids of the same height are to each other as their bases.*

Didn't we just do this? No – we limited the last Theorem to pyramids with triangular bases. Now we are saying: *the bases don't have to be triangles*. They can be any two polygons at all.



Take any two pyramids V and X whose bases (ABCDE and GHKL) are in the same plane and whose vertices are in a parallel plane, and it will follow that

Pyramid V : Pyramid X = ABCDE : GHKL.

[1] Pick any vertex on base ABCDE, say A, and join AC, AD, dividing the base into triangles 1, 2, 3, and thus dividing pyramid X into three triangular pyramids on 1, 2, 3.

Similarly, divide base GHKL into triangles 4, 5, thus dividing pyramid X into two triangular pyramids on 4, 5.

For shorthand, let P_1 mean "The Pyramid on triangular base 1," let P_2 mean "The Pyramid on triangular base 2," and so on with the other triangular bases.

[2] Now $P_1 : P_2$ = Base 1 : Base 2 (Thm.9)

And so, adding each consequent to its antecedent,

$P_1 + P_2 : P_2$ = Base 1 + Base 2 : Base 2 (Ch.5, Thm.15, Remarks)

And the consequents in this proportion are P_2 and Base 2.

[3] But $P_3 : P_2$ = Base 3 : Base 2 (Thm.9)

And the consequents in *this* proportion are also P_2 and Base 2.

Therefore, putting this proportion together with that in Step 2, and leaving out the identical consequents, we have a proportion among the antecedents:

$P_1 + P_2 : P_3$ = Base 1 + Base 2 : Base 3 (Ch.5, Thm.17)

And thus, adding each consequent to its antecedent (Ch.5, Thm.15), we have

$P_1 + P_2 + P_3 : P_3$ = Base 1 + Base 2 + Base 3 : Base 3

[4] Similarly, we can prove that all the triangular pyramids in Pyramid X have to any one of them the same ratio as the whole base to the triangular base of that one, i.e.

$P_4 + P_5 : P_4$ = Base 4 + Base 5 : Base 4

[5] But the consequents of this proportion and in the proportion from Step 3 form a proportion, since

$P_3 : P_4$ = Base 3 : Base 4 (Thm.9)

And therefore the antecedents of the two proportions also form a proportion, and in the same order (Ch.5, Thm.17). That is,

$P_1 + P_2 + P_3 : P_4 + P_5$ = Base 1 + Base 2 + Base 3 : Base 4 + Base 5

which is the same as saying

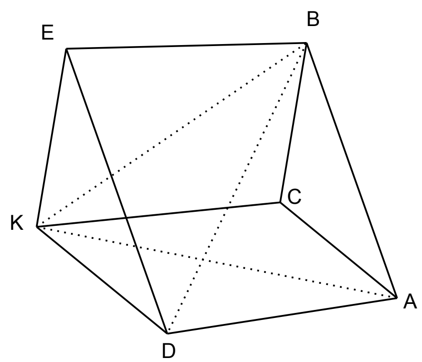
Pyramid V : Pyramid X = ABCDE : GHKL.

Q.E.D.

THEOREM 10 Remarks:

Again, note the particular case that *Pyramids under the same height and on equal bases are equal*.

THEOREM 11: *Any triangular prism is composed of three triangular pyramids of equal volume.*



Let ABC and DEK be the triangular bases of a prism. I say that it is divisible into three equal pyramids having triangular bases.

[1] Join BD, BK, KA.

[2] Thus the prism is divided into three pyramids, namely
one with base DEK and vertex B (DEK*B for short)
one with base DKB and vertex A (DKB*A for short)
one with base AKB and vertex C (AKB*C for short)

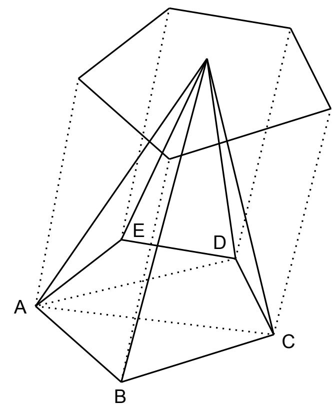
[3] Now DEK*K = DAB*K
since they stand on equal bases (DEB and DAB each being half of parallelogram ABED), and share a vertex, K (Thm.9, Remarks).
But DEB*K is the same as pyramid DEK*B
and <u>DAB*K is the same as pyramid DKB*A</u>
so DEK*B = DKB*A

[4] And DAK*B = AKC*B
since they stand on equal bases (DAK and AKC each being half of parallelogram ACKD), and share a vertex, B (Thm.9, Remarks).
But DAK*B is the same as pyramid DKB*A
and <u>AKC*B is the same as pyramid AKB*C</u>
so DKB*A = AKB*C

[5] Thus DEK*B = AKB*C (Steps 3 and 4)
And so the three pyramids into which the whole prism is divided are equal to each other.

Q.E.D.

THEOREM 11 Remarks:



1. Accordingly, any triangular pyramid is *one third* of a triangular prism sharing its base and under the same height. Using the same diagram above, consider pyramid ABC*K, standing on base ABC. This pyramid is part of the prism contained by ABC and DEK. But imagine some *other* prism, Prism X, also standing on base ABC, whose top triangle is somewhere else in the plane of DEK (just shift DEK over a bit within the same plane, and you get such a prism). So Prism X shares base ABC with prism DEK, and also has the same height as it. Then it must be equal to it. For if we double triangle ABC into a parallelogram, and complete the parallelepipeds on it, prism DEK and Prism X are *half* of these parallelepipeds, which are clearly equal to each other (Ch.9, Thm.23).

But pyramid ABC*K is one third of prism DEK (by this Theorem).
So pyramid ABC*K is one third of Prism X, too.

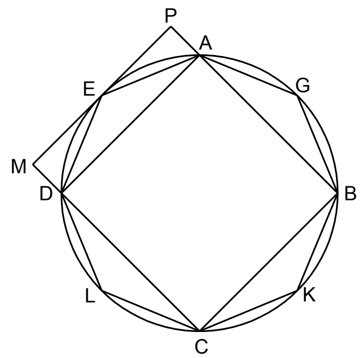
2. More generally, though, *any* pyramid on any polygon as its base is one third of the prism sharing that same polygon base and under the same height.

Take any pyramid, say on base ABCDE. Divide its base into triangles. Thus the pyramid, too, is divided into three pyramids, one on each triangular part of the base. But a prism standing on base ABCDE will likewise be divided into three prisms, one on each triangular part of the base.

So if the prism has the same height as the pyramid, each triangular pyramid will also have the same height as each triangular prism. And therefore, by the above Remark, each triangular pyramid will be one third of the triangular prism sharing its base, and thus *all* the triangular pyramids will be a third of *all* the triangular prisms, i.e. the whole pyramid on ABCDE is one third of the whole prism on that base and under the same height.

3. Since pyramids under the same height, regardless of what kind of bases they stand on, are to each other as their bases (Thm.10), and since the prisms standing on their bases and under the same heights are as those pyramids (being triples of them), it follows also that *Prisms standing under the same height are to each other as their bases*.

THEOREM 12: *Any cone is one third the cylinder sharing its base and height.*



Let circle ABCD be the base of a cone, and also of a cylinder having the same height (we are talking, recall, about *right* cylinders and cones). I say that the cone is exactly one third the volume of the cylinder.

[1] Inscribe a square ABCD in the base circle, and by bisecting the arcs inscribe a regular octagon AGBKCLDE.

[2] As we saw in Theorem 5, each successive polygon (made by bisecting more arcs) takes up more than half of what remained of the circle from the previous polygon, since, for example, △ADE is more than half of the segment of the circle contained by AD and arc DEA.

[3] But since prisms under the same height are as their bases, therefore the right prism on △ADE is *half* the right prism on rectangle DMPA, each one having the height of the cylinder. And since the portion of the cylinder standing on the segment contained by AD and arc DEA is *inside* the right prism standing on rectangle DMPA, therefore it is *less* than that prism. Therefore the prism on △ADE is *more than half* that portion of the cylinder. Thus the prisms standing on ABCD, AGBKCLDE, etc., constitute a series of prisms each taking more than half of what was leftover of the cylinder by the previous prism, and so this series of prisms *approaches* the volume of the cylinder (Thm.1 Remarks).

[4] Again, since pyramids under the same height are as their bases, therefore the pyramid on △ADE is half the pyramid on rectangle DMPA with that same vertex. And since the portion of the cone standing on the segment in AD and arc DEA is *inside* the pyramid on rectangle DMPA, therefore it is *less* than that pyramid. Therefore the pyramid on △ADE is *more than half* that portion of the cone. Thus the pyramids standing on ABCD, AGBKCLDE, etc., constitute a series of pyramids *approaching* the volume of the cone.

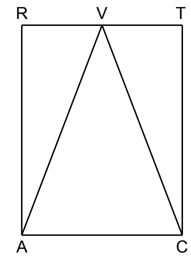
[5] But the pyramids on those polygon bases always have to the prisms on the same bases the ratio of 1 to 3 (since any pyramid is one third of the prism sharing its base and height). Therefore also the magnitude approached by the pyramids has to the magnitude approached by the prisms the ratio of 1 to 3 (Thm.3). That is, the cone is one third of the cylinder.

Q.E.D.

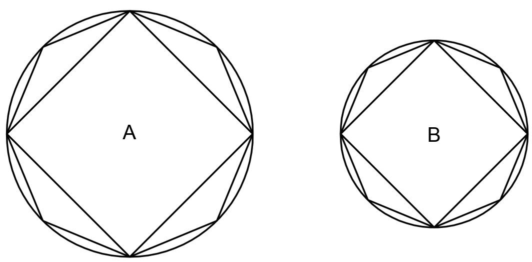
THEOREM 12 Remarks:

If we take a cross-section of the cone and cylinder through their axis, we see that the cross-section of the cone is triangle AVC, and that of the cylinder is rectangle ARTC. The cross-section of the cone is half that of the cylinder. So

Base of Cone : Base of Cylinder = 1 : 1
Cross-Section of Cone : Cross-Section of Cylinder = 1 : 2
Volume of Cone : Volume of Cylinder = 1 : 3



THEOREM 13: *Cones of the same height are to each other as their bases*.



Consider two cones of the same height, one on base A, the other on base B.

[1] Inscribe successive polygons in each as before, beginning with a square, then doubling the number of sides each time by bisecting the arcs of the circles.

[2] Since the corresponding polygons in the two circles are always similar, therefore they always have the ratio of the squares on the diameters (Thm.4).

[3] And since the pyramids on these similar polygons as bases, having the same height, are to each other as their similar bases (Thm.10), therefore they are also always in the ratio of the squares on the diameters (by Step 2).

[4] Now we saw in Theorem 12 that the series of pyramids in each cone *approaches* the volume of the cone, and therefore the cones also have the ratio of the squares on the diameters (Thm.3).

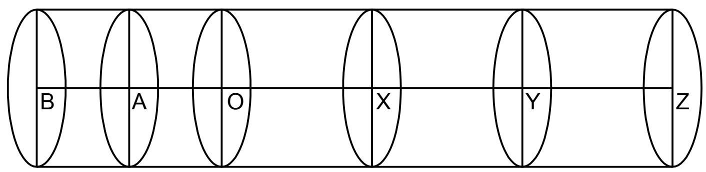
[5] But the squares on the diameters of the circles are in the same ratio as the circular bases themselves (Thm.5). Therefore the cones are in the same ratio as their bases.

Q.E.D.

THEOREM 12 Remarks:

Since each cone is a third of the cylinder standing on its base and under the same height, it likewise follows that *Cones of the same height are to each other as their bases*.

THEOREM 14: *If a cylinder is cut by a plane parallel to its bases, then the two resulting cylinders are to each other as their axes*.



Imagine a cylinder with axis AX, and suppose it is cut by a plane parallel to its bases, resulting in two cylinders, one with axis AO, another with axis OX. I say that

cylinder AO : cylinder OX = AO : OX.

[1] Take any multiple of cylinder AO (and thus of its axis), say double it, as cylinder BO.
Take any multiple of cylinder OX (and thus of its axis), say triple it, as cylinder OZ.

[2]	Clearly, if	cylinder BO	=	cylinder OZ
	then	axis BO	=	axis OZ
	and if	cylinder BO	>	cylinder OZ
	then	axis BO	>	axis OZ
	and if	cylinder BO	<	cylinder OZ
	then	axis BO	<	axis OZ

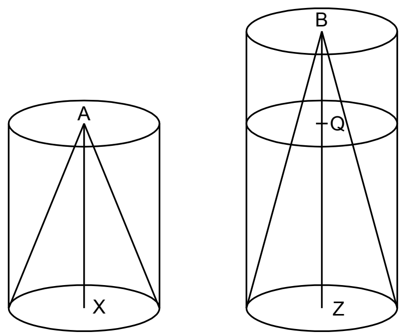
[3] Therefore, taking any multiples of cylinders AO and OX, they will always compare the same way as the corresponding multiples of their axes.

Therefore

cylinder AO : cylinder OX = AO : OX

Q.E.D.

THEOREM 15: *Cylinders on equal bases are to one another as their heights. Likewise for cones.*



Take any two cylinders on equal bases, one with axis (and thus height) AX, another with a greater axis, BZ. I say that

cylinder AX : cylinder BZ = AX : BZ.

[1] Cut off ZQ = AX.
Pass a plane through Q parallel to the bases of cylinder BZ, thus completing cylinder QZ.

[2] Since cylinder QZ has the same height as cylinder AX, and their bases are equal circles, therefore cylinder QZ is congruent to cylinder AX.

[3] Cylinder QZ : Cylinder BZ = QZ : BZ (Thm.14)

[4] But Cylinder QZ = Cylinder AX (Step 2)
and QZ = AX (Step 1)
So let's substitute these in the last proportion from step 3:

[5] Cylinder AX : Cylinder BZ = AX : BZ

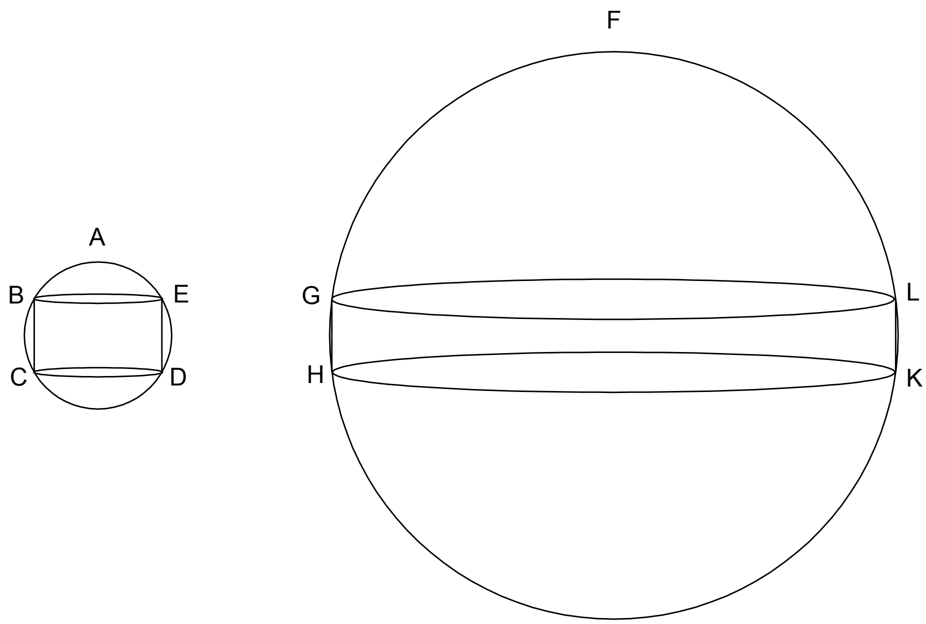
Q.E.D.

[6] As for cones, let AX and BZ be cones. Then if we complete the cylinders on their bases and with heights AX and BZ,
Cone AX : Cone BZ = Cylinder AX : Cylinder BZ
since the cones are each a third of the cylinders (Thm.12).
But cylinders are to each other as their axes (Step 5 above), and so
Cone AX : Cone BZ = AX : BZ.

Q.E.D.

“HOOK”: SPHERICAL RINGS.

Take any pair of spheres, however unequal in size. Next inscribe a right cylinder in the smaller of the two, as BCDE. Excluding the “caps” of the sphere, such as BAE, and also the cylinder itself, what remains is a sort of ring with a bulging exterior face. Next, inscribe a cylinder in the larger sphere, having the same height as that in the smaller (so that GH = BC). Thus we will have formed a “ring” in the larger sphere, too. Now the fun part: The volumes of the two “rings” are the same.



Chapter Eleven

The Five Perfect Solids

DEFINITIONS

1. A PERFECT SOLID is a convex polyhedron whose solid angles are equal and whose faces are congruent regular polygons.

2. A TETRAHEDRON is a perfect solid contained by 4 equilateral triangles.

3. A CUBE is a perfect solid contained by 6 squares.

4. An OCTAHEDRON is a perfect solid contained by 8 equilateral triangles.

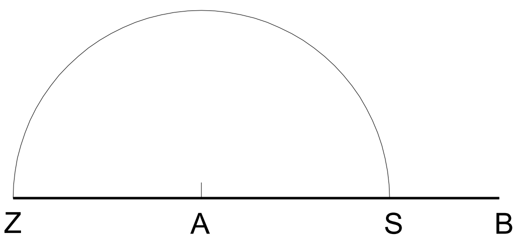
5. An ICOSAHEDRON is a perfect solid contained by 20 equilateral triangles.

6. A DODECAHEDRON is a perfect solid contained by 12 regular pentagons.

THEOREMS

In this Chapter we will make the 5 perfect solids, prove that there are only 5 of them, and learn some of their relationships and properties. We must begin with 7 preliminary Theorems of plane geometry, however, which also happen to be of some interest in themselves.

THEOREM 1: *If a line cut in mean and extreme ratio has added to it its greater segment, the result is a whole line cut in mean and extreme ratio whose greater segment is the original line.*



Let AB be a straight line cut in mean and extreme ratio at S, and let AS be its greater segment. Extend it to Z so that ZA = AS. I say that ZAB is also cut in mean and extreme ratio, and AB is its greater segment.

[1] Clearly $\square AB = AB(AS + BS)$ (AS + BS = AB)
thus $\square AB = AB \cdot AS + AB \cdot BS$ (Ch.2, Thm.1)

[2] But $AB : AS = AS : BS$ (given)
so $AB \cdot BS = \square AS$
since the square on a mean proportional line equals the rectangle contained by the extremes (Ch.6, Thm.12).

[3] Thus $\square AB = AB{\cdot}AS + \square AS$ (Steps 1 & 2)
i.e. $\square AB = AS(AB + AS)$ (Ch.2, Thm.1)
But since the side of a square is a mean proportional between the sides of any rectangle to which it is equal (Ch.6, Thm.12), thus

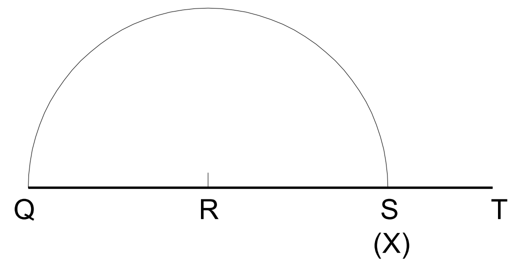
$$(AB + AS) : AB = AB : AS$$

[4] But $AS = ZA$ (we made it so)
thus $AB + ZA : AB = AB : ZA$
i.e. $BZ : AB = AB : ZA$

That is, ZAB is cut in mean and extreme ratio at A, and AB is the greater segment.

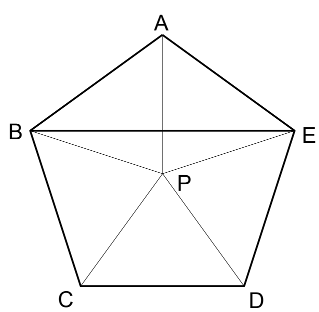
Q.E.D.

Remarks



Conversely, if QRT is cut in mean and extreme ratio at R, and RT is the greater segment, and if we subtract the lesser segment QR from the greater RT, cutting it at S, then RST will also be cut in mean and extreme ratio. That is, the original greater segment RT is now cut in the golden ratio at S, and RS, which was the original lesser segment, is the new greater segment. To see it, we can just use the Theorem above. Cut RT in the golden ratio at X so that RX is the greater segment of RT. Then if we add RX to RT, RT will be the greater segment, and RX the lesser, by the above Theorem. But RT is the greater segment and RQ is the lesser, since that is how the line is given. Therefore RX = RQ. And we cut off RS equal to RQ, so that RS = RX also. Thus RS does in fact cut RT in the golden ratio, and it is the greater segment of RT, just as RX was.

THEOREM 2: *In a regular pentagon, any diagonal is parallel to the opposite side, and the angle of the pentagon is one and one fifth of a right angle* (or 108°).



Given: ABCDE is a regular pentagon.
Prove: BE is parallel to CD, and ∠ABC = 108°.

[1] Because of the regularity of the figure, it is clear by Side-Angle-Side that
$\Delta BCD \cong \Delta EDC$ (BD & CE are not drawn)

[2] Thus these triangles are equal in area.
But they stand on the same base.
Therefore they are in the same parallels. (Ch.1, Thm.33, Remark 3)
Therefore BE is parallel to CD.

[3] Let P be the center of the circumscribed circle. Thus the 5 triangles such as APB are congruent (Side-Side-Side).

[4] Now the total angle-sum of these triangles is 10 right angles (i.e. 2 rights for each triangle times 5, the number of triangles). If we subtract the angles around P, i.e. 4 right angles, the remainder is the sum of the pentagon's angles, namely 6 right angles.

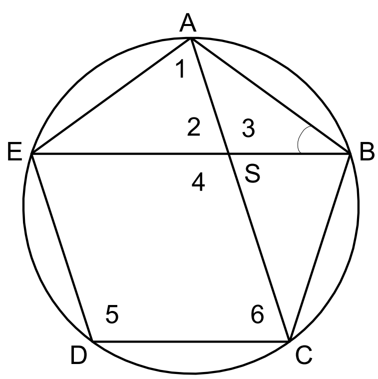
Since the 5 angles of the pentagon are all equal, each one is a fifth of 6 right angles, or one and one fifth of a right angle.

[5] Since a right angle is 90°, therefore a fifth of a right angle is 18°, and so one right angle and a fifth is 108°.

Thus the angle of a regular pentagon is 108°.

Q.E.D.

THEOREM 3: *In a regular pentagon, if two diagonals cut each other, they cut each other in mean and extreme ratio, and their greater segments equal the side of the pentagon.*



Given: Regular pentagon ABCDE. Diagonals AC and EB cut each other at S.

Prove: AC and EB are cut in the golden ratio at S, and ES = EA.

First, circumscribe a circle about ABCDE (Ch.4, Thm.7 Remarks).

[1] Now, by the previous Theorem, EB ∥ DC and AC ∥ ED, so that DESC is a parallelogram. Thus

	$\angle 5 = \angle 4$	(opp. angles in a parallelogram)
but	$\underline{\angle 3 = \angle 4}$	(vertical angles)
so	$\angle 5 = \angle 3$	
or	$\angle EAB = \angle 3$	($\angle 5 = \angle EAB$ since the pentagon is regular)

[2] Notice ∠ABS is common to △EAB and △ABS. Also, ∠EAB = ∠3.
So two angles in △EAB are equal to two angles in △ABS.
Therefore △EAB is similar to △ABS (Ch.6 Thm.4),
i.e. EB : BA = BA : BS

[3] Now since ∠1 and ∠6 are each at the circumference, standing on two of the 5 equal arcs, therefore

	$\angle 1 = \angle 6$	(Ch.3, Thm.21)
but	$\underline{\angle 2 = \angle 6}$	(since ES $\parallel$ DC)
so	$\angle 1 = \angle 2$	(making $\triangle$AES isosceles)
thus	AE = ES	

[4]	So	AE = ES	(Step 3)
	but	$\underline{\text{AE = BA}}$	(being sides of the regular pentagon)
	so	BA = ES	

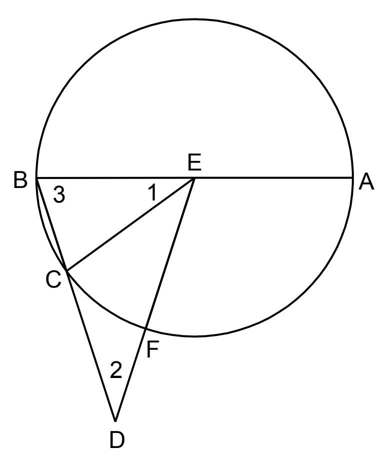
Therefore, substituting ES for BA in our proportion from Step 2, we have

EB : ES = ES : BS

That is, ESB is cut in mean and extreme ratio at S (Ch.6 Def.9), and ES is the greater segment. And we saw in Step 3 that ES is equal to EA, the side of the pentagon. Thus it is clear that in a regular pentagon, the side has to the diagonal the golden ratio.

Q.E.D.

THEOREM 4: *The sides of the hexagon and decagon inscribed in the same circle, when added together, make a whole straight line cut in mean and extreme ratio.*



Given: Straight line BCD, with BC being a side of the regular decagon in circle ABC, CD being equal to the side of the regular hexagon in circle ABC (and thus it is equal to the radius, so that EC = CD, Ch.4 Thm.8).

Prove: BCD is cut in mean and extreme ratio at C.

Take the center of the circle, E (Ch.3, Thm.1). Join EC, join EFD.

[1] Since ∠BEC is one central angle of the regular decagon, of which there are 10 in the whole decagon, and 5 in each semicircle, therefore ∠CEA contains 4 such angles.

Thus 4∠BEC = ∠CEA
but <u>2∠ECB = ∠CEA</u> (∠CEA is exterior to isosceles triangle ECB)
so 2∠ECB = 4∠BEC

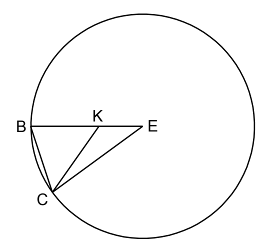
[2] Thus ∠ECB = 2∠BEC (the halves of equals are equal)
but <u>∠ECB = 2∠CDE</u> (∠ECB is exterior to isosceles triangle ECD)
thus 2∠BEC = 2∠CDE

[3] So ∠BEC = ∠CDE (the halves of equals are equal)
i.e. ∠1 = ∠2
but <u>∠3 is common to ΔBEC and ΔEDB</u>
thus ΔBEC is similar to ΔEDB (Ch.6, Thm.4)

[4] So DB : BE = BE : BC (ΔBEC is similar to ΔEDB)
i.e. DB : CD = CD : BC (CD is equal to the radius BE)
i.e. BCD is cut in mean and extreme ratio at C, and CD, the side of the hexagon, is the greater segment.

Q.E.D.

Remarks

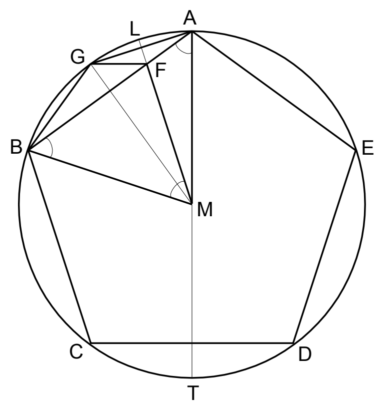


1. Since BC and CD are related as lesser and greater segments of a line cut in mean and extreme ratio, so too are BC and BE (since BE = CD). Therefore if we subtract from the greater segment BE a part equal to the lesser segment BC (say BK), then BKE will also be cut in the golden ratio (by the Remark after the first Theorem of this Chapter), and BK, equal to BC (the side of the decagon), will be the greater segment. *Thus in any circle, the greater segment of the radius, when it is cut in mean and extreme ratio, is equal to the side of the inscribed regular decagon*.

2. Looking back at the diagram for the Theorem, one can see that CE bisects angle BED as follows.

CD = CE (given)
so ∠CED = ∠2 (since △DCE is isosceles)
but <u>∠1 = ∠2</u> (Step 3 in the proof of the Theorem)
thus ∠CED = ∠1
i.e. CE bisects ∠BED, and thus CF is also an arc cut off by a side of the decagon, and BC = CF.

THEOREM 5: *In one circle, the square on the side of the regular hexagon plus that on the side of the decagon equals the square on the side of the pentagon.*



Let ABCDE be a regular pentagon in a circle with center M. Bisect arcs AB, CD at G, T. Thus we get arcs of the decagon, and AG is a side of the decagon. Bisect arc AG at L, join ML, cutting AB at F. Join GB. Now I say that the square on the side of pentagon ABCDE is equal to the square on the regular hexagon in circle M, plus the square on the side of the regular decagon, that is

$$\square AB = \square BM + \square AG$$

[1] $\angle BMF$ (or $\angle BML$) is at the *center* and stands on 1 ½ decagon arcs. Thus it is double an angle at the *circumference* standing on that same arc (Ch.3, Thm.20), and equal to an angle at the circumference standing on double that arc, i.e. standing on 1 ½ pentagon arcs. But $\angle BAM$ (or $\angle BAT$) stands on 1 ½ pentagon arcs, and thus

$$\angle BMF = \angle BAM.$$

[2]	But	$\angle ABM = \angle BAM$	(since BM = MA)
	thus	$\triangle ABM$ is similar to $\triangle BMF$	
	so that	$AB : BM = BM : BF$	(Ch.6, Thm.4)

And since the square on a mean proportional line is equal to the rectangle contained by the extremes (Ch.6, Thm.12), therefore

$$AB \cdot BF = \square BM$$

[3]	Now	$\triangle GMF \cong \triangle AMF$	(Side-Angle-Side)
	so	$GF = AF$	
	thus	$\angle AGF = \angle GAF$	
	but	$\angle GAF = \angle GBA$	(since AG = GB in $\triangle AGB$)
	so	$\triangle AGF$ is similar to $\triangle AGB$	(Ch.6, Thm.4)
	thus	$AB : AG = AG : AF$	
	thus	$AB \cdot AF = \square AG$	(Ch.6, Thm.12)

[4] Now $\square AB = AB \cdot BF + AB \cdot AF$ (Ch.2, Thm.1)

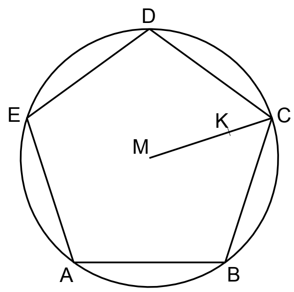
So, replacing these rectangles with the squares shown equal to them in Steps 2 and 3,

$$\square AB = \square BM + \square AG$$

Q.E.D.

Remarks

1. From this fact, by the converse of the Pythagorean Theorem, it follows that BM and AG, if placed at a right angle to each other, will form a right triangle whose hypotenuse is equal to AB.



2. So if we cut the radius MC of a circle in mean and extreme ratio at K, MK being the greater segment, and if ABCDE is a regular pentagon inscribed in the circle, then

MK = the side of the regular decagon in the circle (Thm.4, Remarks)

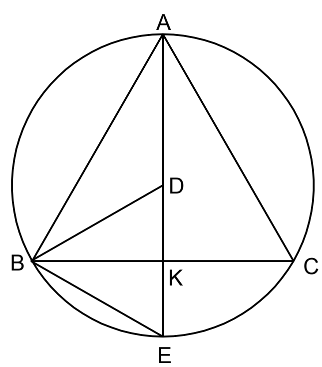
MC = the side of the regular hexagon

and ☐MK + ☐MC = ☐AB (by the present Theorem).

3. Also, since in one circle the side of the decagon is to the side of the hexagon in mean and extreme ratio, i.e. as the greater segment is to the whole (Thm.4, Remarks), and since the side of a regular pentagon is to its diagonal in mean and extreme ratio, i.e. as the greater segment is to the whole (Thm.3), therefore, in a single circle,

Side of decagon : Side of Hexagon = Side of Pentagon : Diagonal of Pentagon.

THEOREM 6: *If an equilateral triangle is inscribed in a circle, the square on its side is three times the square on the radius.*



Imagine an equilateral triangle ABC in a circle with center D and radius DA.
I say that ☐AB = 3 ☐DA.

Extend AD to E. Join BE.

[1] Since arc BEC is one third of the circumference, thus arc BE is one sixth of the circumference, so BE is the side of the hexagon, and BE = DA.

[2] Since AE is a diameter, thus ∠ABE is right (Ch.3, Thm.25), and so

	☐AB = ☐AE – ☐BE	(Pythagorean Theorem)
thus	☐AB = 4☐DA – ☐BE	(AE is bisected at D, so ☐AE = 4☐DA)
thus	☐AB = 4☐DA – ☐DA	(BE = DA, Step 1)

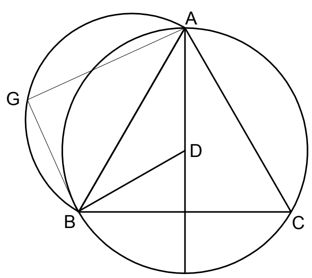
[3] i.e. ☐AB = 3 ☐DA.

Q.E.D.

Remarks

1. Notice that DE is bisected at K. Why?

Because BE = the radius of the circle (Step 1 in the proof above).
Thus BE = BD = DE.
Thus ΔBDE is an equilateral triangle.
Likewise ΔCDE is an equilateral triangle.
Thus ΔBDC ≅ ΔBEC by Side-Side-Side.
Thus ∠DBC = ∠EBC.
Thus ΔDBK ≅ ΔEBK by Side-Angle-Side.
Thus DK = KE.



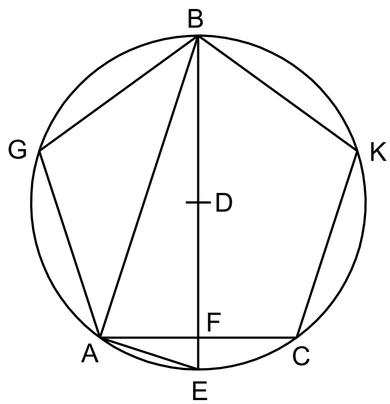
2. It is impossible to make a 1-2-3 triangle in terms of the lengths of the sides, since the sides of lengths 1 and 2 would not, added together, be greater than the side of length 3 (see Ch.1 Thm.17). But it *is* possible to make a triangle the *squares* on whose sides have the ratios of 1, 2, 3. How? Let equilateral ΔABC be inscribed in a circle with radius BD. On AB as diameter, describe a semicircle, and place BG in it equal to BD. Join AG. Thus ∠AGB is a right angle.

So □BG + □AG = □AB
But □AB = 3 □BD (by the present Theorem)
So □BG + □AG = 3 □BD
Thus □AG = 3 □BD – □BG
i.e. □AG = 2 □BG (since □BD = □BG)
but □AB = 3 □BG (since BG = radius BD).

So if we call □BG "1" square unit of area, then we must call □AG "2," and we must call □AB "3."

Notice, also, that since □AG = 2 □BG, it follows that AG is equal to the diagonal of the square on BG. So we could have made ΔAGB that way, namely by placing the side and diagonal of a square at right angles to each other, and joining the hypotenuse. But now we know something else about this 1-2-3 triangle, namely that if its hypotenuse is equal to the side of an equilateral triangle inscribed in a circle, then its shorter leg is equal to the radius of that circle.

THEOREM 7: *The square on the diagonal of a regular pentagon, plus the square on its side, equals five times the square on the radius of the circumscribing circle.*



Take a regular pentagon AGBKC, with AB one of its diagonals, and let D be the center of the circumscribing circle. I say that □BA + □AC = 5□BD.

[1] Join BD, extend it to E. Since AGBKC is a regular pentagon, thus diameter BDE bisects AC at F, and so DF is perpendicular to AC (Ch.3, Thm.3).

Join EA. Thus EA is the side of the decagon.

[2] Now BE = 2DE
so □BE = 4□DE

[3] But □BE = □BA + □AE (∠BAE is right)
so □BE + □DE = □BA + □AE + □DE (+ □DE on both sides)

[4] But □BE = 4□DE (Step 2)
so 4□DE + □DE = □BA + □AE + □DE
i.e. 5□DE = □BA + **□AE + □DE**

[5] But the square on the side of the decagon plus that on the side of the hexagon equals the square on the side of the pentagon,
i.e. **□AE + □DE** = □AC (Thm.5)

[6] So 5□DE = □BA + □AC (Steps 4 & 5)

or □BA + □AC = 5□BD (DE = BD)

Q.E.D.

Remarks

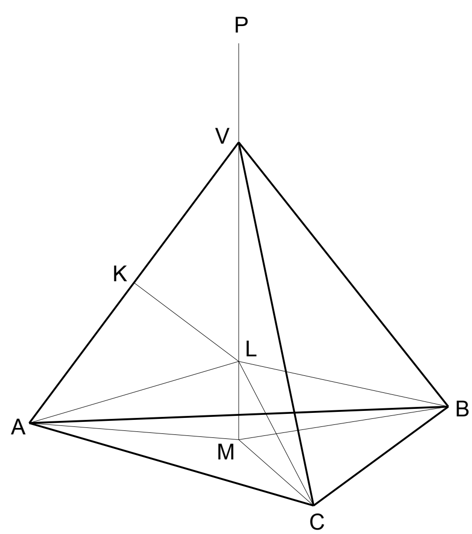
For the sake of brevity, let's use some shorthand in this remark:

R = the radius of a circumscribing circle.
E = the side of the equilateral triangle inscribed in it
P = the side of the regular pentagon inscribed in it
D = the diagonal of the regular pentagon inscribed in it

(a)	Now	$\square E = 3 \ \square R$	(Thm.6)
	so	$5 \ \square E = 15 \ \square R$	(multiplying both sides by 5)
(b)	But	$\square P + \square D = 5 \ \square R$	(the present Theorem)
	so	$3 \ \square P + 3 \ \square D = 15 \ \square R$	(multiplying both sides by 3)
(c)	Thus	$3 \ \square P + 3 \ \square D = 5 \ \square E$	(Steps a & b)

Or, spelled out in words, *Three times the square on the side of the regular pentagon plus three times the square on its diagonal is equal to 5 times the square on the side of the equilateral triangle inscribed in the same circle*.

THEOREM 8: *How to construct a tetrahedron, and contain it in a sphere.*



[1] Draw an equilateral triangle ABC in the base plane. Circumscribe a circle around it (Ch.4, Thm.4). Take its center M. Join MA, MB, MC.

[2] Set up MP perpendicular to the base plane.

Thus angles PMA, PMB, PMC are all right angles.

[3] In the plane PMA, using A as your center and AB as your radius, draw a circle, cutting MP at V. (Since AB is greater than AM, your radius *will* hit MP at some point V above M).

So VA = AB.

[4] Join VB, VC.

Since	∠VMA = ∠VMB	(both are right)
and	MA = MB	(both are radii of the base circle around ABC)
and	<u>VM is common</u>	
thus	ΔVMA ≅ ΔVMB	(Side Angle Side)
so	VA = VB	

[5] Likewise we can show that VA = VC.

So	VA = AB	(Step 3)
and	VA = VB = VC	(Step 4)
so	the lines VA, VB, VC, AB, BC, CA are all equal to each other.	

So the 4 triangular faces of the pyramid are actually 4 congruent equilateral triangles, and therefore we have made a tetrahedron. Now to contain it in a sphere ...

[6] Bisect AV at K.

In plane AVM, draw KL perpendicular to AV, hitting VM at L.

Thus	ΔLVK ≅ ΔLAK	(Side Angle Side)
so	LV = LA.	

[7]

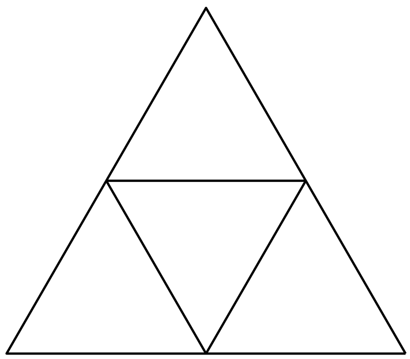
But	∠LMA = ∠LMC	(both are right)
and	MA = MC	(both are radii of the base circle around ABC)
and	<u>LM is common</u>	
thus	ΔLAM ≅ ΔLCM	(Side Angle Side)
so	LA = LC	
and	LC = LB likewise.	

[8] Therefore LV = LA = LB = LC (Steps 6 – 7). So L is the center of the sphere containing the tetrahedron, and VL is the radius of it.

Q.E.F.

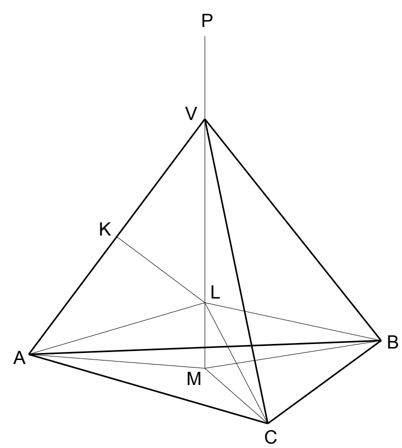
Remarks

1. If it wasn't clear by itself, what I mean by “the sphere containing the tetrahedron” is the sphere whose surface passes through the 4 corners of the tetrahedron.



2. To get a good look at this thing, it's an excellent idea to *make* it in three dimensions. A good material for three dimensional models is the manila that file-folders are made out of. Reproduce the accompanying diagram of an equilateral triangle divided into four equilateral triangles onto a piece of manila. Make your version bigger than the little diagram. Cut it out, and then fold along the three sides of the middle equilateral triangle. Raise up the three outer equilateral triangles all to one point, tape the edges together and … you have a tetrahedron.

THEOREM 9: *If a tetrahedron is contained in a sphere, the square on the diameter is one and a half times the square on the tetrahedron’s side.*



Begin with the same construction as before, in Theorem 8.

[1] Now since both are right,
$\angle VKL = \angle VMA$
and <u>$\angle AVM$ is common</u>
so $\triangle VKL$ is similar to $\triangle VMA$
so $VL : VK = VA : VM$
thus $\square VL : \square VK = \square VA : \square VM$

[2] But VA is the side of an equilateral triangle, and the radius of the circle around it is equal to MA. Therefore

$\square VA = 3\square MA$ (Thm.6)

[3] Now $\square VA - \square MA = \square VM$ (Pythagorean Theorem)
so $3\square MA - \square MA = \square VM$ ($\square VA = 3\square MA$, Step 2)
or $2\square MA = \square VM$

[4] Since $\square VA = 3\square MA$ (Step 2)
and $\square VM = 2\square MA$ (Step 3)
thus $\square VA : \square VM = 3 : 2$

[5] Thus $\square VL : \square VK = 3 : 2$ (Steps 1 & 4)
so $4\square VL : 4\square VK = 3 : 2$
i.e. the square on 2VL (the diameter of the containing sphere) is to the square on 2VK (the side of the tetrahedron) as 3 to 2.

Therefore the square on the diameter of the containing sphere is one and a half times the square on the side of the tetrahedron.

Q.E.D.

Remarks

In Step 1 we assumed that if four straight lines are proportional, then the squares on them are also proportional. Let's prove that. Suppose A, B, C, D are four straight lines, and

$A : B = C : D$

Then I say that $\square A : \square B = \square C : \square D$.

(a) First, make a 3rd proportional straight line, X, to A & B, so that
$A : B = B : X$. (Ch.6, Thm.9)
Thus A : X is the ratio double of A to B, and therefore
$\square A : \square B = A : X$ (Ch.6, Thm.15).
Again, make a 3rd proportional straight line, Z, to C & D, so that
$C : D = D : Z$ (Ch.6, Thm.9)
Thus $\square C : \square D = C : Z$ (Ch.6, Thm.15).

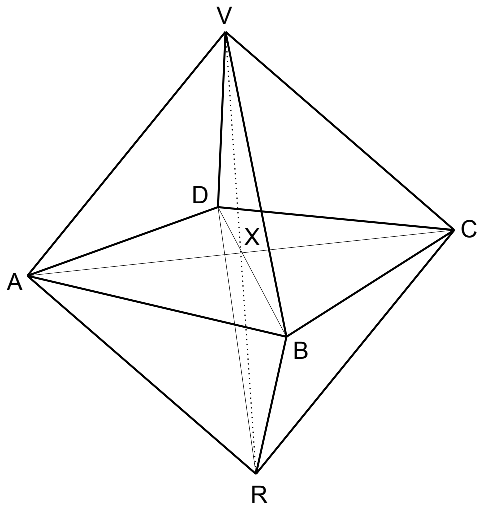
(b) Now $A : B = C : D$ (given)
but $\underline{A : B = B : X}$ (Step *a*)
so $C : D = B : X$

(c) Now $C : D = B : X$ (Step *b*)
but $\underline{C : D = D : Z}$ (Step *a*)
so $B : X = D : Z$

(d) Now $B : X = D : Z$ (Step *c*)
but $\underline{B : A = D : C}$ (inverse ratios of given proportion)
so $A : X = C : Z$ (Ch.5, Thm.16)
thus $\square A : \square B = \square C : \square D$,
since these squares are as $A : X$ and as $C : Z$ by Step (a).

Q.E.D.

THEOREM 10: *How to construct an octahedron, and prove the square on the diameter of the containing sphere is two times the square on the side of the octahedron.*



[1] Make a square ABCD.
Join AC and BD, making X the center of the circle circumscribing the square.

[2] Set up XV perpendicular to the plane of ABCD, cutting off XV = XA.

[3]	So	∠VXA = ∠AXB	(both are right)
	and	VX = XA	(we made it so in Step 2)
	and	XA = XB	(Step 1)
	so	ΔVXA ≅ ΔAXB	(SAS)
	so	VA = AB	

[4] Likewise VB, VC, VD are all equal to AB.
Therefore the four faces of the pyramid are equal equilateral triangles.
So, if we make another pyramid R just like pyramid V on the *other* side of base ABCD, we will have a solid VR contained by 8 equal equilateral triangles, having 6 solid angles each made of 4 plane angles of 60°. Thus we have our octahedron.

[5] And since VX = XA = XR, therefore X is the center of the sphere containing it, and AC is the diameter.

[6] And since AC is the diagonal of square ABCD, therefore

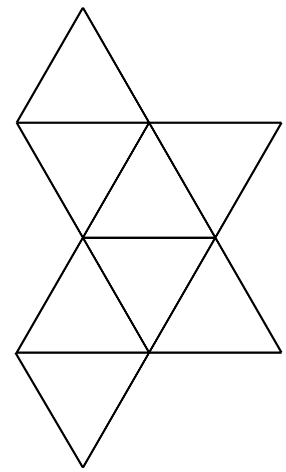
☐AC = 2☐AB

and so the square on the diameter of the containing sphere is two times the square on the side of the octahedron.

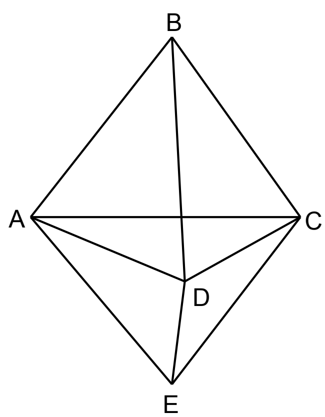
Q.E.D.

Remarks

1. You can make this solid figure out of manila using the accompanying pattern.

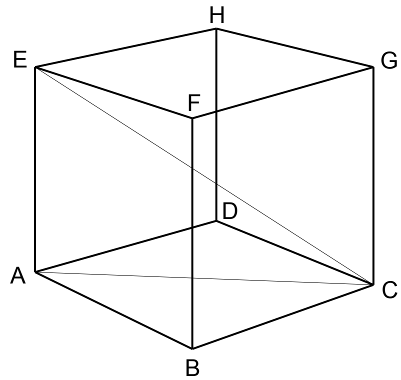


2. Notice that the octahedron is just two identical pyramids joined at their common square base.



3. If we put two tetrahedrons together, would we get a new perfect solid? It would be a solid contained by 6 equal equilateral triangles, as in figure ABCDE, but not all 5 of its solid angles would be identical. Compare the solid angle at B to the one at A, for example. The solid angle at B is made of 3 faces (∠ABD, ∠DBC, ∠CBA), and the one at A is made of 4 faces (∠EAC, ∠CAB, ∠BAD, ∠DAE). Not only that, but the angle at A is "squished," since ∠BAE is much greater than ∠CAD, making solid angle A narrower in one direction and wider in another. Thus it would not be possible for a solid, made with angle A, to be of uniform convexity. Thus solid A is different from the angle of the octahedron, because each angle of the octahedron stands on a square, and (in the figure for Thm.10) ∠DVB = ∠AVC, since both are right angles (which is easily seen since all triangles such as AXV are right isosceles). Hence the solid angle of the octahedron is uniform.

THEOREM 11: *How to construct a cube, and prove the square on the containing sphere's diameter is three times the square face of the cube.*



[1] Make a square ABCD in the base plane.

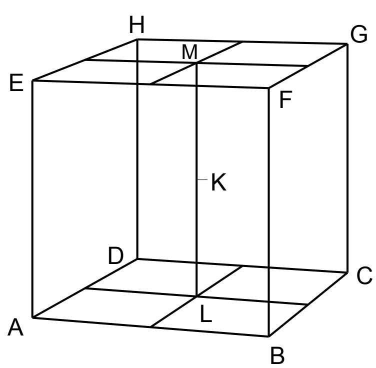
[2] Set up BF perpendicular to the plane ABCD (Ch.9, Thm.7).

Complete the squares ABFE and BCGF (Ch.1, Thm.35).

Drawing parallels, complete the parallelepiped contained by these 3 squares, and you have a cube.

[3] Join EC, CA.

[4]	Now	EC = AG	(being the diagonals of rectangle EACG)
	and	AG = BH	(being the diagonals of rectangle ABGH)
	and	BH = DF	(being the diagonals of rectangle BDHF)
	Thus	all four diagonals of the cube are equal to each other, namely EC = AG = BH = DF.	



[5] Now bisect EF & AB, HG & DC, and pass a plane through these midpoints.

Bisect EH & FG, AD & BC, and pass a plane through these midpoints.

Call the intersection of these 2 planes ML. Each of the 4 diagonals of the cube is bisected by K, the midpoint of ML (Ch.9, Thm.27). Therefore the 4 diagonals of the cube all pass through K, and K is equidistant from the 8 corners of the cube.

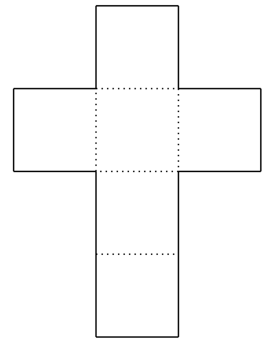
Therefore K is the center of the containing sphere, and EC is its diameter.

[6] In right triangle EAC,

		☐EC = ☐EA + ☐AC	(Pythagorean Theorem)
	but	☐AC = 2☐AB	(since AC is the diagonal of ☐AB)
	i.e.	<u>☐AC = 2☐EA</u>	(since AB = EA)
	thus	☐EC = ☐EA + 2☐EA	
	i.e.	☐EC = 3☐EA	

[7] So the square on the sphere's diameter is three times the square on the side of the cube, which is one face of the cube. *Q.E.D.*

Remarks



To make a model of a cube is easy enough. As with the tetrahedron, copy the accompanying pattern of squares, cut it out, and fold up along the dotted lines, joining edges with tape. Now you have a cube.

THEOREM 12: *How to make an icosahedron*.

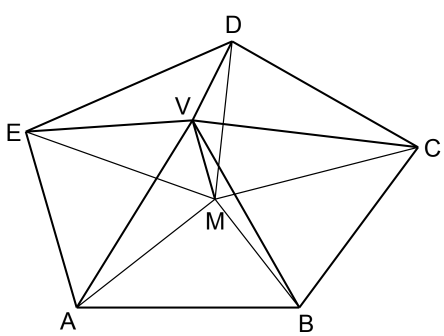
The icosahedron is a bit more complicated, so we are going to build it in separate stages.

STAGE ONE: we will build the top and bottom "caps" of the icosahedron, each being a pyramid built on a regular pentagon as base with 5 equilateral triangles as walls.

STAGE TWO: we will build the midsection or "drum" of the icosahedron, contained by two pentagons and 10 equilateral triangles.

STAGE THREE: we will put these 3 parts together, assembling our icosahedron.

STAGE ONE: The "caps" of the icosahedron



[1] Draw a circle with center M, inscribe regular pentagon ABCDE (Ch.4, Thm.7).

Draw MV perpendicular to the plane of the pentagon.

Cut off MV equal to the side of the *decagon* (Ch.4, Thm.9) inscribed in the circle circumscribing pentagon ABCDE. Of course, we already know that MA, being the radius of that circle, is equal to the side of the hexagon inscribed in it.

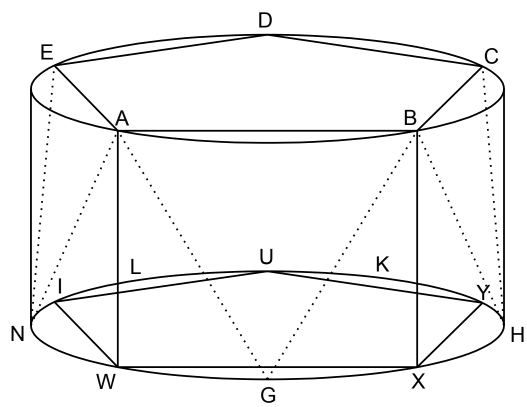
[2] Since MA is the side of the hexagon, and VM is the side of the decagon, and since MA and VM are drawn at right angles to each other, it follows that their hypotenuse, VA, is equal to the side of the *pentagon* in the circle, namely AB (this Chapter, Thm.5).

So VA = AB.

[3] But MB, MC, MD, ME are also sides of the hexagon (i.e. radii of the same circle), and therefore the hypotenuses VB, VC, VD, VE are also all equal to AB, the side of the pentagon.

[4] Therefore ABV, BCV, CDV, DEV, EAV are all equal equilateral triangles (SSS). And that is one of the "caps" to the icosahedron. We will need one for the top, and one for the bottom.

STAGE TWO: The "drum" of the icosahedron.



[5] Using the circle circumscribing ABCDE, and using the radius of that circle (MA) as a height, complete a *cylinder* with circle ABCDE as the top of it.

[6] In the base circle of this cylinder, first draw a pentagon WXYUI identical to ABCDE and with each of its sides parallel to each of the sides of ABCDE. Thus ABXW is a rectangle, and AW and BX lie on the cylinder's surface, parallel and equal to its axis, and perpendicular to the circles of the cylinder.

[7] Bisect arc WX at G.

Thus GX = side of the *decagon* in the base circle.

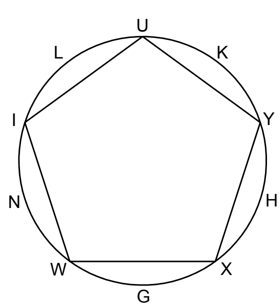
But BX = height of the cylinder = radius of the base circle (Step 5)

i.e. BX = side of the *hexagon* in the base circle.

But BX is perpendicular to GX, since BX is perpendicular to the whole base plane (Step 6; see Ch.9, Thm.3). And therefore BXG is a right triangle, whose legs BX and GX are equal, respectively, to the side of the *hexagon* in the base circle, and the side of the *decagon* in the base circle. Therefore the hypotenuse BG is equal to the side of the *pentagon* in the base circle (this Chapter, Thm.5).

i.e. BG = AB.

Likewise AG = AB. Thus ABG is an equilateral triangle.

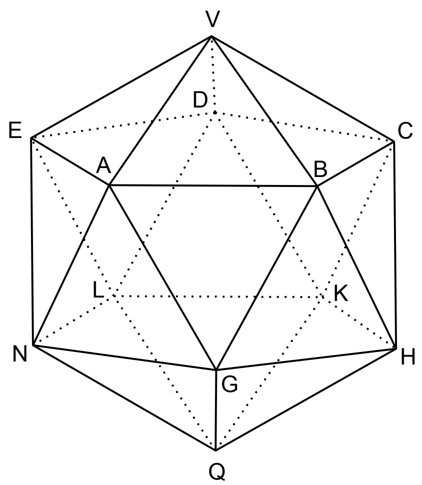


[8] Now, bisect arc XY at H, arc YU at K, arc UI at L, arc IW at N. Obviously GHKLN is another regular pentagon, with its sides equal to WX or AB. It will also follow, just as in Step 7, that

ΔBGH, ΔBHC, ΔCHK, ΔCKD, ΔDKL, ΔDLE, ΔELN, ΔENA, ΔANG are all equilateral triangles, each having sides equal to AB.

[9] Thus we have a solid contained by two identical regular pentagons, namely ABCDE and GHKLN, and by 10 equilateral triangles. The pentagons are in parallel planes, but the sides of the bottom pentagon are not parallel to those of the top one (but rather to its diagonals).

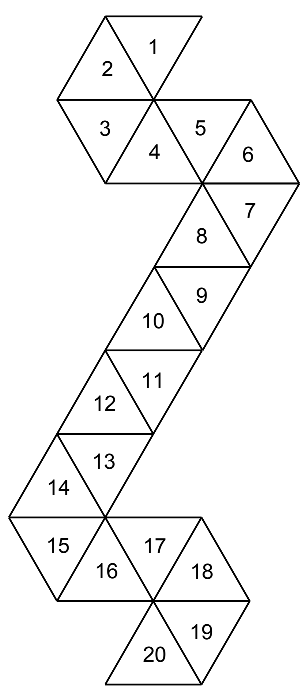
STAGE THREE: Assembly of the icosahedron.



[10] We have made a "cap" in Stage One, and added beneath it a "drum" in Stage Two. Now just add another "cap" identical to the one made in Stage One *underneath* the drum, with vertex Q.

Thus we have a complete solid contained by 20 equal and equilateral triangles, all having a side equal to AB, the side of the pentagon with which we began. And every solid angle on it is formed by 5 plane angles, each of 60° (the angle of the equilateral triangle). Thus the icosahedron is made.

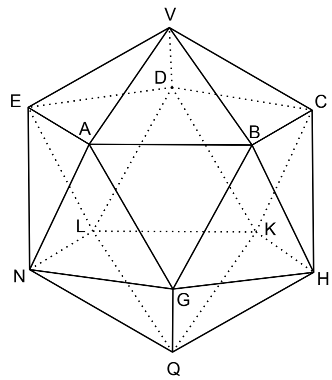
Q.E.F.



Remarks

The accompanying pattern can be used to make an icosahedron. Draw your own larger version of it. Then cut it out, and fold along the lines. Tip: triangles 1 through 5 will make one "cap," just as triangles 16 through 20 will make the other "cap." Triangles 6 through 15 will make the "drum."

THEOREM 13: *How to contain an icosahedron in a sphere.*

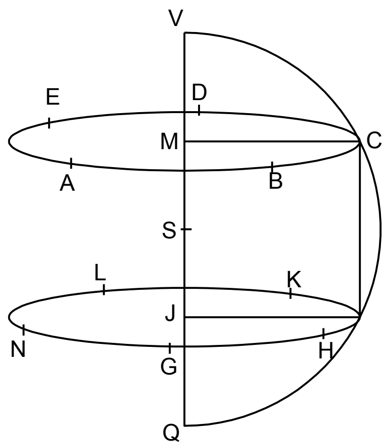


Now let's see how to wrap a sphere around our icosahedron. First "imagine away" all the edges of it, retaining just the 12 vertices ABCDE and GHKLN and V and Q. That simplifies things!

[1] Now, join VQ and bisect it at S.

I say that point S is the same distance from every vertex of the icosahedron.

Choose any vertex, C. I say SC = SV.



[2] Clearly, if M is the center of circle ABCDE, and J is the center of circle GHKLN, then MJ is the axis of the cylinder, and VQ lies in line with it. And because our "caps" are of identical heights,

thus VJ = MQ.

[3] Also, VQ, being in line with the axis of the cylinder, is perpendicular to the planes of the circles, and therefore to every line in them that it cuts. Thus VQ is at right angles to MC.

[4]	Since	VM = side of *decagon* in circle ABCDE (made thus in Step 1 of Thm.12)	
	and	MJ = side of *hexagon* in circle ABCDE (made thus in Step 5 of Thm.12)	
	thus	VMJ is cut in mean and extreme ratio at M (Thm.4 of this Chapter)	
	i.e.	VJ : JM = JM : MV	
[5]	i.e.	MQ : JM = JM : MV	(VJ = MQ, Step 2)
	but	JM = MC	(height of cylinder = radius)
	so	MQ : MC = MC : MV	

[6] So MC is a mean proportional between MQ and MV.

But MC is also at right angles to VQ (Step 3).

Therefore V, C, Q all lie on the circle having diameter VQ and thus center S (by Chapter 6 Thm.10). Therefore

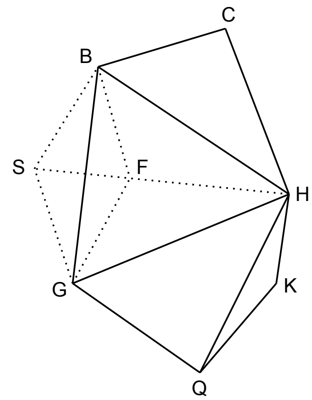
SC = SV

Similarly we can prove that SB, SH, SL etc. are all equal to SV.

Therefore S is equidistant from every vertex of the icosahedron, and therefore it is the center of the sphere containing the icosahedron.

Q.E.F.

Remarks



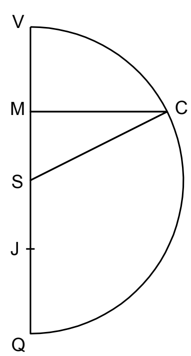
1. Containing the icosahedron in a sphere helps us to see its uniform convexity. It is also helpful, in understanding the figure's radial symmetry, to notice that *every* vertex in it can be regarded as the top of a "cap" with a regular pentagon as base. e.g. H can be seen as the top of a "cap," QGBCK being a regular pentagon. Let's prove this quickly. Since S is the center of the sphere, SB = SG, and since they are sides of the icosahedron, BH = GH, and SH is common. Thus $\triangle SBH \cong \triangle SGH$. So if we drop a perpendicular to SH from B, and another from G, it is clear they will hit SH at the same point, call it F. Likewise the perpendiculars to SH drawn from C, K, and Q will all land on F. But all the perpendiculars to SH through F lie in one plane (Ch.9, Thm.3 Remarks). Therefore Q, G, B, C, K all lie in one plane together, and thus define a pentagon. That it is equilateral is already clear, since QG etc. are all edges of the icosahedron. And since a plane through a sphere cuts out a circle, therefore the plane common to Q, G, B, C, K cuts out a circle on the surface of the sphere containing the icosahedron, and therefore these 5 points all lie on one circle. Now the equal sides of pentagon QGBCK each cut off equal arcs of this circle (Ch.3, Thm.23), making it clear that QGBCK is also equiangular.

2. Notice that the diameter of the containing sphere is VQ,

and $\quad VQ = VM + JQ + MJ$

or $\quad VQ = 2VM + MJ \qquad\qquad (VM = JQ)$

i.e. the diameter of the containing sphere is equal to twice the side of the *decagon* in pentagon ABCDE plus the side of the *hexagon*.



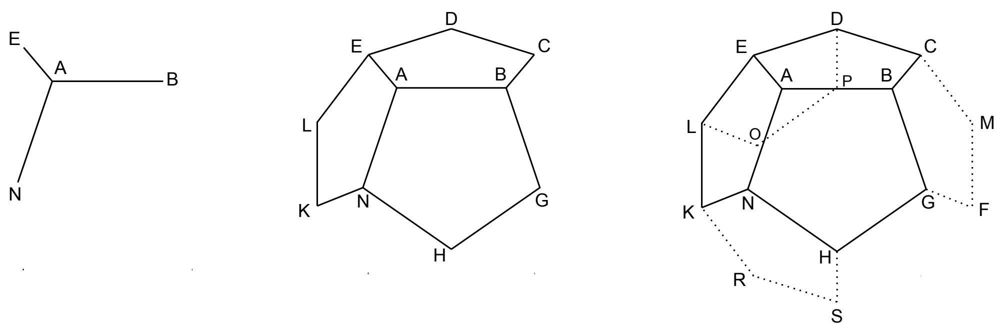
3. Let's compare the radius of the circle around the icosahedron's pentagon (MC) to the diameter of the containing sphere (VQ). Recall, first, that we made the height of the icosahedron's midsection equal to the radius of the circle around the pentagon, i.e. MJ = MC. And since S bisects MJ, it follows that MS is *half* of MC.

Now $\square SC = \square MS + \square MC$ (Pythagorean Theorem)
so $\square SC = \square(\tfrac{1}{2} MC) + \square MC$ (MS is equal to half of MC)
so $\square SC = \tfrac{1}{4} \square MC + \square MC$,
since the square on half of MC is one quarter of the square on the whole of MC.

Thus $\square SC = \dfrac{5}{4} \square MC$ (adding)
or $4\ \square SC = 5\ \square MC$ (multiplying both sides by 4)
i.e. $4\ \square SV = 5\ \square MC$ (SV = SC, being radii of the sphere)
or $\square VQ = 5\ \square MC$ (since VQ = 2SV)

So the square on the diameter of the containing sphere is equal to five times the square on the radius of the circle around the icosahedron's pentagon.

THEOREM 14: *How to make a dodecahedron.*



[1] Since the angle of a regular pentagon is 108° (Thm.2), therefore 3 of these is still less than 4 right angles (i.e. less than 360°). So we can make a solid angle out of three of them (Ch.9, Thm.20). Let it be done: ∠EAB, ∠BAN, ∠NAE are each 108°, the angle of a regular pentagon, and they contain a solid angle at A.

[2] Cut the legs off equally, and complete the regular pentagons in each of our 3 angles, namely ABCDE, ABGHN, ANKLE (Ch.4, Thm.7 Remarks)

[3] Now EC is parallel to NG (since each is parallel to AB by Thm.2)
but <u>EC = NG</u> (being diagonals in congruent pentagons)
Thus the lines joining their endpoints are also equal and parallel (Ch.1, Thm.30),
i.e. EN and CG are equal and parallel.

[4] Thus $\triangle GBC \cong \triangle NAE$ (Side-Side-Side)
and so ∠GBC = 108°.
By the same reasoning again, we can show that
∠KNH = 108°
and ∠LED = 108°

[5] Complete new pentagons in these angles, namely GBCMF, KNHSR, LEDPO.

[6] Now I say that points S, H, G, F are all in one plane.

For, by reasoning similar to that in Step 3,

SG ∥ LE (since SG ∥ KA, and KA ∥ LE)

and GH ∥ LD (since GH ∥ NB, and NB ∥ LD)

Therefore the plane of intersecting lines SG and GH is parallel to the plane of intersecting lines LE and LD (Ch.9, Thm.11). That is, plane SGH is parallel to plane LED.

Again, reasoning in the same way,

FH ∥ DE (since FH ∥ CA, and CA ∥ DE)

and GH ∥ LD (already shown just above)

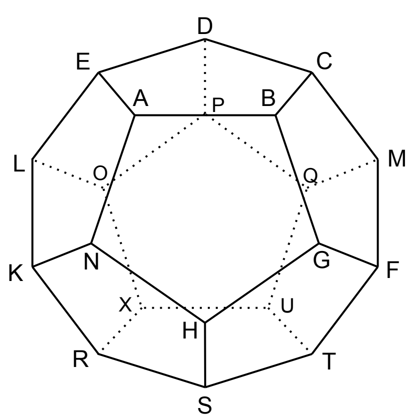
Therefore the plane of intersecting lines FH and GH is parallel to the plane of intersecting lines DE and LD (Ch.9, Thm.11). That is, plane FGH is parallel to plane LED.

But there can be only one plane through straight line GH parallel to the plane of LED (Ch.9, Thm.2 Remark). Therefore the planes of SGH and FGH, both parallel to the plane of LED and both passing through GH, are in fact the same plane. Thus S, H, G, F all lie in one plane parallel to the plane of LEDPO.

And SH = HG = GF, since they are sides of our pentagons,

And ∠SHG = ∠HGF = 108° (by the same reasoning as in Step 4),

So we are free to complete the pentagon in SHGF, namely SHGFT.



[7] Likewise T, F, M, Q are now all in one plane, and we can complete a pentagon TFMQU. Again, by the same argument, R, K, L, O are all in one plane, and we can complete pentagon RKLOX.

[8] By reasoning similar to that in Step 6, we can prove now that points D, C, M, Q, P are all in one plane. Therefore DCMQP is a pentagon. And since PD = DC = CM = MQ (all being sides of our other congruent pentagons), and since ∠PDC = ∠DCM = ∠CMQ = 108° (by the same reasoning used in Step 4), it follows that if we only join PQ, DCMQP is yet another regular pentagon.

[9] Likewise XOPQU is a regular pentagon (reasoning the same way as in Step 8).

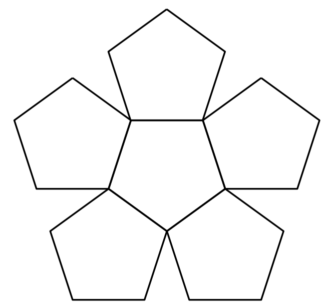
[10]	Now	RS is parallel to KH	(in pentagon KNHSR)
	and	KH is parallel to EB	(since KE = BH, & both parallel to AN)
	so	RS is parallel to EB	
	but	EB is parallel to DC	(in pentagon ABCDE)
	so	RS is parallel to DC	

and likewise *every* side of RSTUX is parallel to a side of ABCDE. Therefore RSTUX, like ABCDE, is a regular pentagon (and it is in a plane parallel to ABCDE).

[11] Therefore we have made a polyhedron contained by 12 equal regular pentagons, all of whose solid angles are made of 3 plane angles each equal to 108° (or the angle of the regular pentagon). Therefore we have made a dodecahedron.

Q.E.F.

Remarks



1. To make a model of a dodecahedron, reproduce and cut out the accompanying figure (preferably enlarged) twice. Fold each up into an open "cup," and place the two together to form a dodecahedron.

2. In Step 8 I said that DCMQP are all in one plane, and that this can be proved similarly to the reasoning in Step 6. Since it is complicated, you might want to see it all done out. Well, all right then! Here goes.

First	PD is parallel to OE	(in PDELO)
next	OE is parallel to RN	(since EN = OR, & each is parallel to KL)
so	PD is parallel to RN	
and	DC is parallel to EB	(in ABCDE)
but	EB is parallel to KH	(since EK = BH, & each is parallel to AN)
so	DC is parallel to KH	

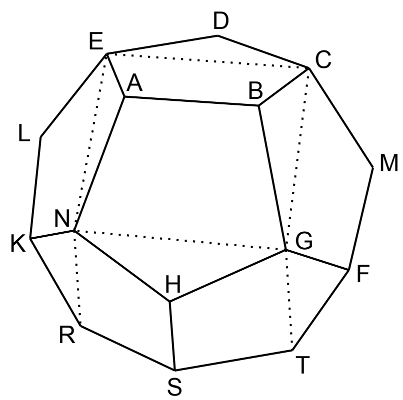
So PD, DC are parallel to RN, KH. But RN, KH intersect in the plane of pentagon KNHSR. Therefore PDC are in a plane parallel to the plane of KNHSR (Ch.9, Thm.11).

Now	QM is parallel to UF	(in QMFTU)
and	UF is parallel to RH	(since HF = RU, & each is parallel to ST)
So	QM is parallel to RH	
And	MC is parallel to BF	(in CMFGB)
and	BF is parallel to NS	(since NB = FS, & each is parallel to HG)
so	MC is parallel to NS	

So QM, MC are parallel to RH, NS. But RH, NS intersect in the plane of pentagon KNHSR. Therefore QMC are in a plane parallel to the plane of KNHSR.

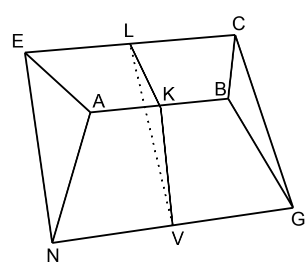
Since PDC and CMQ are all in a plane parallel to KNHSR, and only one plane through C is parallel to the plane KNHSR, therefore D, C, M, Q, P are all in one plane.

THEOREM 15: *There is a cube hidden in every dodecahedron.*



Take the dodecahedron we just made, and join EC, CG, GN, NE.
I say that ECGN is a square.

[1] Bisect EC at L, AB at K, NG at V. Join LK, KV, VL.



[2] From the symmetry of the regular pentagon, it is obvious that

NAKV ≅ GBKV

so ∠AKV = ∠BKV

Thus ∠AKV is right (since ∠AKV & ∠BKV are equal and adjacent). Again, ∠AKL is right by a similar argument.

[3] Thus AK is at right angles to both KL and KV, and therefore
AK is perpendicular to the plane of LKV (Ch.9, Thm.3).
But NV is parallel to AK.
Thus NV is also perpendicular to the plane of LKV (Ch.9, Thm.4).
So ∠NVL is a right angle (Ch.9, Def.1)

[4] But LV is parallel to EN (since EL = NV, and they are parallel)
so ∠ENV is also a right angle (Ch.1, Thm.25).
i.e. ∠ENG is right.

And since ECGN is a parallelogram (for EC = NG and they are parallel), it is now clear that it is also a rectangle. But since its sides are all diagonals of congruent regular pentagons, its sides are also equal. Therefore ECGN is a square.

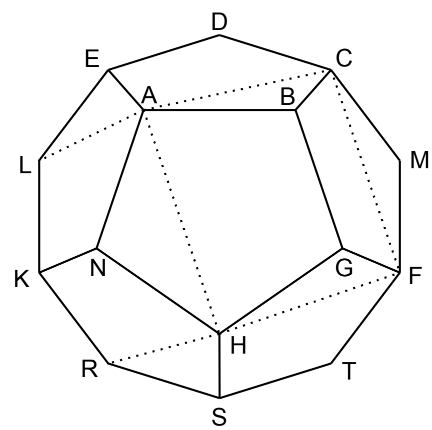
[5] Likewise NGTR is a square for all the same reasons, and it is equal to ECGN, since they share a common side NG.

And thus, in fact, where two faces of the dodecahedron meet, such as at AB or SH, they meet above a square. Since there are 6 pairs of faces on the dodecahedron, therefore there are 6 equal squares that can be traced out along its surface by joining the diagonals of its pentagons, and these 6 squares each share a side with four others, i.e. they form a cube.

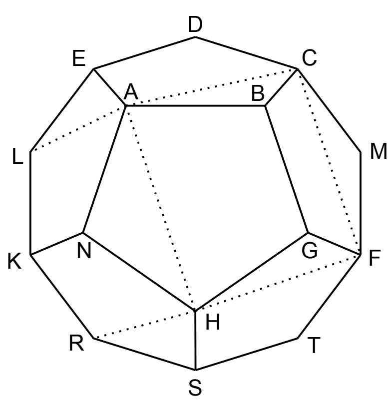
Q.E.D.

Remarks

With equal reason, ACFH is a square, and if we pair off the faces differently, we get another cube. There are, in total, five cubes hiding in the dodecahedron, namely one for every diagonal in pentagon ABCDE.



THEOREM 16: *The diameter of the sphere containing the dodecahedron is the diagonal of the cube hidden in it.*

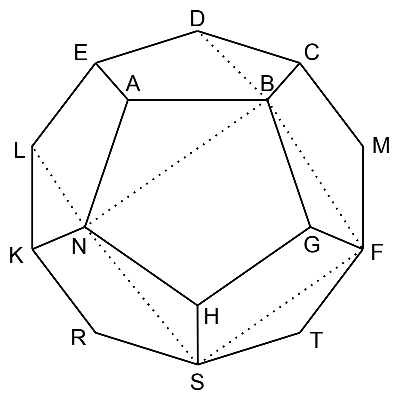


Consider the dodecahedron we have already made. We saw in Theorem 15 that every diagonal in pentagon ABCDE is the side of a cube hidden in the dodecahedron.

[1] Thus RC is a diagonal of such a cube, with AC as its side. But LF is also a diagonal in it, and they bisect each other (Ch.9, Thm.27).

[2] But LF is also a diagonal of the cube with BD as a side, and SD is another diagonal of that cube. Therefore LF and SD also bisect each other (Ch.9, Thm.27).

[3] And since LF has only one midpoint, which is also the midpoint of RC (Step 1), it follows that RC, LF, SD all bisect each other.



[4] Continuing in this way, we will find that all the diagonals of all five cubes share the same midpoint. Call it J. Thus J is equidistant from all the corners of the cubes, i.e. from all the vertices of the dodecahedron. Therefore J is the center of the sphere containing our dodecahedron, and the diameter of that sphere is any diagonal of one of the hidden cubes.

Q.E.D.

THEOREM 17: *There are only 5 perfect solids.*

[1] The fewest number of sides a face of a perfect solid can have is 3 sides, i.e. an equilateral triangle. For there is no rectilineal plane figure with 2 sides or 1.

[2] But the most is 5 sides, i.e. a regular pentagon. For the angle of a regular hexagon, having 6 sides, is 120° (double the angle of an equilateral triangle). And the fewest number of faces which could form a solid angle is 3. But 3 of these angles of the hexagon would add up to 360°, or 4 rights, and no solid angle can be made out of plane angles that add up to 4 rights (Ch.9, Thm.17). So no solid angle can be made out of 3 regular hexagons, and much less could any be made out of more than 3 regular hexagons.

And any regular polygon of *more* than 6 sides would have an angle *more* than 120°. And so no solid angle could be formed out any number of such angles, either. So the greatest number of sides that can be found on the face of a perfect solid is 5.

[3] Therefore a perfect solid can be formed only out of
(a) Equilateral triangles
or (b) Squares
or (c) Regular pentagons

[4] The fewest number of equilateral triangles that form a solid angle is 3, and this is the angle of the TETRAHEDRON.

[5] The next is 4, and 4 equilateral triangles form the angle of the OCTAHEDRON.

[6] The next is 5, and 5 equilateral triangles form the angle of the ICOSAHEDRON.
But it is not possible to form a solid angle out of 6 or more equilateral triangles, since $6 \times 60° = 360°$, or four rights.

[7] The fewest number of squares that form a solid angle is 3, and this is the angle of the CUBE. But 4 or more right angles cannot form a solid angle, since $4 \times 90° = 360°$, or four rights.

[8] The fewest number of pentagons that form a solid angle is 3, and this is the angle of the DODECAHEDRON. But 4 or more cannot form a solid angle, since the angle of the regular pentagon is 108°, and $4 \times 108° = 432°$, which is more than four rights.

[9] Therefore these 5 are the only possible perfect solids.

Q.E.D.

Remarks

It is astonishing that when we limit ourselves to the use of only straight lines and circles, we cannot make all the regular polygons in the world (we cannot, for example, make a 9-sided regular polygon with only straight lines and circles), but we *can* make all the perfect solids!

THEOREM 18: *A comparison of the five perfect solids regarding the number of their vertices, edges, and faces.*

For once, we will not need to prove anything, but only count. If you count the number of vertices, edges, and faces in each solid, you can verify the entries in this table:

FIGURE	VERTICES	EDGES	FACES
Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20

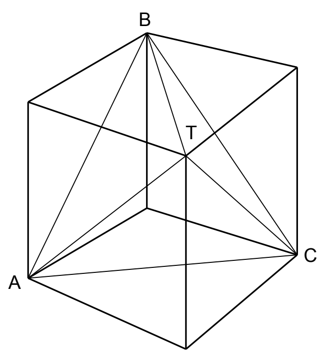
Notice that the cube and the octahedron have the same numbers, but in reverse order, and the dodecahedron and icosahedron also form a symmetry that way. The tetrahedron's numbers are symmetrical by themselves.

This symmetry has an interesting consequence. In one way, we got a cube out of a dodecahedron in Theorem 15. In another way, using the symmetries above, we can also get an octahedron out of a cube. If we find the centers of the circles circumscribed about each square face of the cube, the 6 resulting "centers" of the square faces are also the 6 vertices of an octahedron. This is clear from the uniformity with which those six points are spread out from each other.

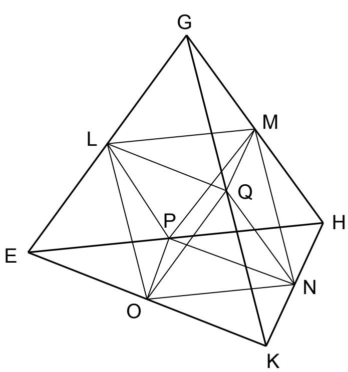
This works in reverse, too. Consider the eight faces of the octahedron, which are equilateral triangles. If we find the "centers" of these triangles, namely the centers of the circles circumscribed about them, we get 8 points spread apart in a uniform way, i.e. we get the 8 vertices of a cube.

If we find the 4 centers of the 4 equilateral triangular faces of the tetrahedron, of course we get the 4 vertices of another smaller tetrahedron.

If we take the 20 centers of the equilateral triangular faces of the icosahedron, we get 20 points uniformly spread out, i.e. we get the 20 vertices of a dodecahedron. And if we take the 12 pentagonal faces of the dodecahedron, finding the center of the circumscribed circle about each one, we get 12 points uniformly spread out, i.e. we get the 12 vertices of an icosahedron.



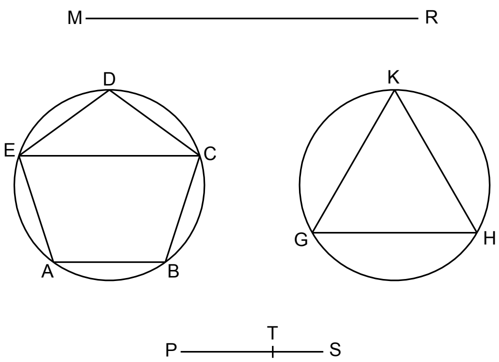
We can also get a tetrahedron out of a cube in much the same way as we found a cube hidden in a dodecahedron. Pick a vertex T on a cube. Across the three squares meeting at T, draw their diagonals from T, namely TA, TB, TC. And AB, BC, CA are three more diagonals on the remaining square faces. Now since the squares are all equal, therefore these six diagonals are all equal, and therefore they contain four equal equilateral triangles. Therefore ABCT is a tetrahedron. The other four vertices of the cube are the vertices of another tetrahedron.



We can also get an octahedron out of a tetrahedron. Bisect the six sides of a tetrahedron, EGHK, at L, M, N, O, P, Q. Joining these six midpoints across each face of the tetrahedron, it is easy to see we have made eight equal equilateral triangles, and therefore L, M, N, O, P, Q are the vertices of an octahedron.

One last note. The Swiss mathematician Leonhard Euler (1707 – 83) showed that if V is the number of vertices in a polyhedron, F the number of its faces, and E the number of its edges, then $V + F = E + 2$. You can verify this in the case of the perfect solids with the table above.

THEOREM 19: *If an icosahedron and a dodecahedron are contained in the same sphere, the circle circumscribing the pentagonal face of the dodecahedron is equal to the circle circumscribing the triangular face of the icosahedron.*



Given: ABCDE is the face of a dodecahedron.
GHK is the face of an icosahedron.
MR is the diameter of the sphere containing both perfect solids.

Prove: The circles around ABCDE and GHK are equal.

[1] Join EC. So EC is the side of the cube inscribed in the sphere (Thms.15 - 16).
Therefore □MR = 3□EC (Thm.11)

[2] Let PS be the radius of the circle around "the pentagon" of the icosahedron, that is, the pentagonal base of one of its "caps," which is a pentagon with side equal to GK.
Therefore □MR = 5□PS (Thm.13, Remark 2)

[3] Thus 5□PS = 3□EC (Steps 1 & 2)

[4] Now cut PS in the golden ratio at T, with PT the greater segment (Ch.6, Thm.22).
Since the diagonal of a regular pentagon is to its side as a whole line is to the greater segment of itself when it is cut in mean and extreme ratio (Thm.3 Conclusion)
Thus EC : CD = SP : PT
Therefore □EC : □CD = □SP : □PT (Ch.11 Thm.9, Remark)
So 3□EC : 3□CD = 5□SP : 5□PT (Ch.5, Thm.12)

[5] Now, taking the last proportion from Step 4, we can add the antecedents to the consequents and still have a proportion (Ch.5, Thm.14, Remark 2). That is,

3□EC : 3□EC + 3□CD = 5□SP : 5□SP + 5□PT

[6] But in Step 3, we saw that 5□PS = 3□EC. So in the proportion of Step 5, we can replace 3□EC with 5□PS. Let's do it:

5□SP : 3□EC + 3□CD = 5□SP : 5□SP + 5□PT

[7] Now PS is the radius of the circle in which GK is a side of the regular *pentagon* (that is how we made PS, in Step 2). And PS is therefore the side of the *hexagon* in that circle. And since PT is the greater segment of PS when it is cut in the golden ratio, PT is the side of the *decagon* in that same circle (Thm.4 Remarks).

Therefore	☐PT + ☐SP = ☐GK	(Thm.5)
or	5☐PT + 5☐SP = 5☐GK	(both sides × 5)

[8] Using this to simplify the proportion from Step 6, we get

5☐SP : 3☐EC + 3☐CD = 5☐SP : 5☐GK

And, looking at this new proportion, it is obvious that

3☐EC + 3☐CD = 5☐GK

[9] Now, in one and the same circle, 3 times the square on the diagonal of the inscribed pentagon plus 3 times the square on the side of that pentagon equals 5 times the square on the inscribed equilateral triangle (Ch.11, Thm.7 Remarks). But EC is the diagonal of a pentagon, and CD is its side, so if Z is the side of the equilateral triangle inscribed in circle ABCDE, then

3☐EC + 3☐CD = 5☐Z

But we have just shown that

3☐EC + 3☐CD = 5☐GK (Step 8)

And therefore it follows that

GK = Z

That is, GK is equal to the side of the equilateral triangle inscribed in circle ABCDE. But GK *is* the side of the equilateral triangle inscribed in circle GKH. Therefore circles ABCDE and GKH must be equal.

Q.E.D.

THEOREM 20: *When the five solids are all inscribed in the same sphere, their order, from longest edge-length to shortest, is: tetrahedron, octahedron, cube, icosahedron, dodecahedron.*

For the sake of brevity, let's use a little shorthand:

Diam	=	the diameter of the containing sphere
Tet	=	the edge of the Tetrahedron
Oct	=	the edge of the Octahedron
Cube	=	the edge of the Cube

[1] Now $\square\text{Diam} = \frac{3}{2}\square\text{Tet}$ (Thm.9)

So $\frac{2}{3}\square\text{Diam} = \square\text{Tet}$ (double both, then one third both)

[2] But $\square\text{Diam} = 2\ \square\text{Oct}$ (Thm.10)

So $\frac{1}{2}\square\text{Diam} = \square\text{Oct}$

[3] So, since the square on the side of the tetrahedron is *two thirds* the square on the diameter (Step 1), whereas the square on the side of the octahedron is only *half* the square on the diameter (Step 2), it follows that

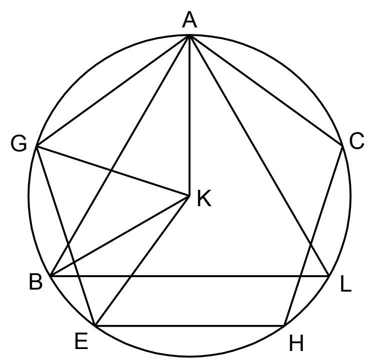
$\square\text{Tet} > \square\text{Oct}$

thus Tet > Oct

i.e. *the side of the Tetrahedron is greater than that of the Octahedron*.

[4] Now $\square\text{Diam} = 3\ \square\text{Cube}$ (Thm.11)

Therefore it is clear that the square on the side of the cube is only *one third* the square on the diameter of the sphere, and so *the side of the Tetrahedron is greater than that of the Cube*.



[5] And we know by Theorems 15 and 16 that the diagonal of any pentagonal face on the dodecahedron is the side of the cube inscribed in the same sphere with the dodecahedron. And we know from Theorem 17 that if we draw a circle around a pentagonal face of the dodecahedron, the side of the equilateral triangle inscribed in that circle is the side of the icosahedron inscribed in the same sphere with that dodecahedron.

Let AGEHC be one pentagonal face of our dodecahedron. Draw a circle around it, with center K, and inscribe equilateral triangle ABL in it. Join KA, KG, KB, KE.

Thus AE, being a diagonal of the dodecahedron's pentagonal face, is the side of the cube inscribed in our sphere, and AB is the side of the icosahedron.

Since ∠AKB stands on one third of the circumference from the center of the circle, therefore ∠AKB = 120°. But ∠AKE stands on two fifths of the circumference from the center of the circle, and therefore ∠AKE = 144°. And yet both stand on less than a semi-circumference, and therefore it follows that

AE > AB

i.e. *the side of the Cube is greater than the side of the Icosahedron*.

[6] But ∠AKG is only one fifth of 360°, i.e. 72°, whereas ∠AKB is 120°. Therefore

AB > AG

i.e. *the side of the Icosahedron is greater than the side of the Dodecahedron*.

Q.E.D.

Remarks

The order of the 5 solids in this Theorem also happens to put them in order of (1) increasing surface area, and (2) increasing volume, when all are inscribed in the same sphere. Two more interesting Theorems, whose proofs I leave to the reader, are these:

(a) *When inscribed all in the same sphere, the surface of the Icosahedron is to that of the Dodecahedron as the edge of the Cube is to the edge of the Icosahedron.*

(b) *When inscribed both in the same sphere, the volumes of the Dodecahedron and Icosahedron have the same ratio as their surface areas.*

"HOOK": A SQUARE SECTION OF A TETRAHEDRON.

Given a tetrahedron ABCV, can you see how to cut it with a plane so that the section is a square?