THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 1

INTRODUCTION

In the first astronomy sequence presented on this site, we saw what it means to call astronomy a "liberal art." That course took us through the principal contributions of Ptolemy, Copernicus, and Kepler. We are now at the next step in the journey: Isaac Newton. This course is a tour through some of the central material in his *Principia Mathematica Philosophiae Naturalis*, or "Mathematical Principles of Natural Philosophy." Although we will be learning plenty about astronomy from Newton, the very title of his work shows that it is not restricted to astronomy. It is nothing less than classical physics in its first form. This prompts the question: "Is physics a liberal art?"

There is one sense in which mathematical physics is not a liberal art. Sometimes this phrase means "one of the seven traditional liberal arts," that is, one of the Trivium or Quadrivium. Mathematical physics is not simply identical with any one of these. But the reasons for that are accidental and historical rather than essential. Despite early intimations of mathematical physics such as we find in Archimedes, the science was not really developed as a whole until Galileo; even in his *Two New Sciences* mathematical physics exists only in its most nascent and embryonic form. The seven traditional liberal arts, on the other hand, come down to us from Plato and before. Among these, the one most similar to mathematical physics is obviously astronomy. Although the principles of mathematical physics apply to the heavens, as we shall see, they also apply to terrestrial phenomena—to sound, light, magnetism, and on and on. We cannot, then, simply reduce mathematical physics to astronomy. That would be to reduce the whole to the part. On the other hand, mathematical physics *does* make models of things for the sake of understanding them, which was the main reason why astronomy was called an "art." Mathematical physics can also bring to light many things worth understanding for their own sake, which is the main reason astronomy can be called "liberal."

The mistaken dichotomy of the ancients between terrestrial and celestial materials and natures is also a factor. Aristotle and Ptolemy, for example, believed that from the Moon upward celestial bodies were immortal, indestructible things constituted of entirely different stuff from the materials of which earthly bodies were made. But that turned out to be wrong. Jupiter is not made of any elements other than those which could be found here on Earth. Newton himself, we shall see, will still be anxious to train us out of the notion that the heavens are fundamentally "other." Once we accept this, we are ready to see that the physical laws governing motions here on Earth apply just as well to the heavenly bodies, and we are prepared to understand their motions in light of physical causes similar to those we find operating on sticks and stones. That would have been an unthinkable thought to the ancients.

Kepler was the first to pave the way to this new astronomical thinking—he thought the motions of the heavenly bodies were due to a power which was magnetic in nature. At any rate, if it is right that the heavens and the earth do not operate on fundamentally distinct natural principles, then astronomy turns out to be nothing but an application of physics to the universe in its large-scale parts—whether the scale is that of suns and planets, or of galaxies and super-clusters and the expansion of all space. The liberal art of astronomy, in other words, is one part of a larger liberal art, mathematical physics. Mathematical physics is nevertheless still essentially astronomical. That is, it is essentially about the universe, since its purpose is to discover the mathematically expressible natural laws that govern all bodies, not just some.

A word or two now about the nature of this course should be helpful to anyone wishing to pursue it.

Our author is Isaac Newton. We shall say more about him when we begin the course proper.

Our text is *Principia*, whose full title I mentioned earlier. The work is divided into three books (more on this later). It is a good idea for the reader to have a copy of the whole text handy, since we will be reading only a slim selection from it in this course, and I will not always quote in full even those texts I will be commenting on. We will pursue the main principles of the book and their principal application, that is, we will be tracing the main steps in Newton's long argument for universal gravitation. Consequently we will skip the entirety of Book 2 of *Principia*, and most of Books 1 and 3. It is good to see just how small a portion of Newton's book we will be reading together, to get a sense of the sheer magnitude of his work. Physicists and mathematicians continue to study it today, and to discover things in it that no one has understood, probably, since Newton himself. It is also good to have a copy handy for those times when Newton refers to things outside our selection, in case the reader wishes to refer to these.

Our translator is Ronald J. Richard, my friend and former colleague, who has accurately rendered into English the portions of Newton's *Principia* which we will study together, and who has generously given me permission to quote his translation at length. If you have another translation, that is good, too. You can see how different translators render Newton's Latin. I will frequently cite the text of the Richard translation in full, and sometimes mention the Latin, in order to see the exact words of Newton we are trying to understand.

Our mode will be to proceed slowly and carefully, often by asking questions about the text and answering them one at a time. The reading to be discussed on a given class day will be listed at the heading of the notes for that class. Probably it is a good idea for you to read the assignment in Newton first, and then read the class notes commenting on the reading afterward, to make my questions and comments more intelligible.

There are several prerequisites to this course. The first of these is elementary geometry, as presented, say, in Euclid's *Elements*, or else as presented on this website. Another is elementary astronomy, as presented on this website (the course on Ptolemy, Copernicus, and Kepler). A further prerequisite is algebraic geometry, as one finds it in the geometry of Descartes—but I will try to supplement the basics of algebraic geometry in our next class, that is, Class 2. Still another prerequisite is a familiarity with the basics of conic sections, as found, for example, in the first three books of *Conics* by Apollonius of Perga. For those unfamiliar with his work, I have supplied my own (so far quite unpolished!) notes on his book here on this site. Knowledge of calculus is *not* a prerequisite, since we will be developing the principles of the calculus with Newton, who is one of its principal discoverers.

That brings us to the fruit we should hope to gather from this course. One of these is to learn the calculus from the ground up. It is rarely taught that way, since in most cases the emphasis is on smooth calculation—"getting the right answer," rather than understanding the philosophical principles underlying the techniques. Here the focus is on understanding the underlying principles. Another fruit we shall reach for is seeing the next phase in the story of astronomy that began with Ptolemy, and progressed through Copernicus and Brahe and Kepler. Still another is to catch a glimpse of an important phase in the history of science. In a way, we will be witnessing the very birth of modern physics. Galileo got the ball rolling, to be sure, but it was only in Newton's *Principia* that the main principles were set down explicitly and in order, and the main elements of the method codified, and an abundance of results discovered, and the fertility of the science abundantly and convincingly demonstrated. Finally, one of the main sights to see will be the amazing argument showing that the same tendency making a stone plummet is also holding the Moon in its orbit—and it is also holding all planets in their orbits around the sun, and shaping and influencing all things in the universe.

Finally, a word of caution. Modern physics retains many things from Newton, although it has altered some, and added much. What vocabulary and notation it retains often comes with some subtle difference from Newton's original ideas. We cannot assume that when we see "force" or "mass," for example, it means exactly what we find it means in a current physics textbook. We must read what Newton himself actually says, and we must read him very carefully.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 2

CRASH COURSE
IN THE BASIC OPERATIONS OF ALGEBRAIC GEOMETRY

In Newton's *Principia* we find him sometimes compounding ratios like the ancients, other times multiplying fractions and irrationals using notations and operational concepts not used by the ancients, but introduced by later thinkers such as Descartes. To make these more modern techniques accessible to anyone unfamiliar with them, I have placed this explanation here at the outset of the course. Please refer to it if the meaning of any algebraic notation or operation later on is obscure to you.

With Euclid's numbers (integers), we can do the following:

ADD	$6 + 3 = 9$
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CUBE	$6^3 = 216$
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TAKE A SQUARE ROOT	$\sqrt{4} = 2$
TAKE A CUBE ROOT	$\sqrt[3]{125} = 5$
TAKE AN n^{th} ROOT	$\sqrt[7]{2187} = 3$

But notice there are many limitations to these operations. We cannot perform each operation given any numbers at all. For example:

• We cannot subtract a greater number from a lesser one, given Euclidean numbers, which are pure numbers with no "direction" associated with them. So "3 – 6" is meaningless and impossible.

• We cannot divide a number by a number which does not measure it exactly, or not without having an answer that involves a remainder. So "5 ÷ 3" is "1 remainder 2."

• We cannot take the square root of a non-square number, or in general any n^{th} root of a number which is not the n^{th} power of an integer. For example, $\sqrt{2}$ means nothing if 2 is a Euclidean number. Even a fraction, like ½ , means nothing in Euclidean numbers, since his numbers are pure multitudes, not multitudes of continuously divisible things like straight lines.

If we think now not of pure numbers, as Euclid did in his *Elements*, but instead of numbered continuous quantities like straight lines, numbered by a unit arbitrarily chosen (but used consistently), then we find we can define operations which will give us answers every time. (For now, we will not worry about taking square roots of negatives, which pertains to complex algebra, which is after Newton.) This is desirable also because geometry and physics and other disciplines are studying continuous quantities, not discrete ones, and yet we find that operations upon them analogous to those on pure numbers are important parts of equations expressing intelligible relationships among them.

Perhaps you wondered exactly what $\sqrt{2}$ meant when you were in high school. I certainly did. I was told it is an infinite and non-repeating decimal. But if it is infinite, we can never "have it all." We can never put down a finite expression and say *there*, *that's* "the square root of two." How do we even know that it exists? Can we ever say anything exactly true about it? And how do we operate on it? How do we add $\sqrt{2}$ and $\sqrt{3}$? It would take literally forever to express either one, and hence it would take forever to add them. Are we stuck with mere approximations? And what would it mean to multiply these by each other? What does $\sqrt{2} \times \sqrt{3}$ mean? What do I do with $\sqrt{2}$ to get the answer? Do I "take it $\sqrt{3}$ times" ? What could that possibly mean?

If we now use our numbered continua, like straight lines, instead of pure numbers, we can define operations to go with these various notations. We will use straight lines for our continua, although we could use any other continuous quantities, such as lengths of time or speeds or areas.

DEFINITION OF ADDITION

This is not difficult. If A and B are two straight lines, and U is our chosen unit line, we "add" A and B by placing them end to end so as to form a new straight line composed of both.

If A is 5 times U,
and B is 3 times U,
then A + B is of course 8 times U.

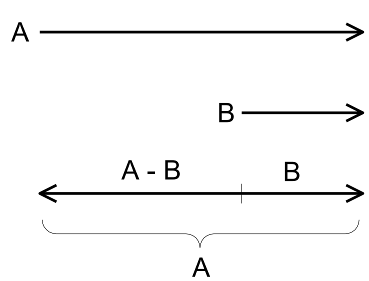
No problem.

DEFINITION OF SUBTRACTION

If A and B are two straight lines, A being the greater, then we "subtract" B from A by cutting off a part of A equal to B. The remainder of A is the difference, or A – B.

If A is 5 times U,
and B is 3 times U,
then A – B is of course 2 times U.

But what if we want to subtract the greater from the lesser? Then we must specify opposite directions as "positive" and "negative." Let's say "toward the right" is positive, and so "toward the left" is negative. And let A and B both be "positive," or "toward the right," each having a direction assigned to them. And let A be greater. Each line is now like an arrow, having both a magnitude and a direction, going from its tail toward its tip. To subtract A from B, now, place the tail of A on the tip of B, and reverse the direction of A, making it negative. The remainder is a "negative" line equal in length to the difference between A and B.



DEFINITION OF MULTIPLICATION

When we multiply two numbers, we add one of them to itself as many times as there are units in the other. Accordingly, the two original numbers and their product are in a proportion starting from the unit and ending with the product.

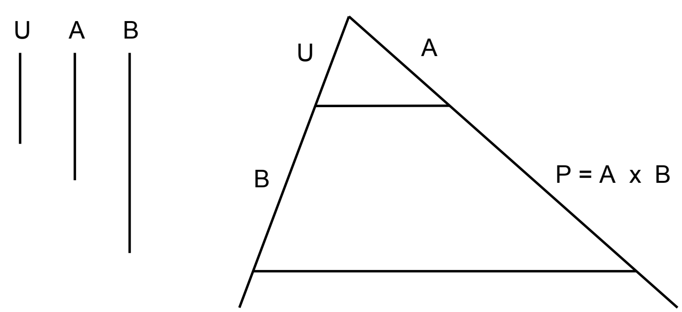
For example: $3 \times 5 = 15$

that is, $5 + 5 + 5 = 15$

just as $1 + 1 + 1 = 3$

thus $1 : 3 = 5 : 15$

But continuous quantities like straight lines can also be in proportions. So if we are given any two straight lines, A and B, and we take their fourth proportional from the unit line U we have chosen, then this fourth proportional line is the "product" of the two given lines, i.e. it is $A \times B$ or $A \cdot B$.



That is, $P = A \times B$

where $U : A = B : P$

Notice that, unlike with numbers, it is possible for the product of two lines to be smaller than them, because a given line can be smaller than the unit line, while no number can be smaller than 1. If A or B are both fractions of the unit, like $\frac{1}{2}$ and $\frac{1}{4}$, then the product will be smaller than them.

$$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

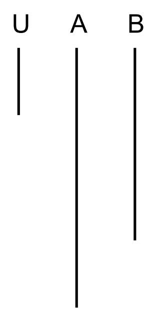
because $$1 : \frac{1}{2} = \frac{1}{4} : \frac{1}{8}$$

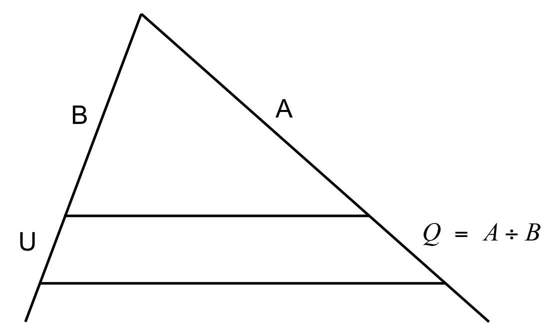
DEFINITION OF DIVISION

When we divide two numbers, we divide one of them (the "dividend") into as many equal parts as there are units in the other (the "divisor"). The number expressing one of the equal parts the dividend has been divided into is called the "quotient." Hence the two original numbers and their quotient are in a proportion, that is, the unit is to the divisor as the quotient is to the dividend.

For example	$15 \div 3 = 5$
that is,	5 is the part which divides 15 into as many parts as there are units in 3.
i.e.	$5 + 5 + 5 = 15$
just as	$1 + 1 + 1 = 3$
thus	$1 : 3 = 5 : 15$
i.e.	unit : divisor = quotient : dividend

But continuous quantities like straight lines can also be in proportions. So if we are given any two straight lines, A and B (like 15 and 3), and we wish to divide A by B, then we must find the fourth proportional to the divisor, the dividend, and the unit, in that order.





That is $Q = A \div B$
where $B : A = U : Q$

Notice that, unlike with numbers, it is possible for the quotient of two lines to be greater than them, because a given line can be smaller than the unit. For example,

if $A = \frac{1}{2}$, and $B = \frac{1}{3}$, then the quotient is $\frac{3}{2}$.

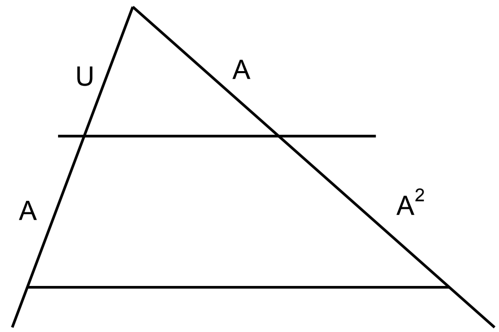
$$\frac{1}{2} \div \frac{1}{3} = \frac{3}{2}$$

since $$\frac{1}{3} : \frac{1}{2} = 1 : \frac{3}{2}$$

DEFINITION OF SQUARING

To "square" any number is just to multiply it by itself. Obviously, then, to "square" a straight line will mean to multiply it by itself. Thus A^2 means the straight line which is a fourth proportional to the unit and A, i.e.

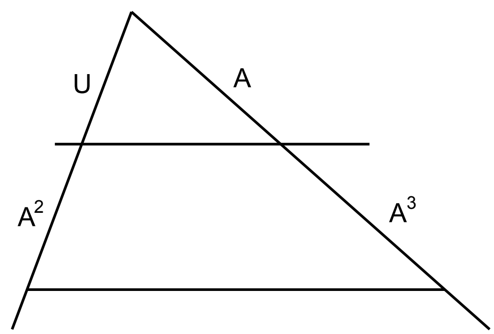
$$1 : A = A : A^2$$



And note that A^2 is not a square figure, but a straight line. And it could be smaller than A, if A is a fraction of 1. For example,

$$1 : ½ = ½ : ¼$$

DEFINITION OF CUBING



To "cube" any number is just to multiply it by its square. Obviously, then, to "cube" a straight line will mean to multiply it by its square, i.e.

$$1 : A = A^2 : A^3$$

And note that A^3 is not a cube figure, but a straight line.

And it is clear that A^4 will mean the fourth proportional such that

$$1 : A = A^3 : A^4$$

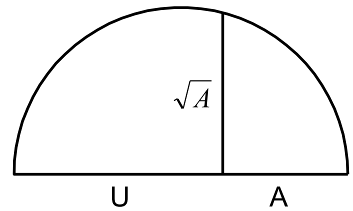
and so on, for higher powers of A.

DEFINITION OF SQUARE ROOT

To take the "square root" of a square number is just to find the number which, when multiplied by itself, produces the given square number. But that means the square root will always be a mean proportional between the unit and the given square number.

For example, 4 is the square root of 16
since $4 \times 4 = 16$
thus $1 : 4 = 4 : 16$

But continuous quantities like straight lines can also have mean proportionals. So if we are given a straight line A, and we find the mean proportional between it and the unit line, this mean proportional is the "square root" of A, or √A. The geometric construction for this is to place the unit and A in one straight line, end to end, then draw a semicircle on their sum as diameter, and set up a perpendicular to the diameter from the point where A and U meet.



DEFINITION OF CUBE ROOT

To take the "cube root" of a cube number is just to find the number which, when cubed, produces the given number. But that means the cube root will always be the first of two mean proportionals between the unit and the given cube number.

For example, 2 is the cube root of 8
since $2 \times 2 \times 2 = 8$
thus $1 : 2 = 2 : 4 = 4 : 8$

So 2 is the first of the two mean proportionals in the continuous proportion between 1 and 8. But continuous quantities like straight lines can also have two mean proportionals between them. So if we are given a straight line like A, and we find the two mean proportionals between it and the unit line, the first of these will be the "cube root" of A, or $\sqrt[3]{A}$.

And it is clear that $\sqrt[n]{A}$ will mean the first of (n - 1) mean proportionals between A and the unit line.

Now the meanings of the following expressions should be clear:

$\sqrt{2}$

$\sqrt{2} \times \sqrt{3}$

$\sqrt{2} \div \sqrt{3}$

$\dfrac{\sqrt[2]{3}}{\sqrt[3]{2}}$

$\left(\sqrt[5]{7}\right)^{13}$

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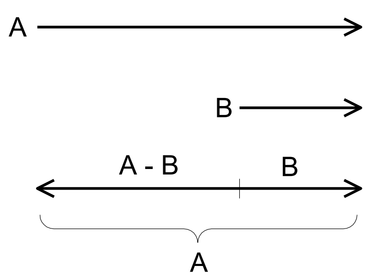

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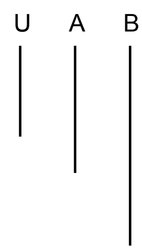

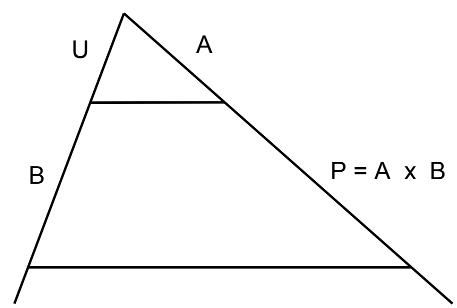

That is, $P = A \times B$
where $U : A = B : P$

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i.e.	$5 + 5 + 5 = 15$
just as	$1 + 1 + 1 = 3$
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i.e.	unit : divisor = quotient : dividend

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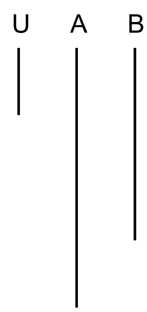

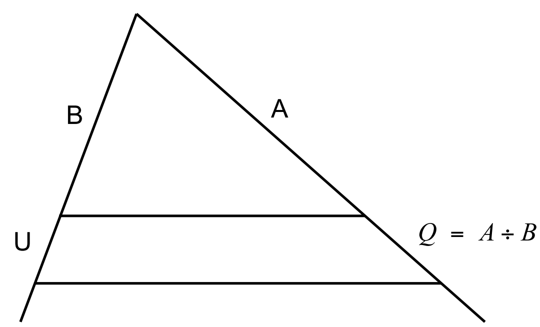

That is $Q = A \div B$
where $B : A = U : Q$

Notice that, unlike with numbers, it is possible for the quotient of two lines to be greater than them, because a given line can be smaller than the unit. For example,

if $A = \frac{1}{2}$, and $B = \frac{1}{3}$, then the quotient is $\frac{3}{2}$.

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since $$\frac{1}{3} : \frac{1}{2} = 1 : \frac{3}{2}$$

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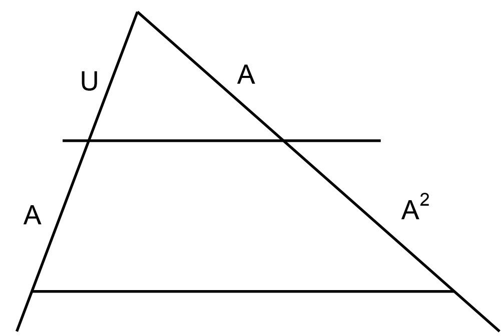
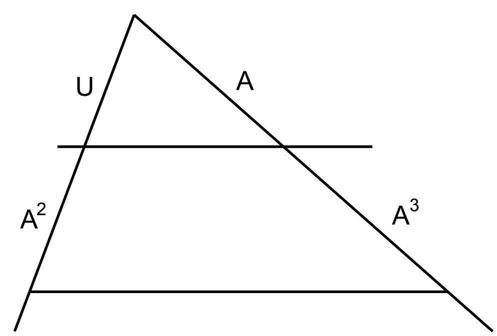


And note that A^2 is not a square figure, but a straight line. And it could be smaller than A, if A is a fraction of 1. For example,

$$1 : \tfrac{1}{2} = \tfrac{1}{2} : \tfrac{1}{4}$$

DEFINITION OF CUBING



To “cube” any number is just to multiply it by its square. Obviously, then, to “cube” a straight line will mean to multiply it by its square, i.e.

$$1 : A = A^2 : A^3$$

And note that A^3 is not a cube figure, but a straight line.

And it is clear that A^4 will mean the fourth proportional such that

$$1 : A = A^3 : A^4$$

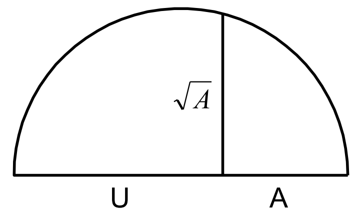
and so on, for higher powers of A.

DEFINITION OF SQUARE ROOT

To take the “square root” of a square number is just to find the number which, when multiplied by itself, produces the given square number. But that means the square root will always be a mean proportional between the unit and the given square number.

For example,	4 is the square root of 16
since	$4 \times 4 = 16$
thus	$1 : 4 = 4 : 16$

But continuous quantities like straight lines can also have mean proportionals. So if we are given a straight line A, and we find the mean proportional between it and the unit line, this mean proportional is the "square root" of A, or $\sqrt{A}$. The geometric construction for this is to place the unit and A in one straight line, end to end, then draw a semicircle on their sum as diameter, and set up a perpendicular to the diameter from the point where A and U meet.



DEFINITION OF CUBE ROOT

To take the "cube root" of a cube number is just to find the number which, when cubed, produces the given number. But that means the cube root will always be the first of two mean proportionals between the unit and the given cube number.

For example,	2 is the cube root of 8
since	$2 \times 2 \times 2 = 8$
thus	$1 : 2 = 2 : 4 = 4 : 8$

So 2 is the first of the two mean proportionals in the continuous proportion between 1 and 8. But continuous quantities like straight lines can also have two mean proportionals between them. So if we are given a straight line like A, and we find the two mean proportionals between it and the unit line, the first of these will be the "cube root" of A, or $\sqrt[3]{A}$.

And it is clear that $\sqrt[n]{A}$ will mean the first of (n - 1) mean proportionals between A and the unit line.

Now the meanings of the following expressions should be clear:

$\sqrt{2}$

$\sqrt{2} \times \sqrt{3}$

$\sqrt{2} \div \sqrt{3}$

$$\frac{\sqrt[2]{3}}{\sqrt[3]{2}}$$

$$\left(\sqrt[5]{7}\right)^{13}$$

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 3

A BRIEF BIO OF NEWTON; THE TITLE OF THE WORK; NEWTON'S PREFACE TO THE *PRINCIPIA*

BRIEF BIOGRAPHY OF ISAAC NEWTON

Isaac Newton was born in Woolsthorpe Manor in Woolsthorpe-by-Colsterworth, a hamlet in Lincolnshire county, December 25, 1642 (in the old dating system, before the Gregorian calendar had been adopted in England), the same year Galileo died.

His father, also named Isaac Newton, died three months before he was born (prematurely). His mother, Hanna Ayscough, remarried to a Reverend Barnabus Smith. Newton disliked this step-father intensely, and one of the sins Newton lists among those he committed before he was 19 was "Threatening my father and mother Smith to burn them and the house over them."

Newton was too in love with study to marry, so he never did, although he was engaged to a Miss Storey when he was in his late teens. From ages 12-16, he was educated at The King's School, Grantham. In June 1661, he was enrolled in Trinity College, Cambridge.

In 1665, he discovered the generalized binomial theorem, and obtained his degree. The university then temporarily closed due to the "Great Plague," and during this time Newton went back home to Woolsthorpe where he privately developed the calculus and his theory of gravitation. This has come to be called his "annum mirabile," his amazing year of rapid, prolific, momentous discovery.

In 1667 he returned to Trinity College as a fellow. As such, he was supposed to be ordained a priest eventually, which he wished to avoid (he was quite heterodox)—but there was no time-limit, so in his case they just postponed the ordination indefinitely. Later he was elected to the Lucasian Chair, which more strictly required ordination, but which he again avoided, this time by special permission from Charles II.

His work significantly advanced pretty much every branch of mathematics at his time (an impossible task to perform today). He and Leibniz independently discovered calculus, and each claimed the priority in discovery. This is a tangled historical question. Newton, together with other members of the Royal Society, accused Leibniz of plagiarism (which appears not to have been true). There was bitterness between these two minds right up to the death of Leibniz in 1716.

Newton did much work in optics. He split and re-composed white light with prisms, and developed the first functioning reflecting telescope, the "Newtonian," in 1668. He developed a particle theory of light. He also had interests in alchemy.

He originally delayed publishing the *Principia* because he was afraid of criticism and controversy (which is to some extent evident in the Rules for Philosophizing at the end). He wanted to be certain he had dotted all his *i*'s and crossed all his *t*'s, as it were. The *Principia* was published on July 5, 1687 with the encouragement of Edmond Halley.

The work drew criticism that Newton was postulating a force (gravity) capable of acting at a distance, even over great distances, and so he, like the ancients, was inventing an "occult quality." In later editions, Newton made very clear that he was not making any claims about the nature of the cause of heaviness or gravity. He was saying only that all bodies were in fact heavy toward each other, whatever the cause of that tendency in them might be. In a later edition, he made these things very clear, as we shall see.

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In 1705, Newton was knighted by Queen Anne during her visit to Trinity College. He was the first scientist ever to be knighted.

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After he died, Newton's body was found to be loaded with mercury! No doubt he inhaled vaporized mercury in his alchemical researches. This probably explains his nervous breakdowns and increasing eccentricity as he got older.

Praise for Newton is abundant and great. He himself wrote of himself (in a letter to Robert Hooke in Feb. 1676) "If I have seen further it is by standing on the shoulders of giants." (The saying is not his own invention, but was attributed to Bernard of Chartres by John of Salisbury in the twelfth century.) Similarly he wrote in a memoir "I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

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THE TEXT OF NEWTON'S PREFACE TO *PRINCIPIA*

PREFACE
OF THE AUTHOR
TO
THE READER
(Ron J. Richard Translation)

Since the ancients specially prized *mechanics* in the investigation of natural things (our authority is *Pappus*), and the moderns, foregoing substantial forms and occult qualities, have undertaken to subject the phenomena of nature to mathematical laws, it seemed right to cultivate *mathematics* in this treatise, insofar as it relates to *philosophy*. In fact, the ancients organized *mechanics* by a two-fold division: *rational*, which accurately proceeds by demonstration, and *practical*. All the manual arts aim at the practical, from which assuredly the name *mechanics* is derived. Since, however, artificers usually work insufficiently accurately, it comes to be that all *mechanics* is so distinguished from *geometry*, that whatever might be accurate is referred to *geometry*, whatever might be less accurate to *mechanics*. But still the errors are not of the art but of the artificers. He who works less accurately is an imperfect mechanic, and if anyone could work most accurately he would be the most perfect mechanic of all. For, the description of right lines and circles, upon which *geometry* is founded, pertains to *mechanics*. *Geometry* does not teach how to describe these lines, but requires [*postulat*] them. For, it requires that the beginner, before he reaches the threshold of *geometry*, should learn how to describe the same accurately; then, it teaches how problems are solved by these operations. To describe right lines and circles are problems, but not geometrical ones. From *mechanics* is required the solution of these problems, in *geometry* is taught the use of these solutions. Moreover, *geometry* glories in that from so few principles brought in from elsewhere it produces so many things. Therefore, *geometry* is founded on mechanical practice, and is nothing other than that part of *universal mechanics* which accurately proposes and demonstrates the art of measuring. Since, however, the manual arts are chiefly involved with moving bodies, it comes to be that *geometry* is commonly referred to magnitude, *mechanics* to motion. In this sense *rational mechanics* will be the science, accurately proposed and demonstrated, of the motions which result from any forces whatsoever, and of the forces which are required for any motions whatsoever. This part of *mechanics* was cultivated by the ancients looking at the *five powers* with respect to manual arts, who scarcely considered heaviness {gravity} (since it is not a manual power) otherwise than for moving weights by those powers. We, however, having regard not to the arts but to philosophy, and writing not about manual but about natural powers, treat especially those things which relate to heaviness {gravity}, lightness {levity}, elastic force, resistance of fluids, and like forces whether attractive or impulsive. And for the sake of those things, we put these forward as our mathematical principles of philosophy. For, the whole burden of philosophy is seen to be involved in this, that from the phenomena of motions we might investigate the forces of nature, then from these forces we might demonstrate the remaining phenomena. And the general propositions which we have treated systematically in the First and Second Books look to this point. In the Third Book, on the other hand, we have put forth an example of this thing by means of the explication of the system of the world. For there, by means of propositions mathematically demonstrated in the prior Books, the forces of heaviness {gravity} by which bodies tend to the Sun and the individual planets are derived from celestial phenomena. Then, from these forces, by means of propositions likewise mathematical, are deduced the motions of the planets, comets, Moon, and sea. If only we were able to derive generally the other phenomena of nature by arguing in the same way from mechanical principles. For many things move me to suspect it to be possible for all these things to depend on certain forces, by which the particles of bodies, by causes not yet known, are either mutually impelled towards each other and cohere according to regular figures, or are driven away and recede from each other: those forces being unknown, philosophers have hitherto probed nature in vain. I expect, however, that either for this mode of philosophizing, or for some truer one, these principles we have laid down will supply some light.

In publishing this work, that man most acute and erudite in all kinds of literature, *Edmund Halley*, rendered assistance not only by correcting typographical errors and attending to the engraving of the figures, but also was the author of the undertaking of this publication. For, when he had obtained from me the demonstration of the figure of the celestial orbits, he did not cease to solicit communication of the same to the *Royal Society*, which thereafter, by their encouragement and benign auspices, effected my beginning to think of letting out the same into the light. But after I had addressed the inequalities of the Moon's motion, and thereafter began to attempt other things which looked to the laws and measures of heaviness {gravity} and of other forces, and the figures described by bodies attracted according to whatever given laws, to the motions of several bodies among themselves, to the motions of bodies in resisting media, to the forces, densities, and motions of media, to the orbits of comets, and to similar things, I resolved to postpone the publication to a different time, so that other cases might be examined and I might present them to the public all together. What relates to lunar motion (it being imperfect) I have brought together in the corollaries to Proposition 66, lest I be constrained to propose and demonstrate separately the individual things by a method more prolix than accords with the worth of the subject matter, and to interrupt the series of the remaining propositions. Some things discovered later I chose to insert in less suitable places rather than change the numbering of propositions and citations. I earnestly beseech that all might be read candidly, and that defects in material so difficult might not be so much reprehended as investigated and benignly made good by new endeavors of the readers.

Given at *Cambridge*, from the College
of *The Most Holy Trinity*, May 8, 1686.

IS. NEWTON

NOTES ON NEWTON'S PREFACE TO THE FIRST EDITION

This preface is an introduction to the whole of the *Principia*, and, in a way, to the whole of modern mathematical physics. Leaving aside the publishing information and editorial information in the last paragraph, this preface consists of five principal parts:

(1) The MODE of the *Principia*, and of modern physics generally.
(2) The TAXIS, i.e. the place or situation of the *Principia* in the larger scheme of human knowledge.
(3) The SKOPOS, i.e. the goal or end of the *Principia*, and of modern physics generally.
(4) The significance of the TITLE.
(5) The DIVISIO TEXTUS, that is, the break-down of the *Principia* into its main parts.

Newton's understanding of the distinction of the disciplines is hardly traditional, and it has not exactly become widely accepted. But it is interesting, and to some extent helpful for understanding Newton's project as he himself saw it. Certainly it is useful to understand his goal—Item (3)—and the main parts and order of the *Principia*—Item (4). For the sake of completeness, however, I will offer here some comments on all the components of the Preface.

(1) THE MODE OF THE *PRINCIPIA*

"The moderns, foregoing substantial forms and occult qualities, have undertaken to subject the phenomena of nature to mathematical laws."

The mode will be to apply mathematics, and to seek out mathematical regularities and intelligibilities in the natural behavior of bodies—as opposed to having recourse to "occult qualities." People accused Newton of introducing another occult quality, or hidden causal power, GRAVITY, but he insists that he does no such thing. He only characterizes the kind of movement toward each other that all bodies exhibit, and shows that there is such a movement or tendency. He nowhere pretends to discover its cause.

Newton remarks on the modern rejection of substantial forms and occult qualities. He does not merely note it, but agrees with it, it seems, since he says later in this preface that, ignorant of certain "forces," philosophers have "hitherto probed nature in vain." His aim is to subject the phenomena of nature to the "laws" of mathematics, as far as possible, and in so doing he speaks as though his discipline were a kind of "mechanics."

(2) THE SITUATION OF THE *PRINCIPIA*

Newton identifies the place of his science, or the nature of his science, in six steps:

(a) He introduces an ancient division (rational vs. practical mechanics), which he accepts, and his science is one member of this division.

(b) He introduces another division which is commonly made, and which he rejects—i.e. the distinction between geometry vs. mechanics as exact vs. inexact.

(c) He introduces a third division commonly made, i.e. geometry concerns magnitude, and mechanics concerns motion.

(d) He gathers the elements of the distinctions he accepts, and concludes that "rational mechanics" is a science exactly demonstrating motions from forces, and forces from motions.

(e) He distinguishes the study of manual forces ("art") from the study of natural forces ("science" or "philosophy," he says both), and places the *Principia* under the latter, i.e. it is a science of natural forces.

(f) He says at the outset that we are not studying math for its own sake, but insofar as it is useful for the study of understanding natural forces.

(a) ANCIENT DIVISION

MECHANICS
RATIONAL = Proceeds by demonstration; exact.
PRACTICAL = Done by hand; inexact.

(b) MISGUIDED DIVISION

GEOMETRY = exact
MECHANICS = inexact

This thinking comes about because "mechanics" first applied to manual arts, and what is done by hand is inaccurate (vs. what is done in the mind). But if someone COULD make perfect straight lines and circles etc., he would be the most perfect mechanic. The inexactness is in the artist, not in the art itself; so this is not a distinction of arts. "Exact" and "inexact" distinguish the capacity of the artist only, not the arts.

["Mechanics" comes from the Greek MECHANAOMAI, like the Latin "machinari," which mean *to make by art, to put together, to construct, to build, to contrive, to devise*. Among the first "machinists" were the theatrical machinists and builders and makers of military engines. And a *MECHOS* was a "means, expedient, remedy," a way of getting this or that on stage or off of it, quickly. Cf. *apparatus*, "what has been prepared for something."]

Again, geometry presupposes accurate description of straight lines and circles, and does not produce those results. Producing those is the work of mechanics. (Cf. Descartes' "machines.") So it is not proper to distinguish between mechanics and geometry as the inexact and the exact, since the exact science of geometry depends upon the prior science of mechanics.

[Mechanics is *prior* to geometry, according to Newton! Is he thinking of forming circles with compasses, and straight lines with straight-edges? Probably not, since that is not exact. He is thinking more of the Divine Mechanic, who produces things exactly in the world and by natural motions. Mechanics shows us how to make the lines and surfaces which geometry needs (e.g. straight lines, circular lines, etc.). We should think of something like

Descartes' "machines" for drawing various curves. That is mechanical drawing—and that is prior to geometry according to Newton. It seems he does not think of geometry as abstract, but concrete, studying things actually drawn (albeit perfectly accurately, if done by a perfect mechanic). This fits with his speaking of geometry as an art of measurement. This is very *English*, and very much the thought of a physicist. Einstein, too, thinks geometry is about physical space (*What other space is there?* he might ask), and hence whether Euclid's geometry is true or not depends on the properties of straight lines in physical space, i.e. on the properties of light rays in a vacuum.]

Since geometry is based upon the products of mechanics, and is about them, it is really just a part of "mechanics" taken generally, i.e. it is a part of "universal mechanics," differing from other parts of mechanics by concerning itself specifically with *measurement*.

(c) THIRD DIVISION

GEOMETRY = about magnitude
MECHANICS = about motion

He says this distinction comes about because manual arts (which people tend to identify with "mechanics") are concerned with moving things—*"Move this over there"* (engineering) problems. Geometry is just about how big things are, relative sizes, proportions, etc., how to measure them.

He accepts this, though, and defines mechanics by its concern for motion.

(d) GATHERING THE DEFINITION OF "RATIONAL MECHANICS"

"Rational mechanics" is the **science of exactly demonstrating motions from forces**, and forces from motions.

"Science" comes from (e) below, where he will say we are pursuing philosophy, not art.

"Exact" comes from (a) and (b).
"Demonstrating" comes from (a) (we use our reason, not our hands)
"Motions" comes from (c)
"Forces" comes out of the blue, seemingly.

Despite its importance, Newton nowhere in the *Principia* (to my knowledge) defines "force" generally, but only certain measures of force, and different types of force.

In his unpublished *De Gravitatione et Aequipondio Fluidorum*, Definition 5 reads:

> **Vis** est motus et quietis causale principium. Estque vel externum quod in aliquod corpus impressum motum ejus vel generat vel destruit, vel aliquo saltem modo mutat, vel est internum principium quo motus vel quies corpori indita conservatur, et quodlibet ens in suo statu perseverare conatur & impeditum reluctatur.

> **Force** is a causal principle of motion and of rest. And it is either an external one, which, impressed upon some body, produces or destroys, or in some way changes its motion, or it is an internal principle by which the motion or rest of a body is conserved, and any being is inclined to persevere in its state and resists impediment.

I think two elements at least enter into the definition of "force" as he intends it. He gives a hint at one of these when he says "forces, whether attractive or impulsive" (bottom of xvii). A "force" is a **cause** of motion, whether by **pushing or pulling**.

Another element becomes clear once he begins to define the different types of force, and especially once he proposes the composition and analysis of forces. A force is or has a **vector quantity**, a magnitude with some direction. This fits with his talk, here in the Preface, about subjecting the phenomena of nature to mathematics, and in particular to geometry. Pure number theory does not involve continuous magnitude, and does not involve direction.

So "mechanics" seems to mean a science of motion as produced by forces, i.e. by certain vector quantities, and hence it is an essentially geometrical study of motion.

There is some connection, too, between the meanings or connotations of "mechanics" and "force." Something mechanical is put together from the outside, with parts compelled to function in some whole which have no interest or inclination of their own to function in such a whole. Their functions or movements within that whole are not natural, then, but forced.

If "force" implies violence or compulsion (contrary to inclination), and "natural" implies agreement with or proceeding from some inner inclination, then "natural force" almost sounds oxymoronic. Certainly for something to be *natural* and for it to be *forced* involves a contradiction, if one is looking to the same thing. But at least a power of acting upon (and even forcing) other things can be natural to the possessor of that power. But since Newton is not interested in studying "occult qualities" like the natures of things, his understanding of "force" does not seem to mean *violent*, but only *some cause productive of motion*. And his understanding of "natural" is merely *as opposed to man-made*.

So MECHANICS seems to mean the science by which one can determine the resultant motion from given quantified forces (e.g. accelerations), and, conversely, by which one can determine the forces given the resultant motion.

NOTE: What are the "*five powers*" of the manual arts? He probably means the lever, pulley, wedge, wheel, inclined plane.

(e) MANUAL vs. NATURAL, ART vs. SCIENCE.

MECHANICAL ART = about manual forces

MECHANICAL SCIENCE = about natural forces

The ancients mainly used natural forces, but did not do much to quantify them, or to learn their quantitative properties and effects, e.g. heaviness. It is true that Archimedes discovered the law of the lever, and certain laws concerning buoyancy and the like.

(f) NOT MATH FOR ITS OWN SAKE.

This is not mathematics for its own sake, but for the understanding of natural forces. So the principles contained herein are not principles of mathematics only, but of natural phenomena. But they are the mathematical principles of natural things, vs. the occult qualities producing them.

(3) THE GOAL OF THE *PRINCIPIA*

What is the job of natural philosophy according to Newton?

To find out the [primary] forces of nature by analyzing the motions we see, and then from those simple natural forces to demonstrate (i.e. predict or explain) all other natural phenomena. So we begin by analysis (and discovery), and then proceed to synthesis (and demonstration)—we reason back to simple causes, then reason forward again to all possible effects.

(4) THE TITLE OF THE *PRINCIPIA*

The title of Newton's famous book is *Principia Mathematica Philosophiae Naturalis*, or "Natural Philosophy's Mathematical Principles."

Even by Newton's time, there was no clear distinction between "science" and "philosophy," in the way people try to draw one today.

But it differs from the first part of natural philosophy by having recourse to mathematics and to very particular forms of sense experience or observation. Although the main conclusion reached, namely the universal acceleration of all bodies towards one another according to an inverse square law, is "universal," it is not founded mainly upon universal experience, nor is it reasoned out from common conceptions of motion.

(5) THE DIVISION OF THE *PRINCIPIA*

The book has two parts:

(1) "The Motion of Bodies" (Books 1-2)

(2) "The System of the World" (Book 3)

The first part (i.e. the first two books) is about finding the basic laws of motion and proving the derivative laws from them.

The second part (i.e. the Third Book) exemplifies, in the case of the structure of the universe and the motions of celestial bodies, how we can derive actual motions of bodies from our laws.

QUESTION: Do Books 1 and 2 prove the truth of the results in book 3? Or do the results in Book 3, by their agreement with observation, prove the truth of the results in Books 1 and 2? In other words, just how "mathematical" is our mode of proceeding in the *Principia*? Do we begin from self-evident things, and deduce the truth of their necessary consequences? Or do we lay down certain hypotheses, and confirm them by the agreement between their consequences and the data of experience?

Cf. Euclid: the axioms and definitions are self-evident, but we learn their *illuminating power* as we progress. To be able to construct the universe from Newton's laws of motion is a significant piece of evidence that they give true insight into nature.

THE NEWTONIAN SUSPICION.

Newton ends his preface by remarking on his suspicion. He suspects that all the phenomena of nature depend on pushes and pulls among **particles**, and could be derived from such forces.

He says all philosophical investigation into nature "hitherto" has been in vain because no one has discovered such forces (or particles). This implies that "substantial forms" and the like offered no genuine insight into nature. This is a facile dismissal of Aristotle and his understanding of nature as explained in his *Physics*. But it is more for his positive contributions than for his dismissals of ancient philosophy that Newton is famous, and so I will refrain from commenting on his attitude toward ancient philosophy.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 3

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THE READER

(Ron J. Richard Translation)

Since the ancients specially prized *mechanics* in the investigation of natural things (our authority is *Pappus*), and the moderns, foregoing substantial forms and occult qualities, have undertaken to subject the phenomena of nature to mathematical laws, it seemed right to cultivate *mathematics* in this treatise, insofar as it relates to *philosophy*. In fact, the ancients organized *mechanics* by a two-fold division: *rational*, which accurately proceeds by demonstration, and *practical*. All the manual arts aim at the practical, from which assuredly the name *mechanics* is derived. Since, however, artificers usually work insufficiently accurately, it comes to be that all *mechanics* is so distinguished from *geometry*, that whatever might be accurate is referred to *geometry*, whatever might be less accurate to *mechanics*. But still the errors are not of the art but of the artificers. He who works less accurately is an imperfect mechanic, and if anyone could work most accurately he would be the most perfect mechanic of all. For, the description of right lines and circles, upon which *geometry* is founded, pertains to *mechanics*. *Geometry* does not teach how to describe these lines, but requires [*postulat*] them. For, it requires that the beginner, before he reaches the threshold of *geometry*, should learn how to describe the same accurately; then, it teaches how problems are solved by these operations. To describe right lines and circles are problems, but not geometrical ones. From *mechanics* is required the solution of these problems, in *geometry* is taught the use of these solutions. Moreover, *geometry* glories in that from so few principles brought in from elsewhere it produces so many things. Therefore, *geometry* is founded on mechanical practice, and is nothing other than that part of *universal mechanics* which accurately proposes and demonstrates the art of measuring. Since, however, the manual arts are chiefly involved with moving bodies, it comes to be that *geometry* is commonly referred to magnitude, *mechanics* to motion. In this sense *rational mechanics* will be the science, accurately proposed and demonstrated, of the motions which result from any forces whatsoever, and of the forces which are required for any motions whatsoever. This part of *mechanics* was cultivated by the ancients looking at the *five powers* with respect to manual arts, who scarcely considered heaviness {gravity} (since it is not a manual power) otherwise than for moving weights by those powers. We, however, having regard not to the arts but to philosophy, and writing not about manual but about natural powers, treat especially those things which relate to heaviness {gravity}, lightness {levity}, elastic force, resistance of fluids, and like forces whether attractive or impulsive. And for the sake of those things, we put these forward as our mathematical principles of philosophy. For, the whole burden of philosophy is seen to be involved in this, that from the phenomena of motions we might investigate the forces of nature, then from these forces we might demonstrate the remaining phenomena. And the general propositions which we have treated systematically in the First and Second Books look to this point. In the Third Book, on the other hand, we have put forth an example of this thing by means of the explication of the system of the world. For there, by means of propositions mathematically demonstrated in the prior Books, the forces of heaviness {gravity} by which bodies tend to the Sun and the individual planets are derived from celestial phenomena. Then, from these forces, by means of propositions likewise mathematical, are deduced the motions of the planets, comets, Moon, and sea. If only we were able to derive generally the other phenomena of nature by arguing in the same way from mechanical principles. For many things move me to suspect it to be possible for all these things to depend on certain forces, by which the particles of bodies, by causes not yet known, are either mutually impelled towards each other and cohere according to regular figures, or are driven away and recede from each other: those forces being unknown, philosophers have hitherto probed nature in vain. I expect, however, that either for this mode of philosophizing, or for some truer one, these principles we have laid down will supply some light.

In publishing this work, that man most acute and erudite in all kinds of literature, *Edmund Halley*, rendered assistance not only by correcting typographical errors and attending to the engraving of the figures, but also was the author of the undertaking of this publication. For, when he had obtained from me the demonstration of the figure of the celestial orbits, he did not cease to solicit communication of the same to the *Royal Society*, which thereafter, by their encouragement and benign auspices, effected my beginning to think of letting out the same into the light. But after I had addressed the inequalities of the Moon's motion, and thereafter began to attempt other things which looked to the laws and measures of heaviness {gravity} and of other forces, and the figures described by bodies attracted according to whatever given laws, to the motions of several bodies among themselves, to the motions of bodies in resisting media, to the forces, densities, and motions of media, to the orbits of comets, and to similar things, I resolved to postpone the publication to a different time, so that other cases might be examined and I might present them to the public all together. What relates to lunar motion (it being imperfect) I have brought together in the corollaries to Proposition 66, lest I be constrained to propose and demonstrate separately the individual things by a method more prolix than accords with the worth of the subject matter, and to interrupt the series of the remaining propositions. Some things discovered later I chose to insert in less suitable places rather than change the numbering of propositions and citations. I earnestly beseech that all might be read candidly, and that defects in material so difficult might not be so much reprehended as investigated and benignly made good by new endeavors of the readers.

Given at *Cambridge*, from the College
of *The Most Holy Trinity*, May 8, 1686.

IS. NEWTON

NOTES ON NEWTON’S PREFACE TO THE FIRST EDITION

This preface is an introduction to the whole of the *Principia*, and, in a way, to the whole of modern mathematical physics. Leaving aside the publishing information and editorial information in the last paragraph, this preface consists of five principal parts:

(1) The MODE of the *Principia*, and of modern physics generally.
(2) The TAXIS, i.e. the place or situation of the *Principia* in the larger scheme of human knowledge.
(3) The SKOPOS, i.e. the goal or end of the *Principia*, and of modern physics generally.
(4) The significance of the TITLE.
(5) The DIVISIO TEXTUS, that is, the break-down of the *Principia* into its main parts.

Newton’s understanding of the distinction of the disciplines is hardly traditional, and it has not exactly become widely accepted. But it is interesting, and to some extent helpful for understanding Newton’s project as he himself saw it. Certainly it is useful to understand his goal—Item (3)—and the main parts and order of the *Principia*—Item (4). For the sake of completeness, however, I will offer here some comments on all the components of the Preface.

(1) THE MODE OF THE *PRINCIPIA*

“The moderns, foregoing substantial forms and occult qualities, have undertaken to subject the phenomena of nature to mathematical laws.”

The mode will be to apply mathematics, and to seek out mathematical regularities and intelligibilities in the natural behavior of bodies—as opposed to having recourse to “occult qualities.” People accused Newton of introducing another occult quality, or hidden causal power, GRAVITY, but he insists that he does no such thing. He only characterizes the kind of movement toward each other that all bodies exhibit, and shows that there is such a movement or tendency. He nowhere pretends to discover its cause.

Newton remarks on the modern rejection of substantial forms and occult qualities. He does not merely note it, but agrees with it, it seems, since he says later in this preface that, ignorant of certain “forces,” philosophers have “hitherto probed nature in vain.” His aim is to subject the phenomena of nature to the “laws” of mathematics, as far as possible, and in so doing he speaks as though his discipline were a kind of “mechanics.”

(2) THE SITUATION OF THE *PRINCIPIA*

Newton identifies the place of his science, or the nature of his science, in six steps:

(a) He introduces an ancient division (rational vs. practical mechanics), which he accepts, and his science is one member of this division.

(b) He introduces another division which is commonly made, and which he rejects—i.e. the distinction between geometry vs. mechanics as exact vs. inexact.

(c) He introduces a third division commonly made, i.e. geometry concerns magnitude, and mechanics concerns motion.

(d) He gathers the elements of the distinctions he accepts, and concludes that "rational mechanics" is a science exactly demonstrating motions from forces, and forces from motions.

(e) He distinguishes the study of manual forces ("art") from the study of natural forces ("science" or "philosophy," he says both), and places the *Principia* under the latter, i.e. it is a science of natural forces.

(f) He says at the outset that we are not studying math for its own sake, but insofar as it is useful for the study of understanding natural forces.

(a) ANCIENT DIVISION

MECHANICS
- RATIONAL = Proceeds by demonstration; exact.
- PRACTICAL = Done by hand; inexact.

(b) MISGUIDED DIVISION

GEOMETRY = exact
MECHANICS = inexact

This thinking comes about because "mechanics" first applied to manual arts, and what is done by hand is inaccurate (vs. what is done in the mind). But if someone COULD make perfect straight lines and circles etc., he would be the most perfect mechanic. The inexactness is in the artist, not in the art itself; so this is not a distinction of arts. "Exact" and "inexact" distinguish the capacity of the artist only, not the arts.

["Mechanics" comes from the Greek MECHANAOMAI, like the Latin "machinari," which mean *to make by art, to put together, to construct, to build, to contrive, to devise*. Among the first "machinists" were the theatrical machinists and builders and makers of military engines. And a *MECHOS* was a "means, expedient, remedy," a way of getting this or that on stage or off of it, quickly. Cf. *apparatus*, "what has been prepared for something."]

Again, geometry presupposes accurate description of straight lines and circles, and does not produce those results. Producing those is the work of mechanics. (Cf. Descartes' "machines.") So it is not proper to distinguish between mechanics and geometry as the inexact and the exact, since the exact science of geometry depends upon the prior science of mechanics.

[Mechanics is *prior* to geometry, according to Newton! Is he thinking of forming circles with compasses, and straight lines with straight-edges? Probably not, since that is not exact. He is thinking more of the Divine Mechanic, who produces things exactly in the world and by natural motions. Mechanics shows us how to make the lines and surfaces which geometry needs (e.g. straight lines, circular lines, etc.). We should think of something like

Descartes' "machines" for drawing various curves. That is mechanical drawing—and that is prior to geometry according to Newton. It seems he does not think of geometry as abstract, but concrete, studying things actually drawn (albeit perfectly accurately, if done by a perfect mechanic). This fits with his speaking of geometry as an art of measurement. This is very *English*, and very much the thought of a physicist. Einstein, too, thinks geometry is about physical space (*What other space is there?* he might ask), and hence whether Euclid's geometry is true or not depends on the properties of straight lines in physical space, i.e. on the properties of light rays in a vacuum.]

Since geometry is based upon the products of mechanics, and is about them, it is really just a part of "mechanics" taken generally, i.e. it is a part of "universal mechanics," differing from other parts of mechanics by concerning itself specifically with *measurement*.

(c) THIRD DIVISION

GEOMETRY = about magnitude
MECHANICS = about motion

He says this distinction comes about because manual arts (which people tend to identify with "mechanics") are concerned with moving things—*"Move this over there"* (engineering) problems. Geometry is just about how big things are, relative sizes, proportions, etc., how to measure them.

He accepts this, though, and defines mechanics by its concern for motion.

(d) GATHERING THE DEFINITION OF "RATIONAL MECHANICS"

"Rational mechanics" is the **science of exactly demonstrating motions from forces**, and forces from motions.

"Science" comes from (e) below, where he will say we are pursuing philosophy, not art.

"Exact" comes from (a) and (b).
"Demonstrating" comes from (a) (we use our reason, not our hands)
"Motions" comes from (c)
"Forces" comes out of the blue, seemingly.

Despite its importance, Newton nowhere in the *Principia* (to my knowledge) defines "force" generally, but only certain measures of force, and different types of force.

In his unpublished *De Gravitatione et Aequipondio Fluidorum*, Definition 5 reads:

> **Vis** est motus et quietis causale principium. Estque vel externum quod in aliquod corpus impressum motum ejus vel generat vel destruit, vel aliquo saltem modo mutat, vel est internum principium quo motus vel quies corpori indita conservatur, et quodlibet ens in suo statu perseverare conatur & impeditum reluctatur.

> **Force** is a causal principle of motion and of rest. And it is either an external one, which, impressed upon some body, produces or destroys, or in some way changes its motion, or it is an internal principle by which the motion or rest of a body is conserved, and any being is inclined to persevere in its state and resists impediment.

I think two elements at least enter into the definition of "force" as he intends it. He gives a hint at one of these when he says "forces, whether attractive or impulsive" (bottom of xvii). A "force" is a **cause** of motion, whether by **pushing or pulling**.

Another element becomes clear once he begins to define the different types of force, and especially once he proposes the composition and analysis of forces. A force is or has a **vector quantity**, a magnitude with some direction. This fits with his talk, here in the Preface, about subjecting the phenomena of nature to mathematics, and in particular to geometry. Pure number theory does not involve continuous magnitude, and does not involve direction.

So "mechanics" seems to mean a science of motion as produced by forces, i.e. by certain vector quantities, and hence it is an essentially geometrical study of motion.

There is some connection, too, between the meanings or connotations of "mechanics" and "force." Something mechanical is put together from the outside, with parts compelled to function in some whole which have no interest or inclination of their own to function in such a whole. Their functions or movements within that whole are not natural, then, but forced.

If "force" implies violence or compulsion (contrary to inclination), and "natural" implies agreement with or proceeding from some inner inclination, then "natural force" almost sounds oxymoronic. Certainly for something to be *natural* and for it to be *forced* involves a contradiction, if one is looking to the same thing. But at least a power of acting upon (and even forcing) other things can be natural to the possessor of that power. But since Newton is not interested in studying "occult qualities" like the natures of things, his understanding of "force" does not seem to mean *violent*, but only *some cause productive of motion*. And his understanding of "natural" is merely *as opposed to man-made*.

So MECHANICS seems to mean the science by which one can determine the resultant motion from given quantified forces (e.g. accelerations), and, conversely, by which one can determine the forces given the resultant motion.

NOTE: What are the "*five powers*" of the manual arts? He probably means the lever, pulley, wedge, wheel, inclined plane.

(e) MANUAL vs. NATURAL, ART vs. SCIENCE.

MECHANICAL ART = about manual forces

MECHANICAL SCIENCE = about natural forces

The ancients mainly used natural forces, but did not do much to quantify them, or to learn their quantitative properties and effects, e.g. heaviness. It is true that Archimedes discovered the law of the lever, and certain laws concerning buoyancy and the like.

(f) NOT MATH FOR ITS OWN SAKE.

This is not mathematics for its own sake, but for the understanding of natural forces. So the principles contained herein are not principles of mathematics only, but of natural phenomena. But they are the mathematical principles of natural things, vs. the occult qualities producing them.

(3) THE GOAL OF THE *PRINCIPIA*

What is the job of natural philosophy according to Newton?

To find out the [primary] forces of nature by analyzing the motions we see, and then from those simple natural forces to demonstrate (i.e. predict or explain) all other natural phenomena. So we begin by analysis (and discovery), and then proceed to synthesis (and demonstration)—we reason back to simple causes, then reason forward again to all possible effects.

(4) THE TITLE OF THE *PRINCIPIA*

The title of Newton's famous book is *Principia Mathematica Philosophiae Naturalis*, or "Natural Philosophy's Mathematical Principles."

Even by Newton's time, there was no clear distinction between "science" and "philosophy," in the way people try to draw one today.

But it differs from the first part of natural philosophy by having recourse to mathematics and to very particular forms of sense experience or observation. Although the main conclusion reached, namely the universal acceleration of all bodies towards one another according to an inverse square law, is "universal," it is not founded mainly upon universal experience, nor is it reasoned out from common conceptions of motion.

(5) THE DIVISION OF THE *PRINCIPIA*

The book has two parts:

(1) "The Motion of Bodies" (Books 1-2)

(2) "The System of the World" (Book 3)

The first part (i.e. the first two books) is about finding the basic laws of motion and proving the derivative laws from them.

The second part (i.e. the Third Book) exemplifies, in the case of the structure of the universe and the motions of celestial bodies, how we can derive actual motions of bodies from our laws.

QUESTION: Do Books 1 and 2 prove the truth of the results in book 3? Or do the results in Book 3, by their agreement with observation, prove the truth of the results in Books 1 and 2? In other words, just how "mathematical" is our mode of proceeding in the *Principia*? Do we begin from self-evident things, and deduce the truth of their necessary consequences? Or do we lay down certain hypotheses, and confirm them by the agreement between their consequences and the data of experience?

Cf. Euclid: the axioms and definitions are self-evident, but we learn their *illuminating power* as we progress. To be able to construct the universe from Newton's laws of motion is a significant piece of evidence that they give true insight into nature.

THE NEWTONIAN SUSPICION.

Newton ends his preface by remarking on his suspicion. He suspects that all the phenomena of nature depend on pushes and pulls among **particles**, and could be derived from such forces.

He says all philosophical investigation into nature "hitherto" has been in vain because no one has discovered such forces (or particles). This implies that "substantial forms" and the like offered no genuine insight into nature. This is a facile dismissal of Aristotle and his understanding of nature as explained in his *Physics*. But it is more for his positive contributions than for his dismissals of ancient philosophy that Newton is famous, and so I will refrain from commenting on his attitude toward ancient philosophy.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 4

DEFINITIONS 1 – 4

Newton begins his *Principia* with eight definitions, followed by a Scholium—a reflection on the significance or application or details of the preceding material, in this case the definitions. The terms he defines are these:

Definition 1	=	Quantity of Matter (also called “Mass”)
Definition 2	=	Quantity of Motion (also called “Momentum”)
Definition 3	=	Innate Force of Matter (also called “Inertia”)
Definition 4	=	Impressed Force
Definition 5	=	Centripetal Force
Definition 6	=	Absolute Quantity of Centripetal Force
Definition 7	=	Accelerative Quantity of Centripetal Force
Definition 8	=	Motive Quantity of Centripetal Force

In this class, we will be looking at the first four of these definitions.
Here is Newton’s text for the first definition:

DEFINITION 1

“QUANTITY OF MATTER” or “MASS.”

Quantity of matter is the measure of the same having arisen from its density and size conjointly.

Air with the density doubled, in double the space as well, becomes quadrupled; in triple, sextupled. The same is to be understood of snow and dust condensed by compression or liquefaction. And alike is the account of all bodies which, by whatever causes, are diversely condensed. Nevertheless, I take no account here of a medium, if there be such, which freely pervades the interstices of the parts. Now, in the following, I generally ***comprehend this quantity under the name body or mass****. It is known by means of the weight of each body; for, as will be shown later, by experiments on pendulums most accurately set up, I have found it to be proportional to the weight.*

NOTES:

(i) For Descartes, who wrote on physics before Newton, "quantity of matter" simply was volume. But Newton is concerned about empty space or a medium which has no resistance to acceleration; that does not deserve to be called "body" or "matter," in his view. It is of no physical significance.

(1) "Quantity of matter" is renamed "mass" later in this definition. And the standard notation for "mass" is the symbol *m*.

(2) "Size" appears to mean "volume," since that is the size of a body.

(3) "Density" cannot mean "mass per volume" here! That would be circular: m = (m/v)v, which is to bring in volume for nothing. So what does "density" mean in this definition?

What about "weight per volume"? He says at the end of the definition that weight is indicative of mass, or proportional to mass, and that this is shown by experiments on pendulums: but this implies that *mass is not defined by weight*. Again, a body could theoretically have mass but have no weight at all, and hence no weight per volume. And the weight of a body can change depending on its location, but not its mass.

Rather, "density" here seems to mean *how closely packed together the bits of matter are*, and so he takes "no account here of the medium" in between all the little bits, meaning he does not count interpenetrating void (which has no mass) in the quantity of matter for a given body. Imagine a sponge or Swiss cheese. In other words, "density" means a ratio of "body-volume to void-volume" in a given object. When this is ratio or percentage is multiplied by the gross volume (which is the sum of body-volume and void-volume), the result is pure body-volume, i.e. pure quantity of matter (vs. quantity of space), i.e. the mass. Hence "mass" or "quantity of matter," for Newton, is the same as **TRUE VOLUME**, or body-volume vs. space-volume.

This understanding of the definition is confirmed in BOOK 3 PROPOSITION 6 COROLLARY 4, where "density" is defined as inertia-per-volume, or shove-resistance-per-volume. He simply assumes that shove-resistance is proportional to body-volume. More on this below.

(4) For example: suppose you have a sponge that is 2" X 5" X 7," so that the total volume is 70 cubic inches. Suppose, further, that the ratio of "sponge-to-air" within that volume is 1 : 8. Then that is our "density." If we multiply the fraction 1/8 by the total volume, we get 8.75 cubic inches of "pure sponge." But this is using "air" instead of "empty space," which is a very different thing. How do we ascertain how much empty space there is in a body, in between all its particles?

(5) Newton's examples of "snow" and "dust" which can be condensed by compression or condensation ("liquefaction") indicate that the thinks of all matter as particulate, with void in between. Newton is in fact an atomist, and probably thinks that all the ultimate particles are the same; hence the mass of a body could be reduced to a number of unit volumes. But alas, that is beyond him, as he notes in his Preface, although he will certainly assume (or perhaps even prove) the homogeneity of matter so far as resistance to acceleration goes. This is why he says **"Alike is the account of all bodies."** Whether matter comes in ultimate particles or not, he presumes that equal volumes of body or matter (excluding empty space) *are equally resistant to acceleration*. Hence RESISTANCE TO ACCELERATION IS PROPORTIONAL TO BODY-VOLUME or MASS. There is no such

thing as two equal volumes of matter with unequal resistance to acceleration. If two volumes of matter unequally resist acceleration, that is because one of them has more empty space in it than the other, and hence one of them contains more matter than the other. So he does away with the distinction between "heavy" and "light" bodies which the ancients introduced, and between terrestrial bodies and celestial ones. All bodies are sluggish, and resistant to acceleration. The ancients thought that some bodies like to move in circles, namely the celestial bodies. Again, all bodies have weight in proportion to their mass (and their proximity to other massive bodies). The ancients thought some bodies were "light," that is, they simply liked to go up, and had no downward tendency or weight.

(6) But now resistance to acceleration is MEASURABLE. Hence the ratios of quantities of matter can be determined. Not only can we measure resistance to acceleration, but we can show that RESISTANCE TO ACCELERATION IS PROPORTIONAL TO WEIGHT. He mentions the experiments with pendulums here which he describes in more detail in Book 3 Proposition 6 (which we will read much later).

(7) So the whole cause for differences in weight among bodies of equal gross-volume is that *there is more empty space (between the atoms) in some than in others*. Equal volumes of gold and water have unequal weights not because gold "has a stronger tendency down," but because in those equal volumes, there is "more stuff" in the volume of gold—more of the same homogeneous matter, more body, and less empty space in between the bits of it.

(8) This frees Newton from ascribing inexplicably different qualities to different bits of matter. Matter is one in kind, the same throughout all the universe, differing only in arrangement (and maybe shape) and relative position of its particles, and in their velocities, etc. He can explain the seemingly qualitative difference of "bodies of different weights" by saying all body is alike, but you have more of it here than there. In this, he is very much like Descartes.

(9) "Having arisen from ... conjointly" means compounding ratios. So

$$m_1 : m_2 = (\text{den}_1 : \text{den}_2) \text{ comp. } (\text{total-vol}_1 : \text{total-vol}_2).$$

But for purposes of MEASUREMENT, since we cannot measure the "densities" in any clear way (how do we get all the void out, and get a body perfectly compressed with no empty space in it? We would have a black hole!), we cannot determine the ratio of the masses by beginning with this formula. So instead we will eventually prove that the masses are proportional to the weights (on the assumption that body-volume is proportional to resistance to acceleration), and then we are given $w_1 : w_2$, and thus we are given $m_1 : m_2$. Can we determine these "masses" absolutely, i.e. get "true pure-body volumes" out of them, in units of volume? Not in any obvious way. We can measure the two total-volumes easily enough, and then by compounding the inverse of these with the known ratio of the masses we will know the ratio of the densities. But that will give us neither density absolutely, and hence we cannot find the masses absolutely just by these means.

(10) The modern idea of "mass" is simply shove-resistance. Newton's idea is that "mass" means volume-of-pure-body, and that the amount of shove-resistance is simply a function of volume.

(11) Newton's examples are of air. Suppose we have two masses of air, m_1 and m_2 . And let m_2 have double the density of m_1, and also let it occupy double the total-volume. Then

$m_1 : m_2 = (1 : 2)\ c\ (1 : 2) = 1 : 4$

If instead we suppose that m_2 has double the density of m_1 and occupies triple the total-volume, then

$m_1 : m_2 = (1 : 2)\ c\ (1 : 3) = 1 : 6$

Here, now, is Newton's text for Definition 2:

DEFINITION 2

"QUANTITY OF MOTION" (or "MOMENTUM")

> ***Quantity of motion is the measure of the same having arisen from the velocity and quantity of matter conjointly.***
>
> *The motion of the whole is the sum of the motions in the individual parts; and so in a body twice as large, with equal velocity, it is doubled, and with double the velocity, quadrupled.*

NOTES:

(1) Like Descartes before him, Newton speaks of both "quantity of matter" and "quantity of motion." So Newton draws from Descartes in a positive way, and is not just a reactionary.

(2) Like Descartes, Newton believes in both the conservation of matter and the conservation of momentum (at least in certain systems, with certain rules about how the bodies interact). Descartes' idea was that God made so much matter, and so much motion, and is too economical to come back and make more as an after-thought, and too attentive to let any pass out of existence.

(3) The "quantity of motion," as here defined, is also called "Momentum," and is symbolized P.

(4) Why define the quantity of motion by the mass of the body rather than the weight? Because the weight differs in different places, as we will learn, whereas mass is always the same regardless of its location or speed (at least for Newton). Also, it is the mass, not the weight, that determines "how much motion" or momentum a thing has when moving (e.g.) sideways. That will not be affected by the absence of gravity; picture a huge I-beam far out in space. A thing can have zero weight, but still a great deal of momentum. If two I-

beams in space were moving toward each other with me in between, even if they were effectively weightless, they would crush me with their momentum.

(5) If v stands for velocity, then the "conjointly" again implies a compounding of ratios:

$$p_1 : p_2 = (m_1 : m_2) \text{ comp } (v_1 : v_2)$$

or $$p = mv$$

or $$\text{Momentum} = (\text{mass})(\text{velocity})$$

(6) "Velocity" is a vector quantity, speed with direction, and hence momentum is, too. The quantity of motion is therefore also a **vector** quantity, a directional quantity (vs. a **scalar**). Similarly "2" is scalar, but "negative 2" and "positive 2" are vectors.

(7) Also, this quantity has an additive property, that is, the vector sums of the motions of the parts should be equal to the quantity of motion of the whole.

(8) QUESTION: Is momentum, or this "quantity of motion," a natural quantity, or a mere human convention? Is it like "length," for example, or is it like inventing a unit of "aardvarks per furlong"? What if we multiply the brightness of a body's color by the speed of the body, and then divide it by the temperature of the body? Does that give us a natural quantity? No.

But momentum seems to be something real, somehow. First, nature conserves momentum, not mere speed, nor even mere velocity. It conserves the total amount of *mv* in a closed system. (Really, what is conserved is energy, but let's not get ahead of ourselves.) Second, if we measure motion by how effective it is, by what it can *do* to things, then the total motion is not just the speed, but also the number of things with that speed. If we are to measure the effectiveness of a stampede to do damage, for example, we need to know not only how fast the animals are moving, but how big each one is and how many there are. So, speaking roughly, "how many things are moving" × "how fast" = "total motion." This is what Newton is getting at when he says "the motion of the whole is the sum of the motions of the individual parts." And this fits with his atomism; the real mobiles are ultimate particles. What we see with our eyes are really crowds of things.

Also, nature seems to obey quantitative rules with respect to this, e.g. the five-ball pendulum on a CEO's desk. If two of the ball bearings are allowed to drop and smack one end, two spring away at the opposite end, and with the same speed (about). If three, then three, and so on.

(9) His examples: if we have two bodies, m_1 and m_2 whose masses are as 1 : 2, and whose speeds are equal, then

$$p_1 : p_2 = (1 : 2) \text{ c } (1 : 1) = 1 : 2$$

but if m_2 also has double the velocity, then

$$p_1 : p_2 = (1 : 2) \text{ c } (1 : 2) = 1 : 4$$

Here, now, is Newton's Third Definition:

DEFINITION 3

"VIS INSITA" or "INNATE FORCE OF MATTER" or "VIS INERTIAE"

> ***Innate force of matter is the power of resisting, by which any body whatever, as much as [the quantity of matter which] is in it {quantum in se est}, perseveres in its state either of resting, or of moving uniformly in a straight line.***
>
> *This force is always proportional to the body, and does not differ in anything from the inactivity {inertia} of the mass, except in the mode of conceiving it. On account of the inertia of matter it happens that every body is dislodged from its state, either of resting or of moving, with difficulty. Whence innate force can be called by the most significant name, "force of inertia." But a body exercises this force only during a changing of its state brought about by another force impressed on it; and the exercise of the former is, under diverse viewpoints, either resistance or impetus: resistance, insofar as the body, in order to conserve its state, opposes the force impressed; impetus, insofar as the same body, by yielding with difficulty to the force of resistance of an obstacle, endeavors to change the state of that obstacle. The public attributes resistance to resting things and impetus to moving things; but motion and rest, as conceived by the public, are distinguished from each other only by viewpoint {relatively}; nor are they always truly resting which are viewed by the public as resting.*

(1) "Innate force of matter": the language makes it seem as though he is bringing out an inherent property of matter.

(2) "any body whatever": terrestrial, celestial, you name it. They all have it, and they have it equally, in equal parts (body-volumes).

(3) Why "power of resisting" vs. "act of resisting"? Because it only shows itself when something tries to move the body (when it was at rest) or to stop the body (when it was moving), or to try to speed it up, or slow it down, or change its direction. And this is sluggishness or laziness—an unwillingness to do anything other than what it was doing. That power of resisting is always there, but it is exercised only when something tries to change its "state."

(4) Motion is here called a "state." As with Descartes, for Newton the significant thing does not seem to be motion, but the change from not moving to moving (or vice versa, or from one speed or direction to another), the change of state. *It is only uniform motion that is here called a "state," i.e. moving at constant velocity in a straight line* is a way of "standing" or "staying." He has a tendency to speak of such motion as though it were not really a motion at all. Certainly it has more in common with rest than accelerated motion has in common with rest, but he seems to go further than this, and say that such motion has more in common with rest than it has in common with accelerated motion.

(5) **"Quantum in se est"** implies that this power of resisting is proportional to the "quantity of matter," the mass. And he brings this out by mentioning "mass" in the

explanation. The innate force of matter is nothing else than “the inactivity of the mass,” i.e. of the quantity of matter. This also excludes the idea that bodies of different natures can have different resistances to acceleration despite the equality of their quantities—rather, the only cause of diverse resistance to acceleration is diverse quantity.

(6) So “force of inertia” means “force of inertness,” or “force of sluggishness” or “of laziness.” He says it is not a positive power in a body, but is just the inactivity of the mass—its sluggishness. (Just as my son’s laziness and slowness to do what I say is not a positive power in him.) In earlier works, Newton spoke of “vis insita” more as though it were a positive power.

(7) He distinguishes between two “viewpoints” or ways of conceiving the exercise of the “force of inertia”: [a] it is *resistance* insofar as the body opposes some impressed force, endeavoring to maintain its state (whether of motion or rest), [b] it is *impetus* insofar as the body, by yielding with difficulty to a resisting obstacle, endeavors to change the state of that obstacle (whether of motion or rest).

(8) He rejects the common distinction which says that “resistance” is what resting things have, and “impetus” is what moving things have. The public way of thinking about what is moving and what is resting is all defined relatively (i.e. they have no grasp of absolute space; and the earth is actually moving, contrary to many everyday expressions).

QUESTION: Is it obvious he is defining a real thing, that this “vis insita” exists? Maybe since this is just a definition, we should hold off the discussion of whether there is such a thing as inertia, as here defined, until the First Law of Motion.

QUESTION: Is INERTIA the same as WEIGHT? No. “Weight” is “downward” only. Inertia is in any direction at all. And things have little or no weight in space, far from other bodies, but maintain the same inertia. And it is much easier to push a car horizontally than to lift it vertically.

QUESTION: Is INERTIA the same as MOMENTUM? No. A body will have different momentums with different speeds, but the inertia of a body is the same at any speed, and even at rest.

Presumably, the inertial resistance offered by a body when you speed it up from 5 mph to 6 mph would be the same as the resistance it offers when you speed it up from 9 mph to 10 mph, or to slow it down from 10 mph to 9, or from 6 mph to 5.

QUESTION: How should one MEASURE the quantity of innate force? He does not here define its quantity, but only says what it is, and what it belongs to. But as soon as he says “quantum in se est,” he implies that the quantity of this resistance is proportional to the quantity of the matter, i.e. to the mass.

But one cannot measure the mass except by measuring the resistance to acceleration, as we said above. How is that done? One can measure how far a spring is compressed by m_1 moving at 5 mph, and then how far the same spring is compressed by m_2 moving at 5 mph. Assuming uniformity in what it takes to compress the spring every inch, one has a direct measure of the resistance to being slowed in each of these bodies. That would not be accurate, of course, but it gives an idea in principle. Newton will develop a more accurate measure with his pendulums in Book 3 Proposition 6.

Here, now, is Newton's Fourth Definition:

DEFINITION 4

"IMPRESSED FORCE"

> ***Impressed force is the action exercised upon a body in order to change its state either of resting or of moving uniformly in a straight line.***
> *This force consists solely in the action, and does not remain in the body after the action. For a body perseveres in every new state by the sole force of inertia. Now, impressed force is of diverse origins, such as from percussion, from pressure, from centripetal force.*

(1) Note that this sort of force is not a quantity, although it *has a quantity*, or can be quantified. He does not quantify this sort of force here, but he will quantify centripetal force in three different ways, in the definitions to follow. More on that later.

(2) "Impressed force" is defined as a principle or cause not of motion, but of *change of state.* One might wonder whether there is such a thing as a "change from rest to motion" or "from motion to rest." There is perhaps a kind of imprecision here, philosophically, which Newton is not very much concerned with. Really there is no such thing as a "change from rest to motion," as though that were some kind of process in things. So nothing is a cause of that change, since that change is not real. Still, there is a cause of accelerated motion, and this is what Newton is interested in. He would say there is a cause of accelerated motion, but not of uniform motion in a straight line which meets with no resistance.

(3) Is he thinking of an impressed force as a species of cause? An agency? A quality? He is not very clear. He is not interested in the nature of the cause producing an acceleration. He is interested more in its effect, the acceleration, and in quantifying the force through the quantity of that effect.

(4) There is something fitting in saying that only "change of state" requires a cause, and in calling that cause "force." At any rate, we seem to be looking at things which are not permitted to remain as they would prefer to be, if left to themselves, and hence they must be *forced*. That makes more sense than calling the tendency of a thing to keep doing its thing an "innate *force*"—but really the innate force of matter is thought of as a "force" only because it acts like one in reaction to other things, e.g. in putting up resistance, or in colliding with something else.

(5) "In order to" does not imply final causation in any strong sense, e.g. toward a good or toward something fitting to the body. It just means the natural effect of the impressed force is to change the state of the body.

(6) As oppose to the "innate force of matter," an impressed force is an *action* rather than a *power*. It is a temporary impulse, and does not abide in a body. It is not like an "indelibly impressed impetus," as one reads about in Galileo's *Two New Sciences*.

(7) Impressed forces have origins, but the origin does not inherently characterize the impressed force itself (similarly the shape in a piece of clay got there by some cause, but that is exterior to the nature of the shape itself).

(8) The three origins he mentions are: percussion (a brief knock), pressure (a sustained push or pull), and "centripetal" force, which is of special importance in nature, and in the *Principia*. So the action of impressed force can either be sudden, and then over with, or else be sustained and kept up (and maybe also varied in strength and direction), but still it is an action which can come to an end, and which is continuously derived from something exterior, as opposed to a permanent impression made upon a body.

QUESTION: Newton does not define "force" in general, but only particular forces, and it is not clear that they are called "force" univocally. Is there one definition of "force" in general for him? One common feature in every meaning is "change of state."

But an "impressed force" is what *results in* a change of state.

And the "innate force of matter" is what *resists* a change of state.

On the other hand, by its "innate force" a body might impress a force upon another body, either by running into it, or by being run into by it. So it is common to everything called a "force" that it can be the cause of a change of state in a body.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 5

DEFINITION 5

Here, now, is the text of Newton's Fifth Definition:

DEFINITION 5

"CENTRIPETAL FORCE"

> *Centripetal force is that by which bodies are drawn, impelled, or one way or another tend, from all directions, toward some point as to a center.*
>
> Of this kind is heaviness {gravity}, by which bodies tend towards the center of the Earth; magnetic force, by which iron makes for a magnet; and that force, whatever it may be, by which planets are constantly drawn aside from rectilinear motion, and compelled to revolve in curved lines. A stone, whirled about in a sling, endeavors to go away from the hand that whirls it; and by the endeavor distends the sling, and that the more strongly the faster it revolves; and, as soon as it is let go, flies away. The force contrary to that endeavor, by which the sling constantly draws aside the stone towards the hand and retains it in orbit, since it is directed towards the hand as if the center of the orbit, I call centripetal. And alike is the account for all bodies which are borne on curves. They all endeavor to recede from the centers of the orbits; and unless some force contrary to this endeavor be present, by which they are confined and restrained in the orbits, and which I therefore call centripetal, they would fly off in right lines with uniform motion. A projectile, if abandoned by the force of heaviness {gravity}, would not be deflected towards the Earth, but would depart in a right line into the heavens; and that with uniform motion, if only the resistance of air be taken away. By its heaviness {gravity} it is drawn aside from a rectilinear course, and constantly turned towards the Earth, and that either more or less according to its heaviness {gravity} and velocity of motion. The less be its heaviness {gravity} for the quantity of matter, or the greater the velocity with which it is projected, the less will it deviate from a rectilinear course and the farther will it proceed. If a lead ball projected by the force of gunpowder from the peak of some mountain with a given velocity according to a horizontal line would proceed in a curved line to a distance of two miles before it descended to Earth, with double the velocity it would proceed twice the distance, and

with ten times the velocity ten times the distance, if only the resistance of air be taken away. And by augmenting the velocity it would be possible to augment at will the distance to which it could be projected, and to diminish the curvature of the line which it would describe, so that it would at length fall at a distance of ten or thirty or ninety degrees, or even go around the whole Earth, or at last depart into the heavens and, by its motion of departing, proceed *in infinitum*. And by the same reasoning, as a projectile can be turned in its orbit by the force of heaviness {gravity} and circle the whole Earth, so can the Moon, by the force of heaviness {gravity}, if only it be heavy, or by whatever other force by which it is urged towards the Earth, be always drawn back from a rectilinear course towards the Earth, and turned into its orbit; and without such a force the Moon cannot be restrained in its orbit. This force, if it be less than what is just right, would not sufficiently turn the Moon from a rectilinear course; if more than what is just right, it would turn it too much, and would draw it down from its orbit toward the Earth. For, it is required that it be of just the right magnitude; and it is for mathematicians to find the force by which a body can be exactly restrained in any given orbit whatsoever with a given velocity, and conversely to find the curvilinear path into which a body, departing from any given place whatever with a given velocity, is turned by a given force. Now the quantity of this centripetal force is of three kinds: absolute, accelerative, and motive.

NOTES

(1) The word "centripetal" means "center-seeking."

(2) Centripetal force is a type of impressed force.

(3) The words "or one way or another tend" in the definition imply that the nature of the cause of the force is irrelevant to his considerations—irrelevant, for example, to what kind of curve the body will trace out, and how fast it will move at each point along that curve.

(4) The phrase "toward some point as to a center" allows the possibility that the seat of the force is *not* at the center. For example, a ball rolling in a circular track; the forces acting on it are from the walls of the track itself, there is no force acting on it from the center. Nonetheless, the forces from the walls are pushing it toward that center, and so we have here an instance of a "centripetal force."

(5) He begins his lengthy explanatory text with the words "of this kind is heaviness." That is clear, since things by their heaviness tend to the center of the earth, to a definite point. The

explanatory text is like an advertisement for what he will show later on in the book, namely that the Moon is heavy, and it, and the other planets, all stay in their orbits, as opposed to flying off in tangents to their orbits by their inertia, because they are heavy towards the bodies at the centers of their orbits.

(6) ". . . alike is the account of all bodies which are borne on curves." This is reminiscent of his definition of mass, where he also says "alike is the account of all bodies." This removes qualitative differences as a reason for the curves, and makes "more of the same stuff" the reason. The explanatory text also makes explicit what Definition 3 (inertia) implied, namely that *no body moves unforced in a curved path*. The "vis insita," the innate force of matter, would keep a body only in one motion: uniform and in a straight line in one direction. Hence all curved motion is in a sense unnatural and forced upon matter from without.

(7) "Alike is the account of all bodies which are borne on curves." Does the curve have to be a circle, since the force is "center-seeking"? No. And the "center" does not have to be the geometric center of a curve, nor does the curved path have to be a closed curve.

(8) The examples Newton draws to our attention in the explanatory text are these:

a. heaviness, toward center of earth
b. magnetism, toward center of magnet
c. planets, "whatever" may be the force which draws them aside
d. sling and stone
e. all bodies moving around curves
f. projectiles (like a lead ball fired from a gun)
g. the Moon in its orbit

a. b. c. are "natural" in the sense that they are not manmade (although they are "forced" in the sense that there is a force on matter from something outside matter). These examples show that centripetal force is something occurring in the world, and on a large scale, and is therefore important for understanding the world.

d. is "artificial," a stone in a sling. This example shows the indifference between manmade things and non-manmade things with regard to forces and their results. Also such an example is easier for us to consider, and convinces us that the stone would indeed fly off in a straight line along the tangent if the string were cut or released. Hence we become convinced by this, together with the idea of the inertial force, that the only reason things fall back toward a center rather than fly off in a straight line is that some force is exerted on them continually.

e. Hence the phrase "all bodies" is mentioned next: "alike is the account for all bodies." There are no bodies which naturally move in some kind of curve.

f. The projectiles lead us by the hand to what would happen if we fired that ball just fast enough so that its inertial motion made it continually "miss" the earth as it fell back toward it. We see that weight makes the thing fall back down toward the Earth, but its inertial speed makes it fly out away from the Earth along the tangent. If these two components could be balanced just so, the ball would orbit the Earth in a circle forever.

g. And the projectiles lead us by the hand to thinking of the Moon as a projectile. (See how the motions of the heavens are being brought down to earth!) And what is the force drawing the Moon aside from its inertial path? Maybe heaviness, he suggests, but that is not clear yet. We might still be attached to the idea that the Moon is a "celestial body" with no weight. But it must have *some* sort of tendency toward the Earth, if we believe in Newton's First Law (although that law has yet to be laid down in Newton's text)!

(9) Notice that centripetal force (and also impressed force, of which it is a type) *has* a quantity (as we see in the next three definitions). Hence it cannot *be* a quantity itself. Gravity, for instance, is a center-seeking force, but it varies in its strength as one moves toward or away from that center (according to an inverse square law).

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 6

DEFINITIONS 6 - 8

Here, now, is Newton's Sixth Definition:

DEFINITION 6

"ABSOLUTE QUANTITY OF CENTRIPETAL FORCE"

> *Absolute quantity of centripetal force is the measure of the same, greater or less in proportion to the efficacy of the cause propagating it from the center through the regions around it.*
>
> As for example, magnetic force, according to the bulk or intensity of the strength of the magnet, is greater in one magnet, less in another.

NOTES:

(1) He now goes on, in Definitions 6, 7, and 8, to distinguish three ways of quantifying a centripetal force. There is something a bit over-specific about this, since there is nothing special about centripetal forces in particular such that we would want to quantify them by accelerations, masses, etc—e.g. the modern formula $F = ma$ is not about centripetal force in particular.

(2) He defines this way of quantifying centripetal force, the "absolute quantity" of a centripetal force, as proportional to the cause which emanates or distributes that force from the center, or to the spaces around that center.

(3) He gives no direct way of measuring the intensity of that cause, but presumably the intensity would be proportional to the acceleration it could produce in a given body at a given distance. Force is defined as a cause of accelerated motion, so it must be proportional to its effect. And in Definition 7 below, he will quantify force again, and do so with reference to acceleration. In *6* he is comparing accelerations to two magnets at one distance, for example, and in *7* he is comparing accelerations to one magnet at two distances.

(4) To be more explicit: If magnet A and magnet B are of the same size, but A produces a greater acceleration in the same paper-clip at the same distance than B does, then A is more intense than B.

(5) But the magnets might also have different sizes. So let C and D have the same intensity, i.e. the same "density" of magnetic-power-per-unit-volume, but suppose C is twice the size of D. Then C will be able to produce twice the acceleration in the same paper-clip at the same distance as D can produce in it.

(6) And if the magnets vary in both size and intensity, then the absolute quantity of the force will arise from the size and intensity conjointly. A bit like the definition of "quantity of matter."

(7) Why call this "absolute"? Because we are now comparing the forces produced toward two centers (e.g. two magnets) *on the same body and at the same distance*. This enables us to compare the two sources, since we have leveled the playing-field by seeing how they compare when given the same job to do: to move this paper-clip at a certain distance away.

(8) He defines this measure of force with reference to a "cause propagating it from the center through the regions around it." Is he, then, talking about causes after all? Only mathematically. He is, as it were, comparing the "force fields" around two magnets (for instance), looking to the accelerations produced in them at equal distances from the center and upon the same body.

On to Newton's Seventh Definition, now:

DEFINITION 7

"ACCELERATIVE QUANTITY OF CENTRIPETAL FORCE"

> *Accelerative quantity of centripetal force is the measure of it proportional to the velocity which it generates in a given time.*
>
> As for example the strength of the same magnet is greater at a lesser distance, less at a greater; or the tending downward force is greater in valleys, less on the peaks of high mountains, and even less (as will be shown later on) at greater distances from the globe of the Earth; at equal distances, however, it is the same in all directions, for the very reason that, when the resistance of air is taken away, it accelerates equally all falling bodies (heavy or light, big or small).

NOTES:

(1) Here we are not comparing different magnets (for example), but only looking to the acceleration which a given magnet produces in some body.

(2) He exemplifies this with a magnet first, saying that it produces greater accelerations at lesser distances, lesser ones at greater distances.

(3) He says the same is true of the "downward" acceleration of heavy bodies, which is greater in valleys than at the peaks of high mountains. This is not obvious or a matter of ordinary experience, and in fact runs contrary to Galileo's assumptions. It was known in Newton's time by the use of accurately made pendulum clocks: Make two identical pendulum clocks, and synchronize them. Bring one to the top of a high mountain; leave the other in the valley. Since the one at the peak is farther from the center of the Earth, the weight of its pendulum is accelerated downward less quickly, and so that clock slows, and gets behind the other one in the valley, as later comparison proves.

(4) It is not part of the definition of "centripetal force" to be stronger near the center. Later he will consider centripetal forces which *increase* with the distance. These may be fictional or artificial, but they are possible, at least to some extent. For example, a rubber-band accelerates me more toward the nail it is wrapped around the more I stretch it.

(5) The accelerative quantity of centripetal force is nothing other than the actual acceleration it produces at a given moment. This can vary from one moment to the next—it can increase, for example, if a body is now further from, now closer to the center of the Earth. For much of the *Principia*, when Newton refers to a "force" as a quantified thing, he is thinking of it as quantified by this or that acceleration it produces (not the total resulting velocity, but the portion of it that was produced in a given time interval), nothing more. Only later in the *Principia* will the mass of the accelerated body become a significant measure of force. And this brings us to the next definition:

DEFINITION 8

"MOTIVE QUANTITY OF CENTRIPETAL FORCE"

Motive quantity of centripetal force is the measure of it proportional to the motion which it generates in a given time.

As for example greater weight is in a greater body, less in a lesser, and, in the same body, greater near the Earth, less in the heavens. This quantity is the centripetancy or propensity of the whole body towards the center, or (as I may accordingly say) the weight; and it is always known by a force contrary and equal to it, by which it is possible to impede the descent of a body.

For the sake of brevity, we can name these three quantities of forces, *motive*, *accelerative*, and *absolute* forces; and for the sake of distinction refer them to the bodies seeking the centers, to the places of the bodies, and to the centers of the forces: certainly the motive force to the body, as the endeavor of the whole towards the center composed of the endeavors of all the parts; and the accelerative force to the place of the body, as a certain efficacy, diffused from the center through the individual places around it, for moving the bodies which are in them; and the absolute force to the center, as endowed with some cause, without which the motive forces would not be propagated through the regions around it, whether that cause be some central body (such as a magnet at the center of a magnetic force, or the Earth at the center of a tending downward {gravitating} force) or something else which is not apparent. This concept is mathematical only; for I am not now considering the physical causes and seats of the forces.

Therefore, the accelerative force is to the motive force as the speed is to the motion. For the quantity of motion arises from the speed and the quantity of matter conjointly. For, the sum of the actions of the accelerative force on each particle of the body is the motive force of the whole. Whence, near the surface of the Earth, where the accelerative heaviness {gravity} or the ending downward {gravitating} force in all bodies is the same, the motive heaviness {gravity} or weight is as the body; but, if it be raised into regions where the accelerative heaviness {gravity} becomes less, the weight will be equally diminished and will always be as the body and the accelerative heaviness {gravity} conjointly. Thus, in regions where the accelerative heaviness {gravity} is twice as small, the weight of a body twice or thrice smaller will be four times or six times smaller.

Furthermore, I name attractions and impulses in the same sense accelerative and motive. The words attraction, impulse, or whatever propensity towards a center, moreover, I employ indifferently and indiscriminately for one another, by considering these forces not physically but only mathematically. Whence, let the reader beware, that he should not, by these words, understand me to define anywhere either the species or the mode or cause of an action or its physical account, or to attribute true and physical forces to centers (which are mathematical points) if perhaps I shall have spoken of either centers as attracting, or forces as being centers.

NOTES

(1) Here he quantifies centripetal force in another way. He says this way of quantifying it is to make it proportional to the momentum (i.e. the "quantity of motion," which he defined earlier by velocity and mass) produced by the force within a given time.

So the force F is proportional to mv over t.

But a velocity over a given time is an acceleration, a.

Hence

F is proportional to ma.

or $f_1 : f_2 = (m_1 : m_2)$ comp. $(a_1 : a_2)$

or $F = ma$

And this he says explicitly: "the motive force arises from the accelerative force and the same quantity of matter conjointly."

(2) He exemplifies this with weight, saying the weight of a body is a motive centripetal force, and he adds again that it is greater at lesser distances, lesser at greater distances, from the center of the force.

(3) He adds that such a force is known (in quantity) by the quantity of an equal and contrary force just sufficient to hinder the body's motion at a given place. (Think of a simple balance, balancing an unknown weight against a known one, or Millikan's famous oil-drop experiment for determining the mass of an electron.) This is an interesting example of how opposites are known through each other. So "weight" is measured by the minimum quantity of impressed force required to keep the body from falling, as with a balance.

(4) Newton has already said "weight" is proportional to "mass." Now he says that "weight" is proportional also to "accelerative force" (which is distance-dependent, i.e. the efficacy of a given force or center at a given distance from it). So, for a given body, since the accelerative force decreases as it is moved away from the center of the Earth, so too its weight.

(5) Newton will not invoke this concept of the quantity of force much until very late in what we will read of the *Principia*.

NEWTON'S REFUSAL TO GET INTO CAUSES

In the explanatory text following Definition 8, Newton says we can speak more briefly, and instead of saying "motive quantity of centripetal force," just say "motive force," and so on with the others. Also, he says we can think of the "motive force" as inhering in the center-seeking body itself, its mass and its acceleration considered together, while we can think of the "accelerative force" as existing in the specific location of the body—the reason the body is accelerated so much toward the center of the Earth, for instance, rather than less or more, is that it is 5000 miles from the center of the Earth. And we can think of the "absolute force" as inhering in the center being sought by the body, e.g. a magnet.

So he distinguishes the three kinds of quantities of centripetal force by their locations or subjects:

- "Motive" is in a whole body, and toward a center, e.g. in a stone toward the center of the earth.
- "Accelerative" is diffused in all the places around a center, acting upon bodies in those places. It is what the field does *here*, and *there*, etc.
- "Absolute" is in the center itself, by which it produces the accelerative force around itself.

But he says this is only "for the sake of distinction," as though this were just a way of imagining these things, or speaking of them, so as not to mix them up, and is not to be taken as a statement about the way things are. He insists that this is for mathematical purposes, or as an aid to the imagination only, and that he is not really positing real causal powers in these things in accord with these ways of quantifying forces:

> The words “attraction,” “impulse,” or whatever propensity toward a center, moreover, I employ indifferently and indiscriminately for one another, by considering these forces not physically but only mathematically. Whence, let the reader beware, that he should not, by these words, understand me to define anywhere either the species or mode or cause of an action or its physical account, or to attribute true and physical forces to centers (which are mathematical points) if perhaps I shall have spoken of either centers as attracting, or forces as being centers.

He is not denying that there are real causal powers at work in nature, responsible for the accelerations we see. But he refrains from any detailed understanding of the natures of those causes—what they are, where they are, and how they work. He is staying true to his promise to address only the “mathematical principles” and causes of motion. He will not, as others before him tried to do, form hypotheses about the natures of the agent or final causes of motion (until, perhaps, his General Scholium at the very end of the *Principia*).

When the *Principia* first came out, certain critics said Newton was positing a “gravitational power” which could act at a distance and reside in a central point. Newton does no such thing. His idea of a center of force is similar to Archimedes “center of weight.” Archimedes did not mean to say that the point actually had weight.

There are others who, reading these words of Newton, say that “force” is a wholly useless idea in physics. It is nothing but an occult quality or an invisible cause. A “hard-headed” physicist needs only the quantitative rules about where something will be when, and this requires no mysterious poetry about causes or forces—only pure algorithms. Newton himself did not seem to think in so positivist a fashion, but he carefully abstains, at least at this point in the *Principia*, from committing himself to any particular views about what things produce centripetal forces, and how, and for what purposes.

A FURTHER PROPORTION:

Newton here also concludes that accelerative force is to motive force as speed is to momentum:

$$a : F = v : p$$

He says that motive force is nothing but accelerative force times mass, and that momentum is nothing but speed times mass, so the proportion is obvious, by definition:

$$a : ma = v : mv$$

STRANGE RATIO? Since Newton is making a ratio between “a” and “ma,” he seems to be thinking of straight lines or numbers representing these quantities more than the quantities themselves.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 7

SCHOLIUM:
ABSOLUTE TIME, SPACE, MOTION

Newton follows his definitions with a lengthy scholium in which he explains his concepts of space, time, and motion. His definitions have served the purpose of explaining technical terms whose meaning is not known to everyone. This scholium serves the purpose of discussing concepts known to everyone, and banishing certain prejudices we all tend to have about them.

A *scholium*, as I have observed earlier, seems to be an explanatory note of something that has gone before. It is not a new definition (or demonstration), for example, but a drawing out of the implications in foregoing definitions (or demonstrations), or an application of them, or else a further explanation of something that was implicit and assumed in them. So it is usually more prosy, less terse.

ABSOLUTE TIME

> *Hitherto, lesser known words have been considered in order to set forth the senses in which they are to be understood in the following. Time, space, place, and motion are best known to all. It is to be noted, however, that the public conceives these quantities not otherwise than from a relation to sensibles. And thence arise certain prejudices, for the abolishing of which it is fitting to distinguish the same into absolute and relative, true and apparent, mathematical and common [vulgares].*
>
> *1. Absolute, true, and mathematical time, in itself and by its nature without relation to anything external whatsoever, flows equably, and by another name is called duration. Relative, apparent, and common time is some sensible and external measure (either accurate or inequable) of duration through motion, which the public use in place of true time; as for example an hour, a day, a month, a year.*

Here Newton will be explaining things implied in the previous definitions which, if not distinguished, could lead to confusions. These are "absolute time," "absolute space," and the like.

He says he will not define time, space, place, and motion—these general ideas are well known to all people, or at least sufficiently well understood for Newton's purposes.

Note that this is an admission by Newton that he relies upon common conceptions. But he will now attempt to purify these ideas of certain prejudices which people commonly suffer from because they conceive of time and place only in relation to sensible bodies, which are mobile.

The first distinction is between "absolute, true, mathematical time" on the one hand and "relative, apparent, common time" on the other. The one that is easier to understand is the second, relative time. When we wish to say how much time some event has taken, we assign to it some number of cycles in some repetitive motion which we regard as uniform—for instance, if we say it took "three days" we mean the event in question took the same amount of time as that for the Earth to spin on its axis (or the Sun to go around us, if you happen to be a geocentrist) three times. In that definition of a quantity of time, however, we include a sensible body, such as the Earth or the Sun. Newton refuses to identify this with time itself—that is, with absolute and true and mathematical time. Why? Because the motion of the Earth or Sun could, conceivably, speed up and slow down, whereas time itself cannot. Again, because there is nothing special about the Earth or Sun such that its uniformity of motion should be the universal time for all things. Third, because we think the motion of the Earth is itself in time and measured by it. Astronomers like Newton are accustomed to noting slight irregularities in things, and correcting for them. How can they think to do this if they have no concept of perfectly uniform time, which is independent of the uniformity or lack of uniformity in the motions of bodies? We can wonder whether the motion of the Earth is perfectly uniform or not—does it speed up? Does it slow down? Plainly, then, we do not think of its motion as time itself. We are holding it up against a more abstract and more perfect standard.

Perhaps Newton's account of "absolute time" is not altogether satisfactory. Is it really conceivable for time to be a sort of "flow," although it is not the flow *of* anything at all? Where does it exist, exactly? Since these questions don't get much more attention from Newton after this scholium, we will not worry about them too much either, or not here.

ABSOLUTE SPACE

> *2. Absolute space, by its nature without relation to anything external whatsoever, remains always similar and immovable; relative space is any movable measure or dimension whatever of this space, which is defined by our senses through its site with respect to a body, and is used by the public for immovable space: as for example the dimension of an underground, aerial, or celestial space defined through its site with respect to the Earth.*

Like "absolute time," "absolute space" is not something *of* another thing. It is not, for example, the location of the Earth, since the location of the Earth can change, but the one part of space the Earth is in at any point in time cannot itself move from one place to another. The parts of space itself are immobile, according to Newton (again, we seem to have a different view today, given that we hear of "expanding space," but that is a can of worms we need not open yet).

What motivates him to believe in such a thing as absolute space? Like the most of us, Newton wants to think some things really are in motion, others really are not, in opposition to the Cartesian idea that "all motion is purely relative." But for this to be true, there must be something ***immobile*** **as a reference frame**, a standard, for saying "this really moved." We need an immobile stage on which motion is to take place. Aristotle, too, thought as Newton does, and indeed puts "immobile" in his definition of "place." But unlike for Aristotle, for Newton there is no more "sphere of fixed stars" or any heavenly sphere, and hence no sensible evidence that the universe is finite; moreover, the philosophical reasons against an infinite magnitude are of no particular force for Newton (who probably was unaware of them). Again, Newton is first and foremost a mathematician, and hence he thinks of things geometrically, and there is no geometrical reason to say the universe is finite, so far as he knows. Furthermore, his First Law of Motion (inertia) seems to require an infinite universe, or at any rate it makes more sense in one. Therefore he does not have the luxury of a finite universe to give him an absolute frame of reference. Therefore he needs "absolute space" to save "absolute motion." He needs an immobile, geometrical reference frame which exists independently of mobile bodies, and through which their real motions are mapped.

So what is "relative space"? This means "the space taken up by this sensible thing," or "the space between these sensible things." As "relative time" is defined by a sensible body, so too "relative space" is defined by a sensible body. "The space which the Earth takes up" is an example. And unlike absolute space, it is not exactly immobile, since the Earth moves, and so the space it is occupying is constantly different.

Clearly, then, a thing can be in absolute rest, but also in relative motion at the same time. If the Earth is truly moving through absolute space, and the Sun is not, then the Sun is at absolute rest, and yet it is still true to say that it is moving in relation to the Earth taken as our frame of reference. The Sun does not stay in the same position *relative to us*. Conversely, a thing can be in absolute motion, but also at relative rest. For example, I am in absolute motion, if the Earth is, since I move along with the Earth through absolute space. But I am at rest in relation to the Earth, since I am not moving with reference to it.

ABSOLUTE PLACE

> *3. Place is the part of space which a body occupies and, according to the notion of the space, is either absolute or relative.*

"Place," for Newton, in general means a part of space which a body takes up.

"Absolute place," then, means a part of absolute space. Imagine a three-dimensional Cartesian coordinate-axes system in absolute space. If I am at rest in that grid, then I have an absolute place in it. "Relative place" would be defined by another grid which is defined by some body, e.g. Earth.

He also adds (although I did not quote it above) that place is *not* "the surrounding surface," thus opposing himself to Aristotle's definition of place. He insists that a "place" is three-dimensional, and has a volume equal to that of the body occupying it, and pervades that body, or coincides with it. He argues thus: the motion of a whole body is the sum of the motions of its parts, and so for the whole to move out of the whole place is the same as for each part to move out of *its* place, and hence the place of the whole *is the sum of the places of the parts*, including those deep inside the body. Hence for Newton place is something "in" a body, not the other way around!

But "immobile" is in the definition of "place" for Newton as well as for Aristotle. We need an unplaced, unmoved place.

ABSOLUTE MOTION

> *4. Absolute motion is the translation of a body out of absolute place into absolute place; relative, out of relative into relative. Thus, in a ship which is borne by sail, the relative place of a body is that region of the ship in which the body is staying, or that part of the cavity of the whole which the body fills up, and which is moved exactly together with the ship: and relative rest is the continuance of the body in that same region or part of the cavity of the ship. But true rest is the continuance of a body in that same part of immovable space into which the same ship together with its cavity and its whole contents is moved. Whence, if the Earth be truly at rest, a body which relatively rests in the ship will be truly and absolutely moving with that velocity with which the ship is in motion on the Earth. But if the Earth also be in motion, the true and absolute motion of the body arises, partly from the true motion of the Earth in unmoving space, partly from the relative motion of the ship on the Earth; and if the body also be in motion relatively in the ship, its true motion arises, partly from the true motion of the Earth in unmoving space, partly from the relative motion, first of the ship on the Earth, then of the body in the ship; and from these relative motions arises the relative motion of the body on the Earth. As, for example, if that part of the Earth where the ship is staying were truly in motion towards the east with a velocity of 10,010 parts, and the ship were borne by sail and wind towards the west with a velocity of 10 parts, but a sailor were walking on the ship toward the east with one part of velocity, the sailor would be in motion truly and absolutely in unmoving space with a velocity of 10,001 parts toward the east, and relatively on the Earth toward the west with nine parts of velocity.*

In opposition to Descartes, Newton is convinced that there is such a thing as "absolute motion," i.e. a motion from absolute place to absolute place in the "true" coordinate-axes system. "Relative motion," by contrast, is just a motion from relative place to relative place, i.e. from a place defined by reference to a body (e.g. "here on the earth") to a different place defined by that body (e.g. "there on the earth").

He gives an illustration of a man walking on a ship, while the ship sails on the earth, and the earth sails through the heavens. He assumes the simple addition of velocities here.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 8

REMAINDER OF THE SCHOLIUM
ON ABSOLUTE TIME, SPACE, MOTION

UNEQUAL DAYS

Newton remarks that "In astronomy, absolute time is distinguished from relative by the equation of common time."

Newton is referring to a correction which astronomers (and also navigators, who in Newton's time needed accurate clocks to determine their longitude, given their speed, direction, and how long they had traveled) made in order to calculate how much time had elapsed from the solar transit of their meridian (noon) to the next solar transit of their meridian. It is not always the same amount of time, but varies, partly because Earth's own axial rotation is not exactly uniform, and partly because it speeds up and slows down in its orbit around the sun. The details are not important here. But astronomers used **accurately made clocks**, and saw that noon-to-noon was not always exactly the same amount of time. So they didn't define a "day" by one such cycle, but by an average of them—and they corrected each "day" by an equation determining how far off from the "true day" that particular day was, depending on what time of year it is.

So he says that the "days" we experience, defined by the sun, are unequal, and the astronomers must correct this inequality to measure the celestial motions by a more accurate time.

UNIFORMITY OF ABSOLUTE TIME

By contrast to a solar day, absolute time is not unequal, but totally uniform. This is something most of us think to be true prior to learning any astronomy or physics, and which Aristotle thought to be true as well. In fact, it might seem impossible and unintelligible to say that time itself could "speed up" and "slow down"—would "slower time" mean a time that *took more time to pass*? Is there some still prior time against which we are comparing times? Then *that* is the true time, or absolute time, and the other "days" or "hours" which were unequal must have been defined relative to some particular motion.

Newton says it "might be" that there is no perfectly uniform motion in the universe by which we can measure time, but rather all motions in fact speed up or slow down or both. Today we use the speed of light as one kind of standard (we say that in a vacuum that is the maximum speed and is the same in all reference frames), whereas Aristotle used the motion of the outermost sphere (fastest motion, and perfectly uniform). We also use the vibrations of cesium atoms (for example) to define a “second,” thus assuming that they are perfectly equal in duration.

IMMOBILITY OF ABSOLUTE SPACE

We can’t sense the parts of absolute space itself, so we must use bodies to measure space (e.g. a yardstick). And in common affairs, such relative places and motions are good enough. (Note again the new precision of science and the inadequacy of common experience for scientific purposes, which began to come to light especially with Kepler.)

But in philosophy, says Newton, we must abstract from our senses! “In philosophical disquisitions, we ought to abstract from our senses, and consider things themselves, distinct from what are only sensible measures of them.”

This is somewhat reminiscent of Descartes, who begins his investigation into reality by assuming that his senses lie to him. It is also reminiscent of Galileo, who dismisses everything but quantity as illusory. It is also reminiscent of Democritus, who also says that while he depends on his senses, much of what they tell him is misleading or false and needs the correction of reason. The idea that we should “abstract from our senses” also fits with the idea that we are learning the *mathematical* principles of natural philosophy, since it is characteristic of mathematics to abstract from sensible matter. But Newton is giving the phrase “abstract from the senses” new meaning. It does not mean “attend to magnitude while ignoring sensible matter and motion,” nor does it mean “attend to the common nature while ignoring this individual matter in which it exists,” but it means “attend to the mathematical uniformity of time and space while ignoring the bodies which make us sensibly aware of time and space and by which we measure them.” We have to get beyond the imprecision and non-uniformity which sensible bodies bring with them if we take them as standards.

But it is also characteristic of mathematics to abstract from motion, and it is characteristic of natural science never to abstract from the senses—even if it must sometimes correct the first ideas to which our sense impressions most easily lead us.

Newton again says it might be that no body to which places and motions can be referred is truly and absolutely at rest, and hence no sensible body can be a marker for a definite spot in “absolute space.” Aristotle thought he had such a point: the center of the earth, which is the center of the universe, and so the earth itself is a big body at rest simply and absolutely. But for Newton there is no sensible evidence of the finitude of the universe (e.g. that it is a moving sphere); the earth is moving, and maybe even our whole solar system is adrift in absolute space.

HOW TO DETECT ABSOLUTE SPACE AND TIME?

Consequently, we cannot determine absolute rest from the position of bodies in our region.

So we must distinguish absolute motion and rest from relative ones by:

(1) Properties
(2) Causes
(3) Effects

(1) PROPERTIES.

It is a property of ABSOLUTE REST that bodies in this condition must rest also relative to each other. This also belongs, I suppose, to things in relative rest—but it does not belong, or at least not necessarily, to things in absolute motion. Things in absolute motion might also be moving relative to each other.

It is a property of ABSOLUTE MOTION that the parts of a body in absolute motion participate in its motion, and share in any tendency that goes with that motion, e.g. to recede from an axis of rotation. (Here he seems to be thinking ahead to his bucket experiment.)

He says if we move a nut, the kernel inside it does not move relative to the shell, but it participates in the motion of the shell, so if the shell is moving absolutely, so is the kernel; but the kernel is not moving with respect to the body in its immediate vicinity. So we have absolute motion of the kernel where there is no motion of it relative to its immediate surroundings. Hence absolute motion cannot be determined "through translation from the vicinity of bodies which are viewed as at rest," contrary to Descartes, who thinks that this is the only way to understand motion.

Again, he argues that we must admit an "UNMOVED PLACE" if there is to be motion at all. If something moves with reference to a moving place (e.g. if a bee flies around inside a jar, but we say the jar itself is moving, too), then the moving place is moving with respect to another place outside itself—is this place also moving? Well, we can't go on like that forever, but must come to a place which simply isn't moving. "No places are unmoved except those which, from infinity to infinity, maintain given positions to each other." And motion with respect to *those* will be absolute.

(2) CAUSES.

Conceptually, we can distinguish between absolute rest and absolute motion by their CAUSES. A body which begins to have relative motion does not by that fact require an impressed force, e.g. if I push A past B, B is in motion relative to A without my acting on B. But a body in true and absolute motion acquires that state only by action upon it, by an impressed force.

Assuming there is absolute space and absolute motion and rest, then, we can say the following: If we act on A and B and move them in exactly the same way through absolute space, we have absolute motion, but no relative motion. Again, if we accelerate A and B through absolute space, they could be moving uniformly in a straight line relative to each other, or resting relative to each other, so that their absolute motion (or state) changes, but their relative motion (or state) to each other does not. Conversely, if we act on A to accelerate it in absolute space, while B just moves uniformly through absolute space, then B might well be accelerating relative to A, but it will not be accelerating absolutely. Hence its absolute motion or state does not change, but its relative motion or state does. "And so true motion does not at all consist in relations of that kind," i.e. if absolute space exists.

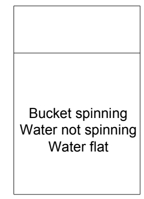
(3) EFFECTS.

Sensibly, we can distinguish between them by their EFFECTS. And to this end he introduces his famous "bucket experiment."

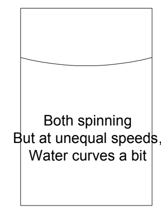
THE BUCKET EXPERIMENT

Hang a bucket of water by a cord, and twist the cord many times until it is tight. Then release the cord and let the bucket spin. You will observe, in order, the following:

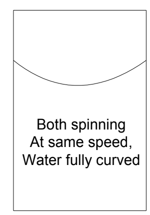
1) The water does not spin with the bucket at first, and the surface of the water is flat. So at this point the water is in a maximum of relative motion with reference to the bucket.



2) Bit by bit, the bucket communicates its motion to the water, and the water begins to revolve, and also its surface grows concave, climbing up the sides of the bucket.



3) The water's motion increases, and catches up with the spinning of the bucket, so that the water is now at rest relative to the bucket. But although the water is spinning together with the bucket, and at rest relative to it, its surface is still concave.



So, when the water is in maximum relative motion, it is FLAT.
When the water is in maximum relative rest, it is CURVED away from the axis of the motion.

Newton is taking the curvature as a sign and measure of the water's true motion or rotation, and he is refuting Descartes, who says that the motion of a body is nothing else than its differing relative position or disposition to a proximate body.

According to Descartes, then, when maximum relative motion is occurring, we should see the maximum curvature on the water, and when there is perfect relative rest (between water and bucket) the surface of the water should be flat. Instead we see the opposite.

(Einstein will later address this.)

Newton remarks: "It is, indeed, very difficult to recognize the true motions of individual bodies, and to discriminate them from apparent actions, since the parts of that immovable space in which bodies are truly in motion do not impinge on the senses. Nevertheless, the case is not wholly desperate."

He follows this up with his example of TWO GLOBES connected by string and spinning around the common center of mass. He says that the TENSION of the string is a sign of their endeavor to recede from the common center of mass, and hence a sign of their revolving about that center. We can push on opposite faces of these globes and see what is required in order to get the string to RELAX, in which case we see we are undoing the circular motion—which tells us which faces the globes are being pushed on, i.e. which tells us the DIRECTION of the circular motion. And we should also be able to determine the SPEED of the circular motion by the tension of the string (he does not mean we should see how to do all this now, but only after reading the *Principia*). And from all that, we can figure out what the absolute circular motion is.

All of this is admittedly rather sketchy. We still have no way of deciding whether a thing is moving uniformly in a straight line in absolute space, or sitting still. Nor will Newton ever give us a way of determining that. Perhaps he doesn't care—if the solar system as a whole is sitting still in absolute space, well and good; if it is moving uniformly in a straight line in absolute space, well, that's not much different anyway. On the other hand, what if the solar system as a whole is accelerating through absolute space? We still would have no way of knowing, if all parts are accelerated equally and in the same direction.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 9

THE FIRST LAW OF MOTION

Newton next introduces his three famous Laws of Motion. He refers to this section as "AXIOMS or LAWS OF MOTION." What does he mean by an "axiom"? And what does he mean by a "law"?

The word "axiom" comes from the Greek word "axios" meaning "worthy." The idea was that only some statements are worthy of our assent, while others are not, and among those that are worthy of our assent, some are worthy because they derive from others that are worthy in themselves. The statements that need no proof through prior "worthy" statements are called "axioms"—those that are worthy of assent in themselves. For example, "When equals are added to equals, the wholes are equal" is an "axiom" in that sense. Even more specifically, sometimes "axiom" means not only a self-evident statement, but one that everyone has heard of, or that gets used in every science. But what does Newton mean by it? He seems to mean by "axiom" a statement which is plausible in itself, and which agrees with all experience, and from which we ought to begin our science. His laws are not quite axioms in the sense that "The whole is greater than the part" is an axiom. They are not things whose opposites are simply inconceivable. At least, his first and third axioms are not like that.

The idea of a "law" of physics, or a natural law, also predates Newton by many centuries. The first meaning of "law" is from human laws, of course. In that case, a law is a rule or standard of human behavior, laid down by someone in a position of power or authority, for the good of a community. A "law of physics" has some likeness to that original idea of "law." It is a rule describing how things are expected to behave, or how they "must" behave, where the "must" has the idea of a natural necessity as opposed to a moral necessity. It is also laid down by someone with authority, such as a scientist—although he merely expresses the law in speech or symbols, whereas nature itself causes the law to be true and "enforced." Or perhaps there is even a "lawmaker" behind the laws of physics, as Newton himself believed, and as we shall see by the end of the *Principia*. The laws of physics also have their efficacy within a community, namely the physical universe, and they somehow unify that multitude of things.

Now let's move on to the particular laws.

LAW 1

Every body perseveres in its state of resting or of moving uniformly in a directed line, except insofar as it is compelled to change its state by impressed forces.

> Projectiles persevere in their motions, except insofar as they are retarded by resistance of the air and are impelled downwards by the force of heaviness. A hoop, the parts of which, cohering, are perpetually drawn aside from rectilinear motions, does not cease to rotate, except insofar as it is retarded by air. Moreover, the larger bodies of the planets and comets conserve still longer their motions, both progressive and circular, made in less resistant space.

Compare this to DEFINITION 3, the definition of "innate force of matter" or "inertia." That definition simply said what is meant by "innate force of matter," and here Newton is basically saying "There IS such a force." Note: "in a DIRECTED line" means not only straight, *but only in one direction along that straight line*. You cannot move back and forth in one straight line simply by inertia.

With this first law of his, Newton sides with Descartes, making rectilinear motion primary, contrary to Galileo and Copernicus and Aristotle. The straight line is mathematically primary, and since we are expecting nature to be mathematically intelligible, we look to mathematical priority. Note: "*except insofar as*" is reminiscent of hypothetical necessity, the kind found most often in the natural world. Also, this law is in some ways reminiscent of Euclid's 2nd postulate, allowing us to continue any straight line as far as needed.

Is this first law of motion self-evidently true? Euclid's Postulate 2 says a straight line can be continued forever in a straight line. That is self-evident. But straight motion is another story.

Law 1 seems to involve two statements, an affirmation and a negation.

AFFIRMATION: A body in a state will continue in that state unless its state is changed by impressed forces (outside causes of acceleration).

NEGATION: Its continuation in that state is not due to any active principle.

• The affirmation is the only thing explicit here in Law 1.

• The negation comes from thinking of Definition 3, the "vis inertiae," which is not mentioned here, but surely Newton means us to think of it. There, we are told that the insistence with which a thing keeps going does not differ from its laziness—so the reason for its continuation in motion is supposed to be nothing else than the *absence of any cause for making it do anything different*.

• The affirmation is more important than the negation, for the purposes of the *Principia*.

• The negation is important for physics only insofar as it denies the need for any physical cause of the continuation of inertial motion.

• There is some experiential support for the affirmation, as Newton's examples illustrate. As for the negation, there is no evidence at all. At best, we see that in many cases *we do not see any cause for the continuation of an inertial motion*. That does not mean that there isn't one.

• The affirmation is not known through the conception of "body" or "mobile" or "straight." There is nothing in our conceptions of these which makes "body which of itself naturally moves in a circle" a contradiction in terms, for example. We know better only by induction.

• Nor can we verify even in one case that a body continues in inertial motion forever, without slowing down even a tiny bit when there are no external causes of it slowing. We can never be sure we have a body in those conditions, even if we see no causes slowing it down—and probably, thanks to universal gravitation and other inescapable forces, we simply can't have a body in that condition.

• So Law 1 does not seem to be self-evident, although there is good reason to accept the affirmation, and perhaps some reason to hold the negation at least as a working hypothesis as regards physical causes. In other words, it is reasonable to assume, based on what all experience suggests, that bodies tend to continue in rectilinear motions at fixed speeds except insofar as they are slowed by external causes, and also to assume that their inherent tendency to continue thus really is not due to the action of any outside bodies.

• But does Law 1 really agree with experience? Is it true about a dog, for example? In one sense, a dog can get up of its own accord, and stop walking of its own accord. On the other hand, when the dog gets up, its muscles must overcome the sluggishness of its mass, and likewise when it stops itself from running. And it cannot start itself or stop itself except by impressed forces from the ground under its feet. If you removed this, it could not start running, and if it were already moving, it could not stop.

What FAVORS the idea that things in such motions continue forever?

• Extrapolation from things like his examples of motions that last a very long time without any obvious source of impressed force.

• The simplicity of uniform motion in a single direction along a straight line, both mathematically, and its resemblance to rest.

• A thing in such motion "feels" as though it were at rest (although the same is true of things undergoing accelerations the same way in all their parts).

What FAVORS the idea that such motion, if it exists, has no cause?

• Only that we see none—which is really just evidence that there is no visible cause, or none which is detectible by our current methods of detection.

NEWTON'S EXAMPLES. Newton's examples are interesting, e.g. the spinning top (or "hoop" in the Ron Richard translation) and the heavenly bodies, which he says would continue spinning forever if we could get rid of friction entirely. These are not straight-line motions! If the law is supposed to be about straight-line motions, why is he using circular motions as examples? He is analyzing them, saying each is in part the result of a centripetal force, such as gravity (for the planets) or cohesion of parts (for the spinning top), and in part the result of the inertial tendency to keep going straight. He cannot find a simple example of what he is talking about, since there is none, as explained above. He must introduce the law not simply as something seen, but as explaining something seen. More than that, with close-to-home examples he must extrapolate, since the top eventually stops. So we must attribute the stopping to friction and irregularities in the table-top and to air resistance. This is why he takes up circular motions. When we roll something in a straight line, it stops very quickly, either because it hits a wall, or because friction stops it. But when we spin something, we reduce the friction to a tiny spot, and the wind resistance is less since the body is simply spinning in its own place. Once we get rid of the air, such *motions go on for a very, very, very long time.* Those in the heavens have been going on for billions of years. The top moving circularly moves within itself, diminishing wind resistance and friction, and approximating the orbits of planets.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 10

THE SECOND LAW OF MOTION

LAW 2

> *Change of motion is proportional to the motive impressed force, and is made according to the straight line in which that force is impressed.*
>
> If some force generate any motion, double will generate the double, triple the triple, whether it will have been impressed all together and at once, or gradually and successively. And, if the body were previously moving, this motion (since it is always determined towards the same region as the generative force) is either (by conspiring with) added to, or (by opposing) subtracted from, or (by being oblique to) adjoined obliquely to its prior motion, and is compounded with it according to the direction of each.

This is like the complement of Law 1, which was about things in their state, what they do without any impressed force. This one says what happens due to impressed force.

"Change of motion" means "change of momentum," of mv. That is plain, since "proportional" implies that motion is quantified, and we have from Newton only one definition for a "quantity of motion," which is mv.

What is "motive impressed force"? This has not been defined, but "motive quantity of centripetal force" has been defined (Def. 8, $F = ma$). Probably Newton wants us to generalize after all, and take "F" to mean not just centripetal force, but any impressed force quantified by ma.

Is this Law true and self-evident? It is true by definition. Since "motive impressed force" is defined as ma, i.e. that is how its quantity is defined, Law 2 is just saying that "change of motion" is proportional to ma, but "change of motion" is change from one momentum to another, from one mv to another mv. So, if we are talking about the same body, and hence the same m, the only thing changing is the v, and a change in v is an a (acceleration). So the Law ends up meaning: the total change of momentum is proportional to ma, i.e. $(mv_1 - mv_2) = ma$. The change in velocity is proportional to (or is equal to, or simply is) the acceleration.

So far Law 2 appears to be true by definition, and is not really experimentally verifiable. But what about the part that says the direction of the change of motion results from the direction of the motive impressed force? If this means the direction of the acceleration in a body is the same as the direction of the body which collided with it or knocked it, that can be verified experimentally, at least in some cases—e.g. in head-on collisions. In other cases, we would have to assume vector addition applies, which would seem to beg the question. Newton's wheel (an illustration he will soon introduce in the Corollaries to the Laws) follows from Law 2, but also from Law 1, so even though that is experimental verification of a kind, it is not a verification of Law 2 independently of assuming any other Laws.

Here for the first time Newton highlights DIRECTION, saying that the direction of the impressed force determines the direction of the change of motion. This is in opposition to Descartes, who says motion is not contrary to motion, and therefore change of direction is *not* a change of state for him. Here we may remind ourselves of the distinction between **vector** quantities and **scalar** ones. A vector quantity is a magnitude (a "how much") plus a direction. A scalar quantity is a mere magnitude, without any direction. It is not merely a matter of human convention to combine magnitudes with directions. Just as it is natural to quantify motion by both speed and mass, since nature preserves that quantity, so too direction must be taken into account, since nature preserves that, too.

If the force is impressed in the same direction as the original motion, then the change in motion is simply added to the original motion. If in the opposite direction, then subtracted. If obliquely, then we need *vector addition*, as specified in Corollary 1 following the Laws.

Again, no special importance is placed on the origin of the force. It can take place "all together and at once" or "gradually and successively," and the resulting quantity (i.e. of change of momentum) will be the same.

QUESTION: Can't I push on something without budging it, and hence impress a force which is not proportional to any change of motion in the body I am pushing against (e.g. a large stone)?

This seems to involve us in more modern considerations of energy. Changes in my muscles and expenditure of their potential energy must translate into some transfer of energy into the stone, but not always in the form of a motion of the whole stone. Perhaps I keep smacking it with my hand, and it does not move—but I do heat it up a little bit. Ignoring these kinds of complications, though, if I shove against the stone, I will accelerate it somewhat, even if just a tiny bit. This kind of difficulty will come up more plainly in Law 3.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 11

THE THIRD LAW OF MOTION

LAW 3

To an action there is always a contrary and equal reaction; or the actions of two bodies between themselves are always equal and directed toward contrary parts.

> Whatever presses or draws another, is pressed or drawn just as much as it. If one presses some stone with a finger, his finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse is drawn back (if I may so speak) equally towards the stone; for the rope, being distended on either side by the same stretching out, endeavors to urge the horse toward the stone, and the stone toward the horse; and as much as it impedes the progress of the one so much will it promote the progress of the other. If a body impinging on another body would by its force change the latter's motion in whatever way, the same, in turn, also will undergo the same change in its own motion towards contrary parts by the other force (on account of the equality of the mutual pressure). By these actions equal changes are made not of velocities, but of motions; that is, in bodies not impeded by something else. For, changes in velocities, being likewise made towards contrary parts, are inversely proportional to the bodies, since the motions are equally changed. This law also obtains in attractions, as will be proven in the next scholium.

This is Newton's star law, and is contrary to Descartes, who said (e.g. in his fourth rule of collision in his *Principles of Philosophy*) that bodies do not always produce accelerations in each other. A smaller body cannot accelerate a larger one, according to him, regardless of how fast the smaller one is going. For Newton, the smaller always accelerates the larger (assuming perfect rigidity), no matter how small it is or how slowly it is going.

Both Descartes' view and Newton's are counter-intuitive. According to Descartes, if a smaller football player runs into a larger, stationary one, the larger one will not budge at all. And if someone is shot by a bullet, then, since the bullet is much smaller than him, he will not be accelerated at all by the bullet. According to Newton, if a fly collides with a freight train, it will slow the freight train down a little bit. Newton's view, while in some ways shocking, is conformable to experience, whereas Descartes' does not appear to be.

What is meant by "ACTION"? Uniform motion in a straight line is not an "action," and so you can have that without having anything equal and contrary going on. So does he mean an acceleration? If so, the "reaction" would be an acceleration, too, and so "an equal and opposite reaction" would mean "an equal and opposite acceleration." But that is false. So by "action" he means "a change of motion," i.e. "a change of momentum," i.e. a mass times its acceleration.

"CONTRARY." Impressed forces, in other words, exist only in contrary pairs. Every one is accompanied by another, of equal magnitude, acting in the contrary direction. A bit like "concave" and "convex."

Every impressing body is in turn impressed by what it impresses. This is reminiscent of the discussion after Definition 3 of "innate force of matter," where he says a body has resistance insofar as it is impressed by another, and impetus insofar as it is impressing another.

"EQUAL." Equal in what way? Are the velocities of the action and reaction equal? No. Are the accelerations necessarily equal? No. The resulting changes of momentums are equal. What equation should we use to express Law 3?

$$m_1a_1 = m_2a_2$$

Is that quite right? Since acceleration is a vector quantity, this means that both bodies are accelerated in the same direction as a result of their interaction—which is false. So we must actually write

$$m_1a_1 = - m_2a_2$$

For every change in momentum, there is an equal and opposite change in momentum in another body.

Impressed forces might arise from the *vis inertiae*, innate force of matter, but that is not necessarily the case.

IS THIS LAW SELF-EVIDENT?

a. If A acts upon B, then A in turn is acted upon by B. That is not self-evident universally, but it might be self-evident if A and B are both bodies, since they can act only by contact.

b. If A pushes B, then B also pushes A. Again, if they are bodies, that seems fairly evident. (But maybe not exactly self-evident, since it requires that the pushed body *resists*, which is perhaps not in the idea of body or of "pushed.")

c. If A pushes B, then B also pushes A *in the contrary direction*. That adds something, but again, seems to be fairly evident.

d. If A pushes B *with so much acceleration in B*, then B also pushes A back, in the contrary direction, accelerating A, such that $m_Aa_A = m_Ba_B$. That is not nearly as evident! But it is subject to experimental verification. Also, it **stands to reason**, since "mass" is proportional to "resistance to acceleration." Hence one should expect masses to be inversely proportional to the accelerations, i.e.

$$m_1 : m_2 = a_2 : a_1$$

EXAMPLES. Imagine **2 books leaning** against each other, each pushing the other. Each pushes the other, but is pushed back by the other equally, so there is balance. A **spring scale** might be placed in line with a rope on which you and I are pulling, at opposite ends. The direction in which the spring scale is turned, whether toward me or toward you, will make no difference to the reading. A **bathroom scale** reads the same either way you turn it. How much do I weigh on the planet Earth? And how much does Earth weigh on the "Planet Me"? The same!

QUESTION. Newton's own example of a **horse** drawing a stone raises a question: How will the two actions not just cancel each other out with zero net motion?

Or how can a HAMMER bang in a nail if the nail bangs back on the hammer just as hard?

Well, the action of the hammer on the nail drives it into the wood, and the equal and opposite reaction of the nail on the hammer slows the hammer down to a stop.

Likewise, the action of the horse on the stone (via the rope) moves the stone (Newton's own example); and the equal and opposite reaction of the stone (via the rope) on the horse compresses its muscles, and slows it down, etc.

So too an ICE SKATER who pushes against a railing—she goes backward away from the railing. She acts on the railing (perhaps causing some little compression or vibration in it or some little acceleration in the large body to which it is fixed, e.g. the Earth), but it also acts on her.

So too a ROCKET in space goes forward because it thrusts gases backward, and they act back on it, throwing it forward.

So too a GUN does not kill the man shooting it. Why not? Because the "*m*" which is accelerated backwards by the explosion of the powder is not that of the bullet, but that of the gun + the mass of my whole body (if I hold the gun correctly and make one piece with it). But the bullet has a much smaller "*m*" than me, so it ends up having a much greater "*a*" than me.

GENERAL REMARKS ON THE LAWS

These are reminiscent of Ptolemy's system of the world, in which certain simple tendencies or motions are induced from certain data, and certain simplicities are presumed to underlie certain complexities, although none of these principles are truly *per se notum* or "axiomatic" in the ancient sense discussed earlier. The soundness of these principles, accordingly, is to be judged from their results, i.e. from their explanatory power and the harmony of all their consequences with the observed facts. So we appear to be beginning not from "axioms" so much as from very reasonable hypotheses suggested by certain preliminary observations.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 12

COROLLARY 1

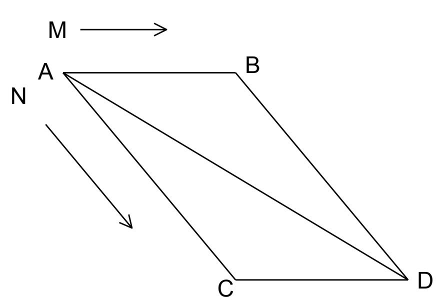
"PARALLELOGRAM OF FORCES"

A body acted on by two forces at once will describe the diagonal of their parallelogram in the same time as it would describe the sides by those forces separately.

Newton now attaches six corollaries to his three laws. They develop certain consequences of his laws, or else illustrating more concretely how they work.

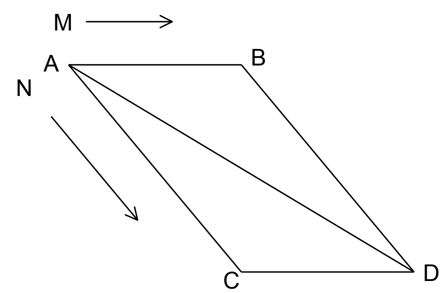
The first corollary is comparable to Heron of Alexandria's "Parallelogram of velocities," which can be found in Heath's *History of Greek Mathematics*, Vol. 2, p. 348. It is also similar to Galileo's procedure in *Two New Sciences*, in the 2nd Proposition of the Fourth Day. Only here we have forces or accelerations, as opposed to mere velocities.

If speed AB = 30 mph, and speed AC = 40 mph, and ∠BAC = 90°, what is the compound speed along AD?



Given: Body at A acted on just by force M instantaneously at A will go to B in time T.
Body at A acted on just by force N instantaneously at A will go to C in time T.

Prove: Body at A acted on by M and N at once will go to D in time T.



By the 2nd Law (an interesting application!), M will do nothing to alter the velocity caused by N, and vice versa, since they act in different directions, and each acts only in its own direction.

Therefore the body acted on by both, since it is acted on by M, must still reach the line BD in time T, and since it is acted on by N, must still reach the line CD in time T. So, after time T, the body will be at both line BD and line CD, hence it must be where they intersect, i.e. at D.

What about in between A and D? Where will the body be during that time? Along AD, because by the 1st Law a body travels only in a straight line after new forces stop acting on it.

So he is assuming an instantaneous action of forces M and N when the body is at A (they "knock" it), and hence there is instantaneous acceleration from rest to some uniform speed. OR he is assuming that M and N act continuously up till the body is at A, then stop acting right there.

So what has Newton done, here? He has proved that the parallelogram is a tool for constructing the net direction of a motion resulting from two simultaneous forces.

NOTE: This same diagram allows us to compose forces, distances, velocities, and accelerations.

Since the forces (accelerations) M and N entirely determine the motions from A to B and from A to C, and those motions take place with uniform velocities and in the same time, those velocities are proportional to the accelerations or forces:

Acceleration M : Acceleration N = Speed (A to B) : Speed (A to C)

But since those motions happen in the same time, therefore the speeds are as the distances,

i.e. Speed(A to B) : Speed(A to C) = AB : AC.

Hence

Acceleration M : Acceleration N = AB : AC

i.e. Accelerative Force M : Accelerative Force N = AB : AC

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 13

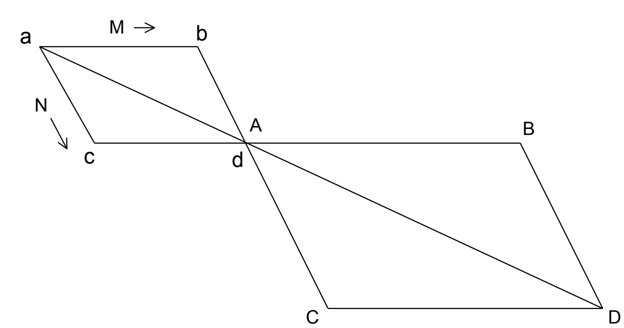
COROLLARY 2

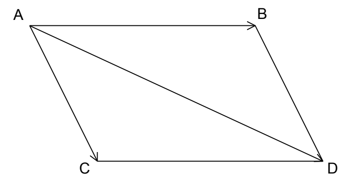
"Newton's Wheel"
or
"The Analysis of Elementary Machines"

> *And in this way, the composition of a direct force AD from any oblique forces at all, AB and BD, and also the resolution of any direct force AD into whatever oblique forces, AB and BD, becomes evident. Which composition and resolution are, in fact, abundantly confirmed from mechanics.*

Q1. How do we compose *forces*? Wasn't Corollary 1 about composing uniform motions? Wasn't AD a *distance* over which there was a compound uniform *motion*? But now he speaks of "direct force AD." What is he thinking of?

The distances AB, AC and AD are as the uniform speeds which cover them in the same time, as we saw. But the accelerations M (or *ab*) and N (or *ac*) are to each other as the speeds they produce, i.e. AB and AC. Hence, too, the compound acceleration *ad* will have to either component, say *ab*, the same ratio as compound speed AD to component speed AB. So we can compound accelerations (or "accelerative forces") by the same sort of diagram used in Cor. 1.





Q2. Given two forces, is the compound force unique? Given the magnitude of two forces, is the magnitude of the compound force unique? How is the magnitude of the compound force given?

• Yes, given two forces, their compound is unique and determined, since we are not given a force unless we are given both its magnitude and direction. Hence we get one parallelogram, one resulting diagonal, and one direction along it.

• But given the magnitudes of two forces, we are not yet given the compound force either in magnitude or direction. two straight line-lengths can be used to make an infinity of different parallelograms, and hence can be used to make an infinity of different diagonals, varying in both length and direction.

• But if the magnitudes of the two component forces are given in units of length, and the angle between them is given in degrees, then the magnitude of the diagonal is given trigonometrically. For example,

Let	$AB = 5$
	$AC = 4$
	$\angle BAC = 80°$
then	$AB = 5$
and	$BD = 4$
and	$\angle ABD = 100°$
but	$a^2 = b^2 + c^2 - 2bc\text{COS } A$ [Law of Cosines]
so	$AD^2 = AB^2 + BD^2 - 2\,(AB.BD)\text{COS } \angle ABD$
i.e.	$AD^2 = 25 + 16 - 2\,(20) \cos 100°$
so	$AD^2 = 41 - 40\,(-.173648177...)$
i.e.	$AD^2 = 41 + 6.945927107...$
so	$AD^2 = 47.945927107...$
so	$AD = 6.92429975...$

Q3. Given a force, is the pair of forces into which it can be resolved unique?

No—a given straight line is the diagonal of an infinity of possible parallelograms.

Q4. Can we compound three forces? Can we resolve a single force into three components? Can we do it in three dimensions? Does the order of vector-addition matter when we have three things to add?

To compound three forces in a plane, the first method that suggests itself is to use Newton's parallelogram-method to compound any two of them, and then compound the compound with the remaining one. And we can likewise resolve a single force into as many as we want, e.g. into three components. Simply resolve it into two, and then resolve one of the components into two. But can we do this in three dimensions as well as in two? And another QUESTION: does the ORDER in which we add make any difference in vector addition? Suppose we have a point P, and three vectors PA, PB, PC. The mere *definitions* of the following operations are not the same:

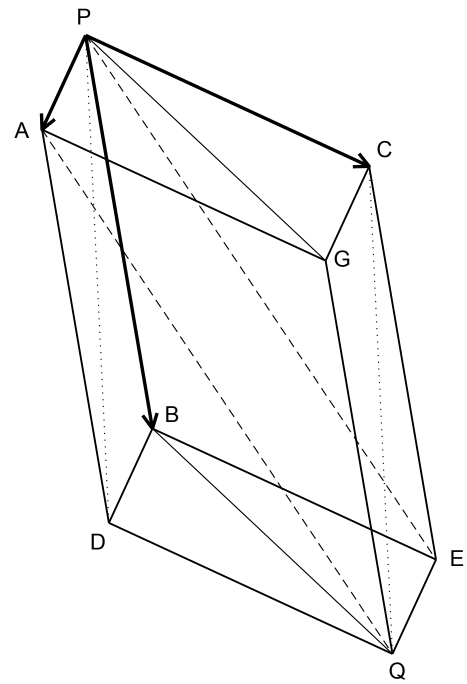
(PA + PB) + PC
(PA + PC) + PB
(PB + PC) + PA

We need to *prove* that the results of these are all one and the same vector, or else we have no unambiguous way of adding three vectors.

In a way, this is easier to see in three dimensions first. What follows is a proof of this, i.e. that the order of vector-addition makes no difference when we are given three vectors to add, whether in three dimensions or in two.

Given: 3 vectors PA, PB, PC
Prove: (PA + PB) + PC
and (PA + PC) + PB
and (PB + PC) + PA
are all the very same vector, i.e. in magnitude and direction.

First let the three vectors be not all in one plane, so that they enclose a solid angle.
Then complete the parallelepipedal solid by making parallelograms
APBD
BPCE
APCG
DAGQ
And join the diagonals
PD, CQ
PE, AQ
PG, BQ



1. Well,	PA + PB =	PD	(prlgrm. APBD)
2. and		PD + PC = PQ	(prlgrm. PCQD)
3. so	(PA + PB) + PC = PQ		
4. Again,	PA + PC =	PG	(prlgrm. APCG)
5. and		PG + PB = PQ	(prlgrm. PBGQ)
6. so	(PA + PC) + PB = PQ		
7. Again,	PB + PC =	PE	(prlgrm. BPCE)
8. and		PE + PA = PQ	(prlgrm. PAQE)
9. so	(PB + PC) + PA = PQ		

So the order in which we add the three given vectors makes no difference: the result is PQ every time.

Q.E.D.

PORISM: *For three vectors from a point, not all in one plane, the compound vector (or vector-sum) is the diagonal PQ of the parallelepipedal solid of which they form the solid angle.*

If we now take PA, PB, PC to be all in one plane, we simply collapse the same diagram into a two-dimensional drawing of parallelograms all in one plane, and the argument works the same way. So instead of making a solid, make the same parallelograms as before, this time all in one plane, and join the diagonals, and all the things called parallelograms in the proof will still be parallelograms. Therefore it is also true that the order of vector addition makes no difference in one plane.

Q.E.D.

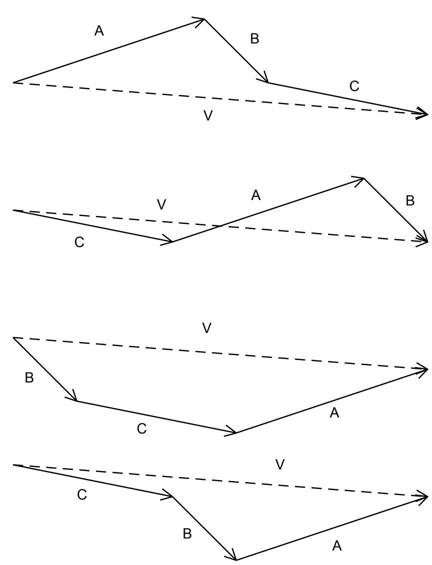
But this leaves us with the question whether we can add four or more vectors, and whether the order of addition will make any difference after three. We don't want to come up with a new and separate proof for each case, so instead we use a new method of adding vectors: the "tip-to-tail method."

Suppose we are given three vectors to add: A, B, C. Starting at any point P, we place the tail of A there, and then we place the tail of B at the tip of A, and the tail of C at the tip of B, and then we join the starting point P to the tip of C. Call this final vector, taking us directly from P to the end of the vector sequence, "V." Obviously if we follow the arrows (vectors) A, then B, then C, we get to the same spot from P as we do by following V . So if A, B, C are motions, or accelerations, or whatever, the result of putting them all together at once, or going through them all, is the same as going through just V. That is,

A + B + C = V (vector addition)

V is the vector-sum of all three given vectors.

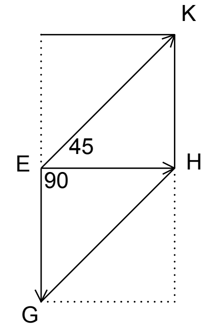
Plainly the order of addition makes no difference to the resulting vector-sum. If I go "10 feet north" first, then "5 feet west," then "3 feet South-East," I will end up in the same place as I would by going through these same steps in any other order. The paths will differ, but the final result will always be the same, and will define the same resultant-vector, whether in two dimensions or in three dimensions, and regardless of how many vectors are added (cf. the "City Block Theorem.").



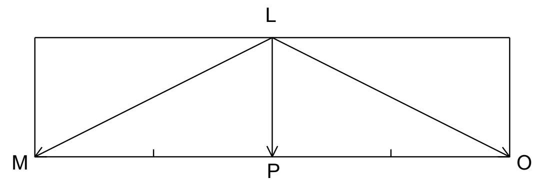
Q5. Will the compound force always be of greater magnitude than the component forces? When will the compound of two forces (which are equal to each other in magnitude) be equal to each component (in magnitude)?

No, the compound will not always be of greater magnitude than the components.

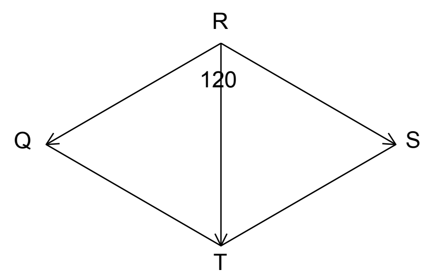
If ∠KEG = 135°, and EG = 1 and EK = √2, then EH, the compound (diagonal) will be 1, which is equal to one of the components, and less than the other.



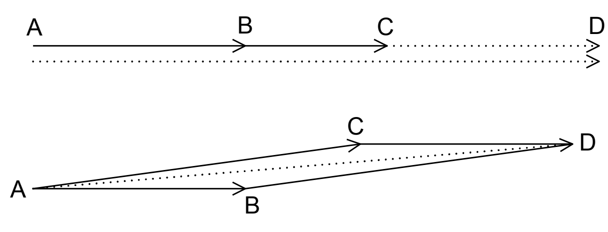
If LM and LO are the diagonals of adjacent rectangles whose common side LP is to the remaining sides MP, PO in the ratio of 1:2, then the compound of LM and LO will be LP, which is 1, whereas each component is √5. So the compound is of much less magnitude than either component.



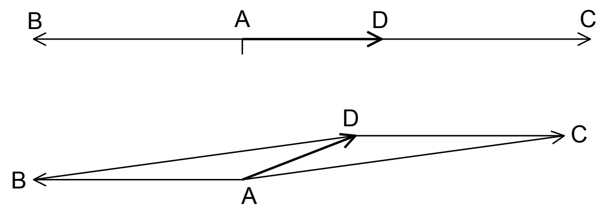
If ∠QRS = 120°, and QR = RS, then the compound force, RT, is equal to each component.



What happens if we want to compound forces AB and AC, and they happen to lie in a straight line, and have the same direction? By looking at cases close to a straight line, i.e. very flat parallelograms, we see that the opposite side CD is always equal to AB; so in the "flat" case, we simply add CD equal to AB, and the resulting AD is the compound force (i.e. just use the **tip-to-tail** method):



And if AB and AC lie in a straight line, but have opposite directions? Then we see that the opposite side of the parallelogram DC must always equal AB, so that in the "flat" case we simply subtract from AC a portion CD = AB, and the resulting AD is the compound force (again, just use **tip-to-tail**):



Q6. What is Newton doing with the wheel, etc., in the explanatory part of the Corollary? Is he proving Cor. 2?

(See NOTES below)

Q7. Given that the weights A and P in Newton's diagram are balancing the wheel, can we prove Archimedes' law of the lever, just using Newton's Laws and Corollaries 1 and 2 ?

(See NOTES below)

NOTES ON COROLLARY 2

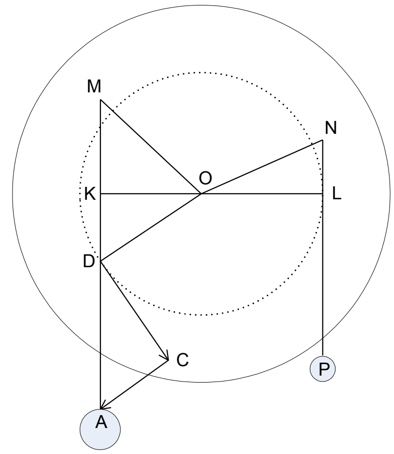
(1) Corollary two states that we now know, by Cor. 1, how to COMPOSE two forces into a new force which will produce the same result as the separate ones do when they work together. Now he adds that we can also RESOLVE any one given force into component forces which produce the same result (when they work together) as the single given one does.

(2) Given two forces, there is only one force which alone will accomplish the same. But given one force, there is an infinity of pairs of forces which will accomplish the same. So the decomposition of forces allows great freedom, and is somewhat more arbitrary or conventional. But, given one actual force, and one other into which to resolve it, the remaining component is determined.

(3) He says that mechanics provides abundant evidence of this.

(4) Newton shows how the composition and analysis of forces gives us a way of calculating the forces of wheels, levers, wedges, mallets, screws, inclined planes, and more complex machines composed of these, including **animals**. He shows us how to determine the forces needed to move a simple machine, or the forces which they exert on other bodies. (Such principles are used, by the way, in certain branches of physical medicine.)

(5) NEWTON'S WHEEL. We will focus just on the *wheel*. It doesn't matter whether it is a vertical wheel moved by hanging weights, or a horizontal wheel moved by shoves. As a vertical wheel, it is really just a balance. It seems here that Newton is ***deriving Archimedes' Law of the Lever (or Balance)*** **from his simple Laws of Motion**. This shows that they are not only consistent with ancient and well-verified mechanics, but also that they give reasons for those ancient ideas.



Given: Wheel with center O
Weight A hung from M
Weight P hung from N
Wheel balanced (this is just one scenario)
KOL horizontal

Prove: A : P = OL : OK

Suppose OL > OK (if OL = OK, obviously P = A to cause balance)
Therefore the circle of radius OL around center O will cut MA below K, as OD.

Newton says that the effect of weight A (on the wheel) is the same anywhere along MA, e.g. at K or at D. So he lets it be attached at D (in order to resolve it into a force tangent to the circle of radius OL). Again, let P be attached at L (for the same reason).

Now, draw tangent DC, and draw AC perpendicular to DC.

Therefore, if we let DA represent the force of weight A straight down, we see it has the same effect as the forces DC and CA together (Cors. 1-2), of which CA does nothing to turn the wheel, and therefore the turning power of weight A is as DC.

But since DC and weight P both act at equal distances from the center (the "obvious" case we dismissed at the beginning), and both at 90° to the radii, therefore both are equally effective in turning the wheel. Hence, since the wheel is balanced, it follows that force DC is equal to the force of weight P.

So: Weight A is as DA,
Weight P is as DC,
and therefore balance happens if
wt.A : wt.P = DA : DC

But $\triangle$ADC is similar to $\triangle$DOK, since each is right, and $\angle$ODC is right, and KDA is one straight line.

Thus DA : DC = OD : OK
or DA : DC = OL : OK [OD = OL]
so balance happens if
wt.A : wt.P = OL : OK

"which is the well-known property of the balance, lever, and wheel".

Q.E.D.

This is a confirmation of Newton's Laws of Motion (and his Definitions), since these are implied in the composition of forces parallelogram (and he relies on Law 1 and Law 2 in his Cor. 1).

Newton next complicates matters by letting weight P be partly supported by an inclined plane pG, causing the string to move aside from the vertical. This will show the power of his method to solve problems not solvable by the simple law of Archimedes. Archimedes needed things to be nice and straight down, but Newton doesn't. He is showing the universal applicability of his Laws, Definitions, etc.

(But perhaps we should skip this more complex case, so long as we understand what he is doing in a general way, and why.)

At the end of this Cor. 2, Newton mentions the other "elementary machines":

wedge
inclined plane
mallet
screw
wheel
drum
pulley
stretched cord
weights (ascending or descending)

He is saying, in effect: "If you accept these things, you should accept my first two Laws, since those Laws give simple, elementary, universally applicable reasons for all the rules by which these machines function."

And he says that "From this corollary" (of resolution of forces into components) "are easily derived the forces of machines which are usually composed out of [all the elementary machines just listed]." So, understand the Laws, and you will understand all simple machines, and thus you will understand all machines whatsoever, since they are either simple, or else composed of the simple.

And then he mentions "the forces of tendons for moving the bones of **animals**." He is showing that his Laws are not limited to artifacts, but apply also to natural "machines."

In the *Principia*, however, he is not going to be interested in the analysis of machines (unless the universe is a machine). He is bringing these up here in order to show the fruitfulness, utility, and explanatory power of his Laws.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 14

COROLLARY 3

"Conservation of Momentum"

> *The [total] quantity of motion (which is obtained by taking the sum of motions in the same directions, and the difference of motions in contrary directions), undergoes no change from the action of bodies among themselves.*

(1) This conservation law (Newton says) follows from Laws 2 and 3, concerning equal and opposite action and reaction. It was discovered experimentally in the mid-17th century, shortly before Newton's time. So this is another way of confirming his Laws, showing their relevance and truth.

(2) If we are not talking about a closed system of bodies, then the conservation law does not hold. If, for example, we look only at bodies A, B, C, the total momentum among them might increase or decrease because of another body D, acting on one or more of them, but which we have failed (or refused) to include in our considerations.

(3) Even within a closed system, momentum is not strictly conserved, but energy is. Momentum can be 'stored' in different forms of energy. But insofar as energy must exist as kinetic energy, as motion, momentum is conserved.

(4) QUESTION: If Body A has 5 units of momentum to the right, and Body B has 10 units of momentum to the left, and they are not on a collision course, what is the total momentum in the system, 5 or 15? (Newton is plainly thinking of vector addition, so the total would be "5 to the left." Oblique motions cannot be added except by vector addition.)

(5) QUESTION: Does this mean nothing can ever slow down? (It means nothing can slow down without something else picking up the lost motion.)

(6) QUESTION: Can a body give to another more momentum than it has itself? (Yes. A body at rest in absolute space has 0 momentum. But it can give [or maybe it is better to say "take"!] new momentum to another which crashes into it.)

(7) We can prove this Conservation Law in a simple case, i.e. for two bodies colliding head on, and assuming perfect elasticity (so no momentum is lost by being transformed into some other kind of energy). Let perfectly rigid Sphere A collide with perfectly rigid Sphere B, head on. By the 2nd Law, an impressed force is proportional to (or is equal to or measured by) the change in the quantity of motion resulting from it, i.e. by the change in momentum produced. So consider the force exerted by Ball A on Ball B (written F_{AB}):

$$F_{AB} = m_B v_B - m_B v'_B$$

i.e., the force exerted by A on B in the head-on collision is equal to the difference between B's initial momentum (prior to collision) and B's resulting momentum (after collision). Since B's mass does not change (we assume), the only new thing is B's velocity. Likewise the force of B on A can be written (by the 2nd Law):

$$F_{BA} = m_A v_A - m_A v'_A$$

Now by the 3rd Law, these forces are equal and opposite. Hence

$$F_{AB} = - F_{BA}$$

i.e. $m_B v_B - m_B v'_B = - (m_A v_A - m_A v'_A)$

or $m_B v_B - m_B v'_B = - m_A v_A + m_A v'_A$

i.e. $m_B v_B + m_A v_A = m_B v'_B + m_A v'_A$

...which is the law of the conservation of momentum.

Since the 3rd Law is pretty much a logical equivalent of the Conservation of Momentum, this reasoning is not so much a proof of the Conservation Law as a show of equivalence and consistency—Newton is connecting his Laws to other recognized and venerated ideas. Since we can work these last four steps in reverse, we see that the 3rd Law is entailed in the Conservation Law (and this is perhaps how Newton first discovered the 3rd Law), and we see that **Descartes** was inconsistent in affirming the Conservation Law while effectively denying the 3rd Law (with his odd rules of collision).

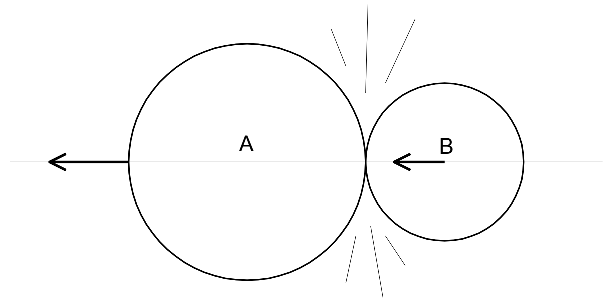
(8) Suppose our Spheres have unequal masses and unequal (initial) velocities. Can we say what the new velocities are by this law? For example, let A have a mass of three and a velocity of two, while B has a mass of two but a velocity of three. What happens when they run head on? What will the new velocities be? Newton does not get into this here. What we know, by this Corollary, is only this:

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2$$

We know that the **sum** of the momentums in the system is a constant, but how the individual velocities change has not yet been explained here. Nor is he interested in getting into details on this, since he brings it up only as a sign of the validity of his laws.

How does Newton illustrate the conservation law in the examples with 2 SPHERES colliding head-on?

Suppose we have two spheres, A and B, and



$m_A = 3$
$m_B = 1$
$v_A = 2$
$v_B = 10$

thus $p_A = m_A v_A = 3 \times 2 \quad = \quad 6$
and $p_B = m_B v_B = 1 \times 10 \quad = \quad 10$

Therefore in the whole system, prior to collision, the total momentum is 16, since the motion of the whole is the vector-sum of the momentums of the parts (as he stated in his definition of "quantity of motion" or momentum).

NOTE: Thus if mass in a closed system is constant, so is net velocity.

Now, AFTER COLLISION,

suppose $p_A = 6 + 3 = 9$ [a gain of 3]
then $p_B = 10 - 3 = 7$ [a loss of 3]

And so, in the whole system, after collision, the total momentum is **still 16**.

Or, AFTER COLLISION,

suppose $p_A = 6 + 12 = 18$ [a gain of 12]
then $p_B = 10 - 12 = -2$ [a loss of 12]

And so, in the whole system, after collision, the total momentum is **still 16** (in the original direction in which A and B were moving, though now B is moving with a velocity of 2 in the contrary of its original direction).

Note that the mere initial momentums and the rule of conserving total momentum do not enable us to say what the individual new momentums will be. But if we assume what one of them will be, we can say what the other will be.

But if we know the individual resulting momentums, then we also know the individual resulting velocities (assuming the masses have not changed). For instance,

if $p_A = 18$ after collision, as in the last case,
then $18 = m_A v_A = 3 v_A$ [since $m_A = 3$]
thus $v_A = 6$

What does Newton say about cases in which the bodies are NOT SPHERES, or which COLLIDE OBLIQUELY? What are the steps to finding the new motions resulting from the collisions?

He is very sketchy here, but gives us two basic steps if we wish to find the momentums after collision when we are given the prior momentums of the two bodies (and, presumably, the resulting momentum of one of them).

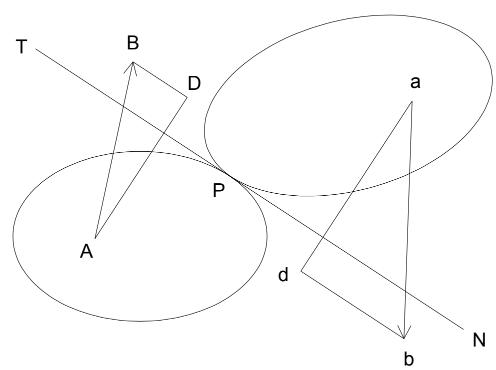
(1) FIND THE ORIENTATION OF THE PLANE TANGENT TO BOTH BODIES AT THE POINT OF COLLISION. He does not explain what this means, or define this plane. What if two cubes meet at their corners? What is the plane "tangent" to each of them through that point? He does not say.

(2) DECOMPOSE THE MOTION OF EACH BODY INTO TWO COMPONENTS: ONE PERPENDICULAR TO THE PLANE, THE OTHER PARALLEL TO IT. This is a geometry problem, and he does not explain how to solve it. But he says that the components acting at right angles to the plane are the ones which we must add or subtract in order to determine the new momentum in the body for which this is unknown.

He excuses himself from further consideration of this, saying it is a long and tedious thing to get into the details.

If two ideally elastic and simply convex solids with centers of mass A, *a* and momentums as AB, *ab* collide at P, then the way to determine the results of their collision is:

(1) Draw TPN, the plane tangent to both solids at P.
(2) Drop AD and ad at right angles to TPN.
(3) Draw DB and db parallel to TPN.
(4) Whatever the vector-sum (AD + ad) is before collision, it must also be after collision, although each of the components AD, ad will be different. But components BD, bd will be unaffected (by Laws 1 and 2 and Cors. 1 and 2).



THE NEWTONIAN ART OF CLASSICAL PHYSICS

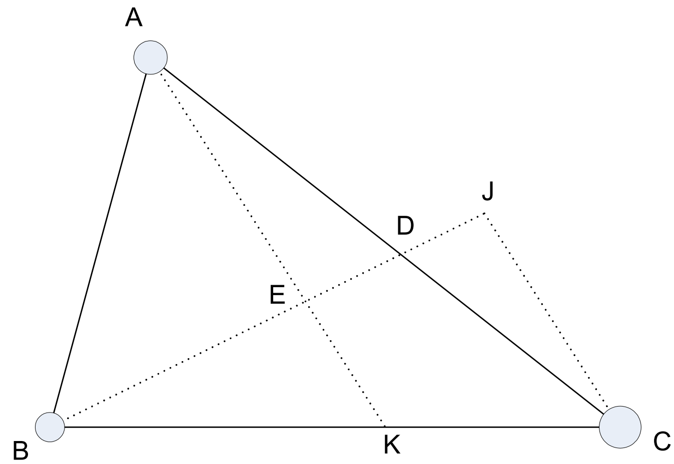
CLASS 15

COROLLARY 4

The common center of gravity of two or more bodies does not alter its state of motion or rest by the actions of the bodies among themselves; and therefore the common center of gravity of all bodies acting upon each other (excluding external actions and impediments) is either at rest, or moves uniformly in a straight line.

• Newton gives a sketchy geometrical argument for this Corollary after enunciating it, letting two points move with uniform motion along straight lines, and saying their center of gravity will divide the line between them in a fixed ratio, and hence either sit still, or move uniformly in a straight line.
• This is his first use of "point-masses." We will continue to speak of moving masses as points for quite a while.
• He refers to Lemma 23 (which we will not read), in which he proves this Corollary more generally.
• It is fairly obvious anyway, but we will explain it more fully below. Lemma 23 does not apply in any obvious way to 3-D cases in which the motions of two bodies are not in the same plane, so we will not bother with Lemma 23.
• One can now add a third body, and if it is moving uniformly in a straight line or resting, so will be the common center of gravity between it and the center of the previous two. And so on, *ad infinitum*.
• But you can also take Cor. 4 as a CASE OF LAW 1 in which the "body" happens to be a crowd of bodies, and the thing moving by its inertia is the center of mass of all the bodies. Still, he prefers to argue for this, and concludes at the end: "There is, therefore, the same law for a system of many bodies as for solitary bodies, as far as perseverance in its state of motion or of rest." i.e. LAW 1 applies equally well to systems as to individual bodies.
• Even a single body, like a thrown wrench, behaves like a "system" of many bodies in which only the common center of gravity either rests or moves in a straight line uniformly, in the absence of impressed forces, as when you slide it across a friction-free surface and the parts of the wrench orbit the center of weight, which point is the only one moving uniformly in a straight line. If you slide a wrench across a table-top, and give it some spin, the center of mass of the wrench will not spin, and it will move in a straight line. The only reason it slows down is because of the friction of the table top and wind resistance.

NOTE ON THE COMMON CENTER OF WEIGHT
FOR 3 OR MORE BODIES



Suppose we have three weights A, B, C and we want to find their common center of weight. With Archimedes we learned how to find the center of weight for one triangle which is made of uniformly heavy material (namely by finding the point of intersection of the three lines dropped from the vertices to the midpoints of the opposite sides), and also for two bodies (namely by finding the point dividing the distance between the centers of weight for each body in such a way that the segments of this distance are inversely proportional to the weights; i.e. the Lever Law). But how do we find the center of weight for three weights at different distances from each other? (Let us suppose the centers of weight for each body are identified at points A, B, C.)

Well, one method suggests itself. We can find the center of weight of A and C by taking point D such that

A : C = CD : DA

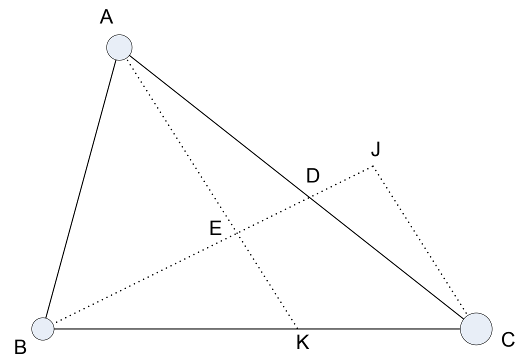
and then we can join BD, and divide it at E so that

B : D = DE : EB

Since point D "weighs" the same as A + C, as it were, because it is the common center of their weight, then it seems logical to conclude that E is the common center of weight for all 3 bodies, since it is the common center of weight for B and D.

But this should make us wonder whether the ORDER OF PAIRING WEIGHTS makes any difference. What if we found the common center of weight for B and C first, and then found the common center of weight for this center and the remaining body A? Would we still get point E as the common center of weight for all 3 bodies? Put another way: If we join AE and produce it to K on BC, will K be the common center of weight for B and C? And will E be the common center of weight for A and K ?

It will. But this requires proof. So here it goes.



Grouping A and C first, then

Given:	A : C = CD : DA	
	B : D = DE : EB	(D = A + C)
	AE joined and extended to K	
Prove:	B : C = CK : KB	
	B + C : A = AE : EK	

Note that E is the "crotch" of a Menelaos figure (see the Ptolemy course for a proof of the Menelaos Theorem).

Hence	CK : KB = (DE : EB) c (AC : DA)	[Menelaos Theorem]
But	CD : DA = A : C	[Given]
so	CD + DA : DA = A + C : C	[Componendo]
so	AC : DA = A + C : C	
And	DE : EB = B : D	[Given]
so	CK : KB = (B : D) c (A + C : C)	[Substituting from above]
so	CK : KB = (B : A + C) c (A + C : C)	[weight D = A + C]
so	CK : KB = B : C	

Hence K is indeed the common center of weight for weights B and C.

Again	CD : DA = (KE : EA) c (BC : BK)	[Menelaos Theorem]
so	AE : EK = (DA : CD) c (BC : BK)	[rearranged]
But	CK : KB = B : C	[just proved]
so	CK + KB : KB = B + C : C	[Componendo]
i.e.	BC : BK = B + C : C	
And	DA : CD = C : A	[Given]
So	AE : EK = (C : A) c (B + C : C)	[Substituting from above]
so	AE : EK = B + C : A	
i.e.	AE : EK = K : A	[weight K = B + C]

Hence E is indeed the common center of weight for weights K and A.

So E is once again the center of weight for A, B, C even though this time we began by finding the common center of weight for B and C, rather than for A and C.

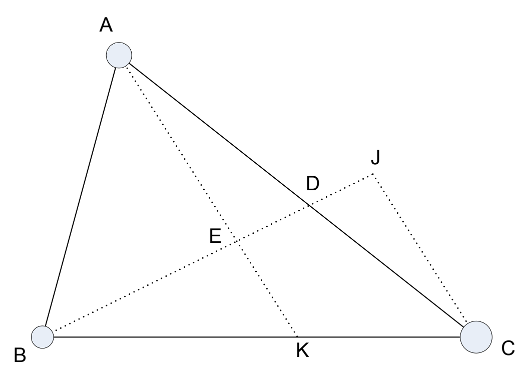
Hence it makes no difference, when we are looking for the common center of weight for three bodies, which two we pair up first.

Q.E.D.

I hate trying to remember the Menelaos Theorem, so I end up reproving it all the time. In this case, extend BD until it meets CJ drawn parallel to AEK, and we reason like this:

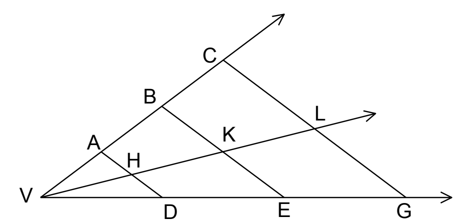
	CK : KB = JE : EB	[parallels]
so	CK : KB = (DE : EB) c (JE : DE)	[DE new term, compounding]
i.e.	CK : KB = (DE : EB) c (DE + DJ : DE)	
Now	DJ : DE = DC : DA	[similar triangles]
	DE + DJ : DE = DA + DC : DA	[componendo]
so	CK : KB = (DE : EB) c (DA + DC : DA)	
so	CK : KB = (DE : EB) c (AC : DA)	

Q.E.D.



Now let's take up some of the cases of Cor.4, and verify them one at a time:

(1 Body) Consider first the case of a single body, although it has many parts. If you throw an axe, it spins end over end in the air, but if you watch it in slow motion, the center of its weight (around which it spins) describes a nearly straight line, curving down only because of the outside influence of gravity. Or if you slide a wrench across a flat table top, it may spin, but it spins around its center of mass, which moves in a straight line. This is, in a way, just an application of Law 1: Once you stop acting on the wrench, it moves uniformly in a straight line—and although not every part of the wrench does, there is one point associated with the wrench which does, namely its center of mass. All the other points, too, are trying to move uniformly in straight lines, but they are drawn back around the center of the wrench's mass by the cohesive forces among its parts, a bit like planets in orbit.



(2 Bodies, not colliding or attracting, moving to/from a point simultaneously occupied) Now consider two bodies, but not acting on each other, and let their 2 motions lie in one plane, intersecting at V, and let them begin from (or arrive at) V at the same time. Suppose the one body moves uniformly through the equal distances VA, AB, BC in three successive and equal time increments, while the center of gravity of another body moves uniformly (though with a different speed, if you like) through the equal distances VD, DE, EG in the same increments of time. Then the common center of gravity for both bodies taken together must either be at rest or travel in a straight line.

Obviously if VC and VG are equal and in contrary directions and the bodies have equal masses, then V will be the common center of weight, since it bisects the distance between the two centers of weight, and in that case it will be at rest.

But suppose instead CVG forms some angle. Join AD, BE, and CG. Obviously the common center of weight, when the bodies' individual centers are at A and D, must lie along the line AD, say at H. Similarly, when the centers are at B and E, the common center must lie along BE, say at K. And when the centers are at C and G, the common center must lie along CG, say at L. But given the Archimedian property of the balance, the mass of the body with its center at A must be to the mass of the body with the center at D as DH is to AH. So too EK : BK and GL : CL must be in the inverse ratio of the masses of the bodies. But that ratio is constant. So VHKL is a straight line. Hence the common center of weight moves in a straight line. Also, by the parallels, VH = HK = KL, so the common center is also moving at uniform speed.

(3 Bodies whose motions are all to or from a point) Adding a third body into this same type of system will make no difference to this rule, nor even to the argument, really, so long as the three motions have a common point of intersection which the bodies did (or will) occupy at the same time. Since H, K, L locates the common center of our original two bodies, it is the same as a center of weight of one body, of which our original two are considered parts—so adding a third body is just like the two-body problem, taking the original two as one.

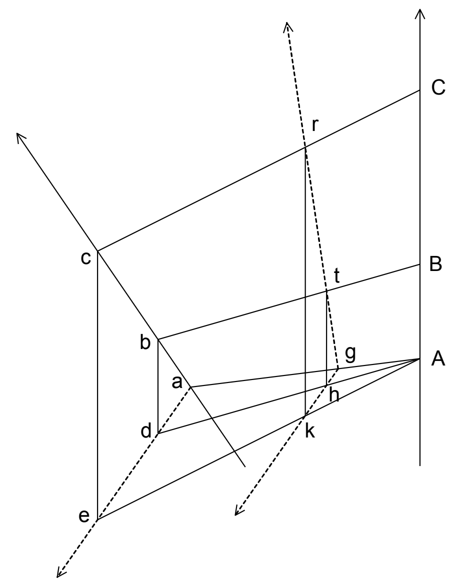
(Not Simultaneously at Point of Intersection) But what if the bodies are not simultaneously at the point where their lines of motion intersect? Or what if there is no point at which their lines of motion intersect, but they are askew? Newton handles these cases by referring us to his Lemma 23 in Book 1 of the *Principia*; but his argument there is terse and difficult, and needlessly so. Also, it is not at all clear how his argument there will apply to the non-coplanar case, since his Lemma seems to rely upon a point of intersection for the lines of motion. So I prefer to demonstrate this case as follows:

Given: One body of mass M is at A, then B, then C with ABC collinear
when the other, of mass m, is at a, then b, then c with abc collinear
and each body is travelling at uniform speed (hence AB : BC = ab : bc)
(ABC and abc can either intersect or not; it makes no difference)

Prove: The common center of mass is also travelling in a straight line at uniform speed.

Join Aa, Bb, Cc.
Draw ce equal and parallel to CA, thus completing parallelogram CceA.
Draw bd equal and parallel to BA, thus completing parallelogram BbdA.

Now ce : bd = CA : BA [ce = CA, bd = BA]
so ce : bd = ca : ba [ca : ba = CA : BA, from the givens]
but ce is parallel to bd [each is parallel to ABC]
so △abd is similar to △ace
so ade is collinear, and ad : ae = ab : ac



Join Ad, Ae.
Cut Aa at g so that Ag : ga = m : M
Cut Ad at h so that Ah : hd = m : M
Cut Ae at k so that Ak : ke = m : M

Thus g, h, k collinear, and ghk is parallel to ade.
Thus gh : gk = ad : ae = ab : ac = AB : AC

Now in parallelogram ABbd, draw ht parallel to AB, cutting Bb at t.
In parallelogram ACce, draw kr parallel to AC, cutting Cc at r.

Thus ht : kr = AB : AC [ht = AB, kr = AC]
so ht : kr = gh : gk [gh : gk = AB : AC, shown just above]
but ht is parallel to kr [each is parallel to ABC]
so △ght is similar to △gkr
so gtr is collinear, and gt : gr = gh : gk
i.e. gt : gr = AB : AC [since gh : gk = AB : AC]

But g is the common center of mass when the bodies are at A, a
since Ag : ga = m : M [construction]

and t is the common center of mass when the bodies are at B, b
since Bt : tb = Ah : hd = m : M

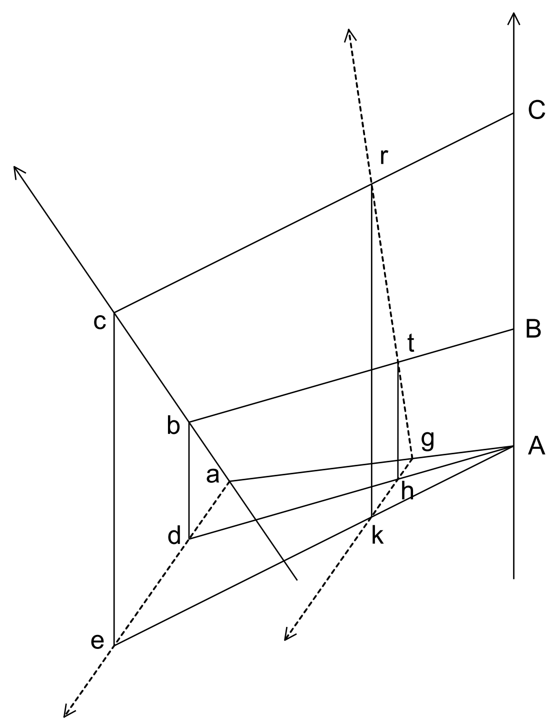
and r is the common center of mass when the bodies are at C, c
since Cr : rc = Ak : ke = m : M

Hence g, t, r are the three locations of the common center of mass when the bodies are at A, B, C and a, b, c respectively; and g, t, r have been shown collinear, and also gt : gr = AB : AC, i.e. gt : tr = AB : BC, and therefore the common center of mass is moving uniformly in a straight line.

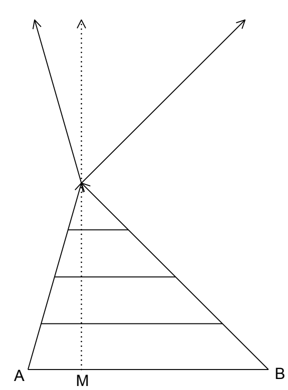
Q.E.D.

PORISM:

Since Ad is equal and parallel to Bb,
and Ae is equal and parallel to Cc,
thus Aa, Ad, Ae show us how mass *m* appears to move from the point of view of mass *M*, if mass *M* considers itself at rest. Hence *ade* describes the motion of mass *m* relative to mass *M*. But this shows that *when two bodies are moving uniformly in straight lines with reference to absolute space, each is also moving uniformly in a straight line relative to the other considered at rest.* And by the exact same argument we can say, conversely, *when two bodies are such that one is moving relative to the other uniformly in a straight line, and the other is moving uniformly in a straight line in absolute space, then the one is also moving uniformly in a straight line in absolute space*. (Resting is also an option, of course.) Accordingly, since *ghk* describes the motion of the common center of mass relative to mass *M*, and this is uniform motion in a straight line, therefore it follows that the common center of mass is also moving uniformly in a straight line in absolute space (motion gtr).



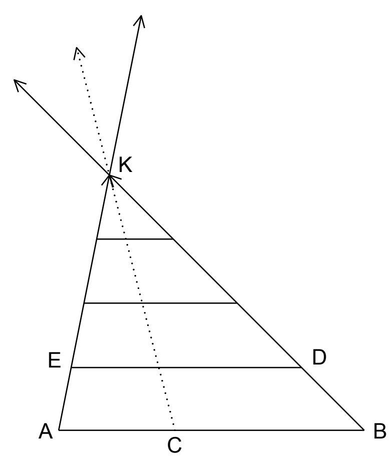
(Colliding Bodies) Even if the bodies in our system collide and change direction and velocity and the like, the common center of mass of all of them must either be at rest, or must be in uniform motion in a straight line, and cannot change from one of these "states" to the other (unless something outside the system acts on the bodies in the system). We can see this by Cor. 3, the Conservation of Momentum, which says the total momentum of the whole system must always remain the same, regardless of collisions. We can also see it by Law 1, as hinted at in the enunciation of Cor. 4: If things don't change their "state" unless acted upon,



then the common center of mass in a system cannot change its state, either, unless something from outside the system is introduced.

(Easy case: if angle of Incidence = Reflection) If two bodies, A and B, move uniformly in straight lines on a collision course, then their center of mass M does, too. If it happens that the bodies, after colliding, retain their former speeds, and each of them bounces off the line traced by their center of mass (produced) so as to hit it and bounce off it with equal angles, then the center of mass itself must continue in its original line and at its original speed.

(Easy case: if momenta of the bodies are equal) Also, if $m_A v_A = m_B v_B$, then A and B will make the same angle with the line of the center of mass, even if $m_A > m_B$ and $v_A < v_B$. For: AE is as the velocity of body A, and BD is as that of body B, and AC is as the mass of B, and BC is as the mass of A. So if the momentums are equal, it follows that AC·BD = BC·AE, i.e. AC : AE = BC : BD. But since AK and BK are just n times AE and BD, it is also true that AC : AK = BC : BK. But from this it follows that ∠AKB is bisected by KC (Eucl. 6.3)

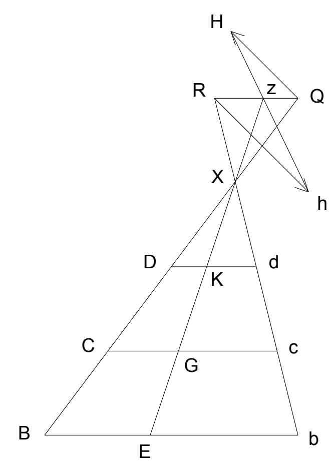


But now let's not make any such suppositions, but simply let our two bodies, B and b, collide at X. Each body has a uniform velocity up to X, and in equal time increments B is at B, C, D, X, and so would be at Q in the next increment if there were no collision, and b is at b, c, d, X, and would be at R if the two bodies passed through each other at X like ghosts. Let the common center of mass be at E, G, K, X in the same time increments, so that EGKX must also be collinear and EG = GK = KX.

Given:	$m_B : m_b = bE : EB = 3 : 2$
	$v_B : v_b = BC : bc = 5 : 4$
	B after collision is not at Q, but at H
Prove:	The common center of mass after collision continues to move uniformly in the same straight line, at the same speed, as before collision

Since B would have been at Q were it not for the collision, but it is actually at H, then vector QH is representative of the change in its velocity due to the collision. Hence the vector QH is as the acceleration of body B due to collision.

Now Law 3 tells us that



$$m_B a_B = -\, m_b a_b$$

So $\quad m_B[QH] = -\, m_b a_b$

so $\quad a_b = -\dfrac{m_B}{m_b} QH$

so $\quad a_b = -\dfrac{bE}{BE} QH$

so $\quad a_b = -\dfrac{3}{2} QH$

So now draw Rh parallel to QH and equal to 3/2 QH. Therefore the vector Rh is as the acceleration of body b due to collision. Hence, instead of being at R after one time increment following collision, b is at h. So when B is at H, b is at h.

Now cut Hh, the distance between the bodies, at Z so that

$$HZ : Zh = m_b : m_B = 2 : 3$$

Hence Z is the location of the center of mass one time increment after collision.

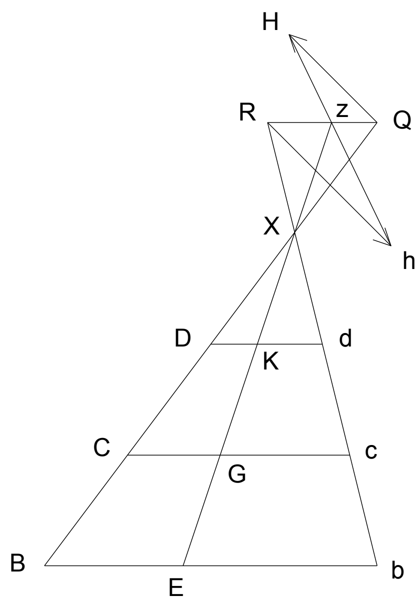
But now, since

	$HZ : Zh = 2 : 3$	
and	$QH : Rh = 2 : 3$	
thus	$QH : HZ = Rh : Zh$	
but	$\angle QHZ = \angle RhZ$	(since Rh is parallel to QH)
so	$\triangle QHZ$ is similar to $\triangle RhZ$	
so	R, Z, Q are collinear	

Also $\quad RZ : ZQ = HZ : Zh = 2 : 3$

so that Z is also where the center of mass *would* have been *without* collision. Therefore the center of mass after collision continues to move uniformly along the same straight line, and at the same speed, as it was moving on prior to collision.

Q.E.D.



NOTE: There is a certain indeterminacy in Newton's Three Laws of Motion. They do not enable us to say how two given bodies will react after a collision. This is in part because they tell us nothing about the rules governing the cohesion of bodies. Will the two colliding bodies be smashed to bits? Or will they stay together and each bounce off in some direction? The Three Laws do not tell us. Again, even if we suppose the two colliding bodies will not be destroyed or deformed, or stick together in a clump, but will instead bounce back in some direction, nothing about the laws specifies the direction. Law Three tells us only that *given* the original directions and masses and velocities of the two bodies, and given the resulting direction of one of them, we can determine the direction of the other. But we cannot, from the initial conditions prior to collision, and from the Three Laws alone, determine the outcome of the collision.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 16

COROLLARIES 5 AND 6
AND SCHOLIUM TO THE LAWS

COROLLARY 5

> *The motions of bodies included in a given space are the same among themselves, whether that space is at rest, or moves uniformly forwards in a straight line without any circular motion.*

In other words: If you are on a plane or ship that moves uniformly in a straight line, you can drink a hot cup of coffee and not spill it all over yourself.

If you are inside a big box, you can't tell, by any number of physics experiments performed inside there, whether the box is sitting still in absolute space or moving uniformly in a straight line, even if you are given that it is doing one of these.

COROLLARY 6

> *If bodies, moved in any manner among themselves, are urged in the direction of parallel lines by equal accelerative forces, they will all continue to move among themselves, after the same manner as if they had not been urged by those forces.*

In other words: If you are acted on by forces accelerating every particle of your body at the same time, in the same direction, and with the same power or "umph," you won't feel a thing. You can drink that cup of coffee and not worry about spilling it all over yourself. There will be no motion of your parts relative to each other due to these forces.

If I am twirling a rock around my head with a string, that motion will not be affected at all by equal forces acting on all parts of me, the string, and the rock, in parallel lines in the same direction.

Cor. 6 will be important in certain simple-case three-body problems, e.g. the Sun-Earth-Moon problem. For certain considerations, we can think of the Moon or the Earth as being accelerated more or less uniformly toward the Sun along parallel lines, given the distance of the Sun from the Earth-Moon system.

QUESTION: When an elevator plummets, aren't you in this situation, i.e. where every part of you and everything in the elevator is accelerated equally in the same direction, and yet you do feel something? Don't you feel one part of your body press against another? Why is that? Is that because of something like what happens to Wile E. Coyote? When he falls, his body goes down first and his neck stretches; then his head follows. This is reminiscent of the "spaghettification" you would experience as you get sucked into a black hole.

Indeed the lower parts of a human body are closer to the center of the earth than the upper parts of the body, and therefore do get accelerated faster than the upper parts do, as we shall see—but not to a sufficient degree that the lower parts are *noticeably* accelerated faster. It is not a difference that you can feel. So the reason we have the experience of our stomach lurching up as we plummet down, as if our legs were leaving our stomach behind, must be something else. The reason seems to be that we began by standing on a platform that pressed against our legs, and our legs pressed against our hips, our hips against our guts etc., so that we are always feeling a gentle pressure from the supportive parts of our bodies below. When we leave the pressing floor, or the floor stops pressing against our feet, that pressure departs, and we feel it. But we do not continue to feel it. If we begin in a weightless environment, and then come within the influence of a gravitating body, we would feel no such lurch. Instead, all parts of your body would be accelerated at more or less the same rate and in pretty much the same direction, so that there would be no compression or stretching of your body parts. You would not feel a thing.

SCHOLIUM

Here Newton confirms his laws by experimental results, and a little bit by authority, too, citing other eminent physicists and their findings, and how they agree with this laws. This further testimony to his laws falls into four parts:

(1) The testimony of Galileo, and the discovery of the parabolic path of projectiles.

(2) The testimony of Christopher Wren, John Wallis, Christian Huygens, and the laws of pendulums.

(3) The laws of mutual attraction among bodies, e.g. heaviness and magnetism.

(4) The laws governing simple machines.

He is particularly interested in showing the THIRD LAW is true, as one sees in his preoccupation with it in (2), (3), (4). The first two laws are ratified by Galileo, but then again they were also enunciated, or nearly enunciated, by Galileo himself, and by Descartes, and others. The 3rd Law (equal and opposite reaction) seems to be Newton's most distinctive contribution. At any rate, he treats it as something more novel and more in need of manifestation and support.

(1)
GALILEO

He mentions that by Laws 1 and 2 Galileo discovered that the descent of bodies varies as the square of the time, and that the motion of projectiles is in the curve of a parabola (ignoring wind resistance).

"When a body is falling, the uniform force of its gravity acting equally, impresses, in equal intervals of time, equal forces upon that body, and therefore generates equal velocities; and in the whole time impresses a whole force, and generates a whole velocity proportional to the time." So, double the time means double the added velocity, etc.

"And if a body be projected in any direction, the motion arising from its projection is compounded with the motion arising from its gravity."

He shows how the parallelogrammic composition of motions agrees with the rule that projected bodies describe parabolas. The motion due to falling alone (in some time T) is the diameter of the parabola, that due to projection alone (during T) is the tangent to the parabola at the vertex of that diameter, and it is parallel and equal to the ordinate of the parabola actually described (during T) joining the body to the diameter.

(2)
PENDULUMS
(Collisions)

More confirmation of Laws 1 and 2: "On the same Laws and Corollaries depend those things that have been demonstrated concerning the times of the vibration of pendulums."

More confirmation: "By the same, together with Law 3, Sir Christopher Wren, Dr. Wallis, and Mr. Huygens, the greatest geometers of our times, did severally determine the rules of the impact and reflection of hard bodies." He then goes into detail about these results.

NOTE: Newton describes these gentlemen as "easily the master geometricians of the prior generation" (as Ron Richard has it). **Descartes is noticeably absent**—is this just because he is not a countryman?

Much of the detail to follow is dedicated to showing how, experimentally, one removes the "retardation" caused by air resistance. He enunciates the law of pendulums: "Thus trying the thing with pendulums of 10 feet, in unequal as well as equal bodies, and making the bodies to collide after a descent through large spaces, . . . I found always . . . that when the bodies collided together directly, equal changes towards the contrary directions were produced in their motions" (as the Cajori translation has it). Hence Law 3 is confirmed by experience.

In the course of this discussion, he acknowledges that no bodies are perfectly hard or elastic, and says that this does not disprove Law 3, but is simply another fact to be taken into account. Hence two bodies which smash into each other do not entirely transmit their motion to each other, nor do they bounce back with the same velocity with which they hit each other, but the loss is in a definite ratio for a given material. For example, if we use two WOOLEN balls, the ratio of the speeds they had to the speeds they have after collision is about 5 to 9. STEEL balls return with almost the same velocity they had. CORK balls have somewhat less. GLASS balls bounce back in such a way that their original velocity is to their reflected velocity in the ratio of about 15 to 16. (Note: Newton does not get into how "loss" of speed due to the degree of non-elasticity is compatible with the Conservation of Momentum. But we have here a hint that momentum can be "used up" or "stored" by causing deformation in something, e.g. a spring, and that it is then potential energy, or else it can be turned into heat energy.)

"And thus the third Law, so far as it regards percussions and reflections, is proved by a theory exactly agreeing with experience."

(3)
ATTRACTIONS

ARGUMENT FOR LAW 3 IN THE CASE OF ATTRACTIONS, GOING BACK TO LAW 1. Newton goes on to confirm the 3rd Law again, this time in the case of attractions. He says imagine you have two bodies attracted to each other, and yet attractor A is more attracted to B than B is to A (like unrequited love!). Then if you put your hand (or some obstacle) between them, A will press the hand more (toward B) than B presses it back (toward A). Hence the hand, and the two attractors, will be out of equilibrium, and in that state, will accelerate in the direction of A to B, and will do so forever—contrary to Law 1.

NOTE: Law 1 is considered more certain than Law 3, and is used to confirm it. Law 1 is more fundamental, it seems.

ILLUSTRATION OF LAW 3 IN THE CASE OF ATTRACTION TO EARTH. "So too the gravitation between the earth and its parts is mutual." So the 3rd Law applies in that kind of attraction, too. He says that a small part of the earth presses as hard, by its heaviness, against a larger, as the larger against the smaller. This seems counter-intuitive, but is again an application of Law 3. And really it is clear that my feet press against the Earth just as hard as the Earth presses back against my feet.

QUESTION: Is Newton saying that a tiny portion of the Earth is just as heavy as the remaining portion of the earth it is pressing against? (No—only that their "weights toward each other" are the same. If one brought these unequal parts of Earth to a much larger planet, each would weigh toward that planet in proportion to its mass, and they would not have equal weights. But the weight of the little one toward the big one is the same as the weight of the big one toward the little one. My weight on Earth is equal to the Earth's weight toward me. This is already an eye-opener for us readers, the idea that the Earth has "weight toward me." But Law 3 demands it, if Law 3 is true! If I am attracted to the Earth, then it must be equally attracted to me, such that my *ma* is equal and opposite to its *ma*. Obviously, its acceleration toward me is much smaller than mine toward it, since its mass is so much greater than mine.)

EXPERIMENTAL CONFIRMATION OF LAW 3 IN THE CASE OF ATTRACTIONS USING MAGNETS. Newton next says that a magnet and a bit of iron, each allowed to float on water in a little vessel (like a wax-paper "boat"), to reduce friction, will attract each other equally, because when they snap together after release, their boats might spin, but they will not drift to one side with an accelerated motion—probably not even with a uniform motion. If we thought that the iron was attracted to the magnet, but the magnet was not attracted back to the iron, or not as much, then there would be a net acceleration of the "iron boat" toward the "magnet boat", and the system of the two boats should accelerate in the iron-to-magnet direction. But they do not. Hence the attraction is mutual and equal and opposite.

(4)
MACHINES

Next, he confirms or explains the meaning of Law 3 in regard to the elementary machines: balance (lever), pulley, screw, wedge.

So, if there is an equal and opposite reaction to every action, how does anything happen at all? "The power and use of machines consist only in this, that by diminishing the velocity we may augment the force, and the contrary; from whence, in all sorts of proper machines, we have the solution of this problem: *To move a given weight with a given power*, or with a given force to overcome any other given resistance. For if machines are so contrived that the velocities of the agent and resistant are inversely as their forces, the agent will just sustain the resistant, but with a greater disparity of velocity will overcome it" (p 27 Cajori).

For example, we gain "having to go through less distance" at the cost of "putting in more force" if we close a door by pushing near the hinges, and we gain the reverse way if we push out near the edge of the door.

He then says "But to treat of mechanics is not my present business. **I was aiming only to show by those examples the great extent and certainty of the third Law of Motion**."

NOTE ON LAW 3

Throughout the Corollaries and this second scholium, Newton seems preoccupied with Law 3. Why is that?

(1) Newton defines himself in opposition to Descartes, who was all the rage at the time. So he entitles his work "Principia *Mathematica* Philosophiae" as a kind of in-your-face reaction to "Principia Philosophiae." And he refuses to mention Descartes as a great mathematician of the prior generation. One key place Newton disagrees with Descartes is where the latter says (in his Fourth Rule of Collision) that a smaller body can never move a bigger one, as if one acts with greater force on the one than the other acts with on the other. Hence the 3^{rd} Law is a locus of the disagreement.

(2) The First Law was already well accepted by Newton's time, and had been recognized by Descartes and Galileo, or very nearly. This law is not Newton's contribution.

(3) The Second Law is more original with Newton. Newton and Descartes have disagreements about the importance of direction. Descartes says motion is not contrary to motion, but only to rest, and so the conservation of momentum does not require the conservation of net direction in a system. Hence the little body can be flying at 30 mph to the right, hit the larger resting body, and then bounce off at 30 mph to the left, leaving the resting body at rest, unaffected. Hence there was a net change in the direction of the system. Newton thinks this is impossible. Descartes would have the impact of the little body, going rightward, not cause any rightward change of motion! That is against Law 2.

(3) The Third Law is more original with Newton, too. And Descartes' Fourth Rule of Collision is clearly contrary to Law 3. But Law 3 is also counterintuitive, at least at first. Students resist it.

(4) The 3rd is close to the purpose of his book. He wants to show that all bodies weigh towards each other. In the 2nd Scholium he is already saying that "weighing" is a mutual thing—although he keeps it here on earth, he is saying one part of earth weighs toward another, and the other weighs back just as much. Simply in dividing the earth, he is getting us to think of weight as tending to the center of massive bodies, not just to the center of the earth. And the 3rd law will play a decisive role in the argument for universal gravitation.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 17

On The Motion of Bodies
Book 1
Section 1
Lemma 1

Introductory Notes:

• Newton intends to present us with mathematical propositions about motions, and so Books 1 and 2 are "On the Motion of Bodies" in a very general, abstract way. In Book 3 we will find concrete application of some of these motion theorems. So Newton's Books 1 and 2 are to his Book 3 a little bit like Euclid's Book 5 is to his Book 6. In physics, we study motion first in the abstract, and then find some of its forms in various bodies, much as in geometry we study proportion first in the abstract, and then discover it in various things.

• We have now finished the Newtonian analog of Euclid's Definitions, Postulates, and Common Notions, namely Newton's Definitions, Laws, and Corollaries (with some explanatory scholia). But the Newtonian analog of Euclid's Theorems will begin only once we have equipped ourselves with some "Lemmas."

• The word "**Lemma**" comes from the Greek "*lemma, -atos, to*" = "that which is peeled off, a peel, a husk, a skin, a scale, bark." This in turn comes from the verb "lepo" = "to strip off the rind or husk, to peel, to debark" (cf. "leper"). The idea is that a "Lemma" in a science is something you have to go through before you get to the real substance, i.e. something necessary and preliminary but not desired for its own sake (like the peel of an orange). It is something relatively insipid.

• Sometimes a "lemma" is simply uninteresting, but is necessary. We see some of these in Euclid. Other times it is something interesting in itself, but is somehow foreign to the matter at hand, and so it is not essentially part of the thing aimed at, but is nonetheless necessary to go through in order to get to it. We see this in Ptolemy's *Almagest*, when he develops trigonometry, because it had not been developed sufficiently by others so that he could safely assume it for his purposes, and yet he was not interested in trigonometry as such, but astronomical applications of trigonometry. This is the sense in which Newton's Lemmas are "Lemmas" (or "Lemmata"). They are very interesting in themselves, and perhaps they are not even foreign to the science of motion (although the terms "approaching" and "becoming" in them could be reinterpreted so that they are purely geometrical things), *but there is no use of Newton's Laws of Motion in these Lemmas, and no consideration even of force except in Lemma 10.* Hence these "Lemmas" are foreign to, or prior to, the science of motion built on Newton's Laws.

• If Ptolemy's mathematical lemmas are "**trigonometry**," one might say that Newton's Lemmas are the historical roots of what today is called "**calculus**." We need this mathematical equipment in order to apply Newton's Laws so as to derive the motions of bodies in curves from given rectilineal forces (or, given their motions in certain curves, to derive the rectilineal forces producing these motions).

• Historical Note: Newton is actually one of the inventors/discoverers of the calculus. He invented it at the same time as, and independently of, Leibniz. Each accused the other of intellectual theft. It was a bitter affair, and some of the historical details are matter for debate.

• What we have here is nevertheless not quite full-fledged CALCULUS, but differs from it in the following ways:

(1) It is very geometrical, rather than having the generality of "functions," and so it is too concrete and specific to be calculus. Nonetheless, the underlying principles herein are of broader application than that to which they are put by Newton in this book, and it is evident that he knew that.

(2) It lacks a specific notation, and hence one does not get to learn mechanical rules whereby to "calculate," mechanically, what the right answers are.

(3) There are no general definitions of certain limits of particular interest, such as derivatives and integrals.

• We are not doing pure math, even in these "lemmas," insofar as we have the idea of motion. Pure math abstracts from matter and motion. One could reinterpret the Lemmas so that they are not about "changing quantities" but about the different quantities that could be taken out of a continuum in a series of some kind, and then indeed they would be theorems of pure mathematics. But there is no need to do that, here, since this is a book "On the Motion of Bodies."

ON THE MOTION OF BODIES
BOOK ONE

SECTION 1

On the method of primary and ultimate ratios, by the use of which the following are demonstrated.

Newton's section-title here (the text in italics above) explains that the Lemmas are presenting us with a new mathematical technique. He is trying to make up for the defects in GALILEO's composition of continuous things out of indivisibles—and although he basically succeeds, he himself falls back into that sort of thinking often enough. Galileo had said that a line is composed of an infinity of points, and a surface is made out of an infinity of lines, and an accelerated motion is composed out of an infinity of instantaneous speeds. But it is impossible to put together two indivisible points in any way so as to get anything other than two points at a distance from each other or else a single point in which the two coincide. In neither way do the two points constitute a *line*. The same is true of all indivisibles and continuous things. But consideration of things continuously different, such as an accelerating motion, requires us to consider the presence of the indivisible in the continuous in some way. For example, if a body accelerates uniformly from 0 mph up to 10 mph over the course of one hour, how far does it go? One's gut feeling is "5 miles"—and that is correct. But how does one prove that, exactly? We know if the body moves at 10 mph uniformly for the whole hour, then it will go 10 miles. But the body has an infinity of different speeds during its acceleration, and none of them for any length of time! We need to see that what the accelerating body does can be approximated as nearly as we please by a series of uniform speeds for short lengths of time. The shorter the lengths of time, and the greater the number of uniform speeds we take, the closer the total distance accomplished will be to the distance actually accomplished by the accelerating body. If we can see that the approximation can be made as accurate as we like, and if we can calculate somehow what the limit of the approximations is, then we will also have discovered how far the accelerating body has moved. The details of this illustration are not important at the moment. The point is only this: In the study of motion, and especially in the study of accelerated motion, we often find ourselves needing a way to determine the limit of some infinite process of refined approximations. It is for that reason that Newton gives us these Lemmas prior to beginning his treatise on motion proper.

NOTE: When he says "by the use of which the following are demonstrated," he does not mean "the following Lemmas," but "the Propositions of Book 1, following these Lemmas." ("Sequentia" might be translated, here, as "the things afterward," rather than as "the following.")

LEMMA 1

Quantities, and also ratios of quantities, which for any finite time continually tend towards equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal.

If not, let them become ultimately unequal, and let their ultimate difference be *D*. Therefore, they are not able to come nearer to equality than by the given difference *D*, contrary to the hypothesis.

QUESTIONS:

Q1. **"approach nearer each other ..."** How can one quantity "approach" another? Can 2 "approach" 3?

Q2. What is an example of two quantities "continually tending towards equality," and which "before the end of a finite time approach nearer to each other than by any given difference"?

Q3. Do both quantities have to be changing? Can both be changing?

Q4. Newton lists three conditions for two things to be "ultimately equal":
(1) They "continually tend towards equality"
(2) They do so for some "finite time"
(3) Before the end of that time they "approach nearer to each other than by any given difference"
What does each of these mean?

Q5. Are all three necessary? What things does each one exclude from being ultimately equal?
What if we have (1) and (2) but not (3)?
What if we have (1) and (3) but not (2)?
What if we have (2) and (3) but not (1)?

Q6. Must the results after the finite time is finished be still the same kinds of things that were "tending towards equality"? Must two lines, for example, end up as two equal lines in order for them to be "ultimately equal"? Does "ultimately equal" ("ultimo aequales") mean the same as "equal in the end"?

Q7. If two variable and comparable quantities in a finite time both shrink until they simultaneously vanish, does it follow that they are ultimately equal?

Q8. If two variable and comparable quantities in a finite time both shrink until they vanish into two comparable quantities, and throughout the process the two variable quantities were continuously tending toward equality by nearer than any given difference, does it follow that the remaining quantities are equal? What if the variable quantities were always equal throughout the process?

Q9. If two variable quantities in a finite time differ by less than any given difference, does it follow that they are ultimately equal?

Q10. Is Lemma 1 a demonstration or a definition?

Q1. **“approach nearer each other ...”** How can one quantity “approach” another? Can 2 “approach” 3?

We must be thinking of the quantity *of* something, not just a quantity itself. For example, the height *of* someone can approach the height of someone else, because “Joe’s height,” even if his height changes, is still Joe’s height. So it is in some sense the same quantity, i.e. it is the quantity of the same subject.

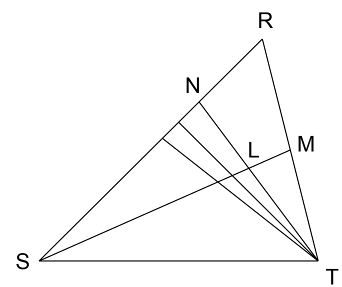
So we are not really thinking of one quantity, simply speaking, approaching another, but *the many quantities of (or associated with) one thing, one subject*—e.g. the successive polygons inscribed in a circle.

In modern terminology, we speak of a **variable quantity**. This really means *the quantity of a thing which can have various quantities*.

Q2. What is an example of two quantities “continually tending towards equality,” and which “before the end of a finite time approach nearer to each other than by any given difference”?

The major and minor radii rotating in an ellipse. One is getting continuously bigger, the other continuously smaller. And within a finite time that process is over, and the one that was getting bigger is now getting smaller, and the one that was getting smaller is now getting bigger. This is a case where the two quantities actually become equal at some point.

Or suppose △RST has side RT bisected at M. Then if we were to rotate TS about T it would be divided by SM into segments like TL and LN which are not in fact equal, but ultimately become equal as TM and MR. That is another case in which the unequal things become actually equal by the end.



Q3. Do both quantities have to be changing? Can both be changing?

Do both quantities have to be changing? No, not necessarily, but it is possible. In the diagram above, TL and LN are both changing, and they are ultimately equal. On the other hand, LN is ultimately equal to MR, and LN is changing while MR is not.

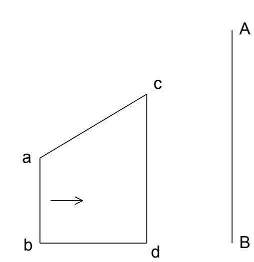
Q4. Newton lists three conditions for two things to be “ultimately equal”:
(1) They “continually tend towards equality”
(2) They do so for some “finite time”
(3) “Before the end of that time” they “approach nearer to each other than by any given difference”
What does each of these mean?

Condition (1) means the two quantities tend towards equality continuously, as opposed to discretely, and (more importantly) that they do so monotonically, instead of getting now closer to equality, now further away, etc. (The word “continuously,” i.e. “constanter,” does *not* mean “at a constant rate” here.)

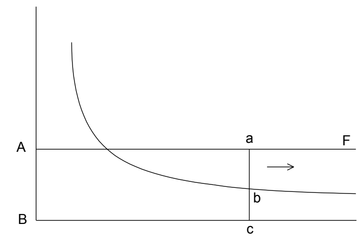
Condition (2) means the process by which they are tending towards equality has some definite end.

Condition (3) means that if the quantities are A and B, and we are given any challenge difference, D, we can find a place in the process where A and B differ by less than D.

Q5. Are all three necessary? What things does each one exclude from being “ultimately equal?
What if we have (1) and (2) but not (3)?
What if we have (1) and (3) but not (2)?
What if we have (2) and (3) but not (1)?

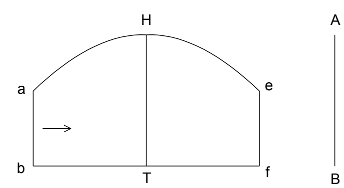


Let *ab* grow continuously during finite time *t*, up to size *cd*, and then stop. And let *cd* be less than AB. Then we have conditions (1) and (2) but not (3), and Newton will not call ab and AB "ultimately equal." They are ultimately *unequal*.



Let AB, Bc be asymptotes of an hyperbola, and AF a secant parallel to BC, and "*a*" some point moving right, with *abc* drawn parallel to AB. Then *ab* continually tends toward equality with AB, and nearer than any given difference. So *ab* and AB meet conditions (1) and (3). But they don't meet condition (2), since this process cannot be completed in a finite time. So we cannot say AB and *ab* are "equal in the end," or that they tend toward any definite thing by the end of the process, since there is no end to this process. AB and *ab* are not "ultimately equal" since they are not "ultimately" anything.

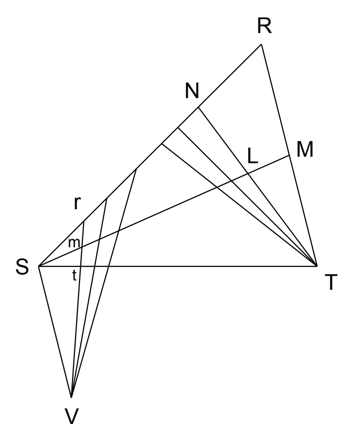
NOTE: The finitude of time is not really what is important to Newton. He uses this as a way of expressing that the process has some definite terminus. But if one looks ahead to Lemma 2, one sees that the infinity of rectangles he wants us to draw cannot be drawn in any finite time, and yet he will call the figure they compose "ultimately equal" to the fixed curvilinear figure. But the fixed curvilinear figure is itself a terminus of the process, so that the process is finite in that sense. One can point to a definite *spot*, the curved line, and say what the step-figure is doing as the polygonal perimeter moves toward that finish-line. Similarly there is a sort of "finish line" in the case of *ab* in the accompanying figure, namely the asymptote, although there is no fixed point on it where point *b* is tending! Newton seems to want to exclude these cases, in order to understand what is in flux by something that is real and fixed. Nevertheless, one could still say that "the first ratio which AB : *ab* will not attain by this process, carried out indefinitely, is the ratio of equality." But Newton's language of "ultimate equality" does not make much sense there, since there is no "ultimate condition" in such a process, no "there" where the line *ab* is going.



Let *ab* grow as it moves to the right, up to a maximum height of HT = AB, and then shrink back down to *ef*, something less than AB. Then *ab* and AB meet conditions (2) and (3), but not (1). Hence they are not ultimately equal, but ultimately unequal. This is the sense in which continuous tendency toward equality is necessary.

Strictly, though, the quantities can get closer to and further from equality as they shrink or change, and still be "ultimately equal," so long as there comes a time when they do nothing else but tend more and more toward equality, or so long as each remains always sandwiched between other things which purely tend toward, never away from, equality. But these are secondary things.

Q6. Must the results after the finite time is finished be still the same kinds of things that were "tending towards equality"? Must two lines, for example, end up as two equal lines in order for them to be "ultimately equal"? Does "ultimately equal" ("ultimo aequales") mean the same as "equal in the end"?



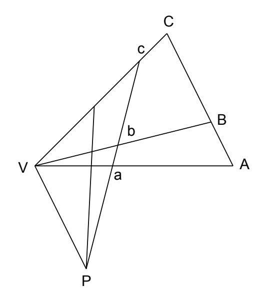
Recall the diagram with △RST, side RT bisected at M. Draw SV (of any length) parallel to RT and draw through a secant Vtmr, and rotate this about V toward S. Then *tm* and *mr* are not equal, nor will they ever be equal (there is no *rm* which is actually equal to *mt*). But within the finite time it takes to get to S these two lines will tend continuously toward equality and their ratio will differ from the ratio of equality by less than any assigned difference. Hence all three conditions of Lemma 1 are met. So we must say that *rm* and *mt* are "ultimately equal." And Newton will in fact say this for such a case. But at the end of this process we do not have "a pair of equal straight lines," nor lines or magnitudes at all, but only the one point S.

So it is not necessary, in order for magnitudes to be "ultimately equal," that the remainders of the process be magnitudes of the same kind as those which underwent the process, nor that they be comparable magnitudes, nor magnitudes at all, nor even distinct from each other.

Newton's language both here and later suggests that by "ultimately equal" he means "equal in the end," i.e. the two things actually end up as a pair of equal things, though perhaps infinitely small. Throughout the *Principia* he seems to waffle on this, sometimes talking like Galileo about such things, and other times tending more toward the modern rigor about them. We should not try to make him more consistent than he really is, but we should ourselves distinguish what he leaves confused.

Since it makes no sense to say S is "a pair of equal lines", and yet it is all that remains after a finite process in which two lines become as close to equal as we please, therefore, by saying the lines are "ultimately equal," we should not always assume this means the same thing as "they are a pair of equal lines when the process is over." Instead, we should take "the lines are ultimately equal" to mean nothing else than that they meet the three conditions of Lemma 1. Modern terminology would say that the ratio of equality is the "limit" of their ratio as they go through this process, since they tend continuously toward that ratio, and get closer to it than any given difference. And while that ratio might not, in some cases, be the last thing attained by the variable quantities, when that happens it will be the first ratio not attained by them.

Q7. If two variable and comparable quantities in a finite time both shrink until they simultaneously vanish, does it follow that they are ultimately equal?



Not necessarily. For example, let there be a triangle VAC, in which AC is divided at any point B other than the midpoint, and VB joined. Now draw VP of any length so that it is parallel to AC. If we draw through secants from P such as Pabc, and rotate this about P toward V, plainly *ab* and *bc* ultimately vanish, and simultaneously become nothing. But they are not ultimately in the ratio of equality—that is not the ratio toward which they tend nearer than any given difference. Rather, it is the ratio AB : BC.

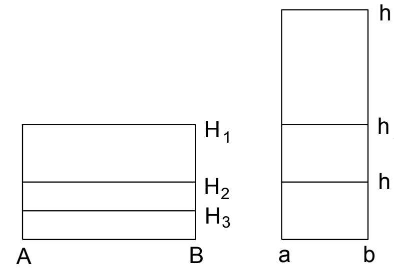
So *ab* : *bc* ult.= AB : BC.

But what condition of Lemma 1 is missing, such that we cannot say *ab* and *bc* are "ultimately equal"? An IMPLIED condition, which Newton intimates when he says "and ratios of quantities." Here we have a situation in which *ab* and *bc* continually tend towards equality (the ratio *ab* : *bc* becomes more like the ratio of equality throughout the process), and *ab* and *bc* in a finite time come to differ from each other by less than any given difference. Nevertheless, they are not ultimately equal, since their ratio does not itself tend toward the ratio of equality by less than any given difference. So when Newton says "continuously tend towards equality," we are given to understand not only that the ratio differs less and less from equality, but that it will differ from it by less than any given difference during the process. In the case above, *ab* : *bc* never gets any closer to the ratio of equality than AB : BC.

Q8. If two variable and comparable quantities in a finite time both shrink until they vanish into two comparable quantities, and throughout the process the two variable quantities were continuously tending toward equality by nearer than any given difference, does it follow that the remaining quantities are equal? What if the variable quantities were always equal throughout the process?

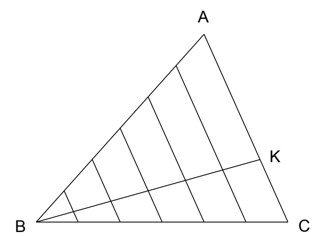
Not even when there *is* a pair of things after the process is over, and those things have comparable quantity, do they have to be equal, in order for the previous magnitudes which vanished into them to be "ultimately equal." Suppose you have two equal rectangles, ABH and *abh*, of unequal bases AB, *ab*. We can shrink them by diminishing their heights in such a way that they always remains equal. Hence, by the definition, they are "ultimately equal" (or perhaps *more* than just ultimately equal, but always equal). We could also make it so that they are never actually equal, but tend towards equality by nearer than any given difference as they go to nothing. But the remainders of the process are the comparable, yet unequal, bases AB, *ab*.

• NEVERTHELESS, if the remainders of things "ultimately equal" are of the same kind as those that were changing, it **is** necessary that these remainders be actually equal, and in such cases the changing quantities are simply "equal in the end." And when this is not so, but they vanish somehow, it is necessary that their ratio can be made to differ from that of equality by less than any assigned amount, and that this remains true in all times ending the time of change (e.g. in the "last second" or "last nanosecond"), no matter how short.



Q9. If two variable quantities in a finite time differ by less than any given difference, does it follow that they are ultimately equal?

NO. This was already touched on in Question 7 above, but it is worth saying again. For example, let there be △ABC in which AC is divided unequally at K, e.g. so that CK : KA = 1 : 3, and join BK. Now, if we start moving AKC, always parallel to itself, toward vertex B, it is manifest that the difference between CK and KA will become less than anything. And yet CK and KA are not ultimately equal, but are ultimately (and in fact always) in the ratio of 1 : 3. They are not "tending toward equality" at all, much less is their ratio tending toward it in such a way as to differ from it by less than any given difference during the process.



QUESTION: What does it mean for a ratio to "differ from equality" (or from any fixed ratio) "by less than any given difference"?

That could be defined any number of ways, but I'll define it "Cartesian-style," as follows.

If two ratios *r* : *t* and R : T be both changing (or one of them changing and the other fixed), they are said "to differ from each other by less than any given difference" during the change if, during the change, the difference of the lengths *r/t* and R/T becomes continuously less, and if, given any line length D, the difference of the lengths *r/t* and R/T becomes less than D at some point during the change.

Q10. Is Lemma 1 a demonstration or a definition?

Newton presents it as though it were a little demonstration. This is an indication that he wishes us to understand the phrase "ultimately equal" not as a new term being defined so much as the familiar expression "equal things at the end of the process."

But this is part of his inconsistency, and is not really tenable. It is only in special cases that things which meet the three givens of the Lemma are, when the process is finished, a pair of equal magnitudes (or ratios), as examples above have illustrated. If we try to take it as an argument, it fails because things can meet the three conditions, yet not become "ultimately unequal" (in the privative sense), and hence have some "ultimate difference, D," but simply become "ultimately NOT equal" (in the purely negative sense), and hence not have any ultimate difference. For example, two areas that vanish into a pair of lines, are "ultimately [i.e. at the end of the process] *not equal areas*." They are not "unequal areas," either. They are not areas at all. If the only alternative to their being equal areas were for them to be unequal areas, the argument would follow. But that is not the only alternative.

So Lemma 1 really has only the force of a definition, and Newton does not need it for more than that. As a definition, and with some of the implications made explicit, Lemma 1 would read like this:

> *Two ratios which in any finite process continually change towards equality, and, given any difference between them, D, come to differ from each other by less than D at some point during the process, I call "ultimately equal."*
>
> *Two quantities which in any finite process continually change towards equality, and whose ratio comes to differ from the ratio of equality by less than any given difference D at some point during the process, I call "ultimately equal."*

The modern analog is the definition of "**limit**," which we will not enter into here.

REMARKS

Suppose you have a triangle ACD, in which D is joined to B somewhere along AC, and you draw a line Cgh through at random, with g being where the secant cuts DB, and h being where it cuts DA. If we start rotating Cgh toward CBA, will the ratio ab : bc get nearer than any given difference to AB : BC ?

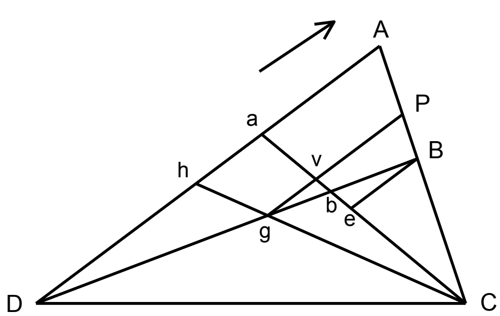
Yes it will. And we might think that is obvious just because Cgh and CBA will coincide at the end of the rotation. The ultimate version of Cg : gh *is* CB : BA. But although that is true, it is not quite the same thing as saying that the ratio Cg : gh gets as close as we please to CB : BA on the way to the end of the process of rotation. This latter statement means that *prior* to ultimate coincidence of the rotating line with CBA, we can find an instance of Cg : gh which is closer to CB : BA than any "challenge ratio." To prove this: Choose any point P along AC, as near to B as you please, so that the ratio CP : PA is as near as you like to CB : BA. I can find a place in the rotation of my secant such that it will be divided in a ratio which is *closer to* CB : BA than CP : PA is. Draw Pg parallel to AD, cutting BD at a point we will call g. Join Cg and extend it through to h. Choose any point *a* along hA and join Ca, cutting gB at b and gP at v. Also, draw Be parallel to AD, cutting Cb at e.

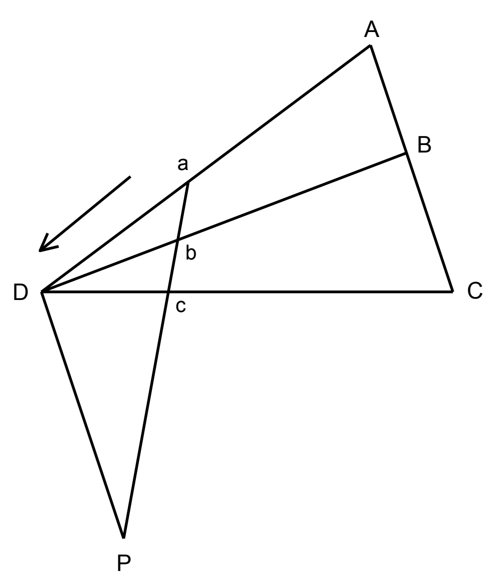
Obviously,	av : vC = AP : PC	[by the parallels]
thus	ab : bC > AP : PC	
Again,	ae : ec = AB : BC	[by the parallels]
thus	ab : bC < AB : BC	
So	AP : PC < ab : bC < AB : BC	

And so we have found a secant Cba, the ratio of whose segments is closer to AB : BC than the challenge ratio. *Q.E.F.*

The point of all this is merely to illustrate that it is not quite the same thing to say that "in the end the ratio of the parts of the secant will be the same as CB : BA" on the one hand, and to say, on the other hand, that "the ratio Cb : ba can be made to differ from CB : BA as little as we please during the process of rotating the secant."

That kind of distinction is very important, since sometimes it will NOT be the case that the changing quantity will coincide with the fixed one in the end, and nevertheless it remains true that the changing quantity can be made to differ from the fixed one by as little as we please during the process. For example, if we draw DP parallel to AC, and draw secants from P through, like Pcba, and rotate the secant toward PD, then obviously ab and bc will not ultimately coincide with or become AB and BC. Rather, they will vanish into point D. Still, it is true that ab : bc can be made to differ from AB : BC by as little as we please during this rotation—we can beat any challenge ratio. Hence there is reason to call AB : BC the "limit" ratio being approached by the changing ratio ab : bc.



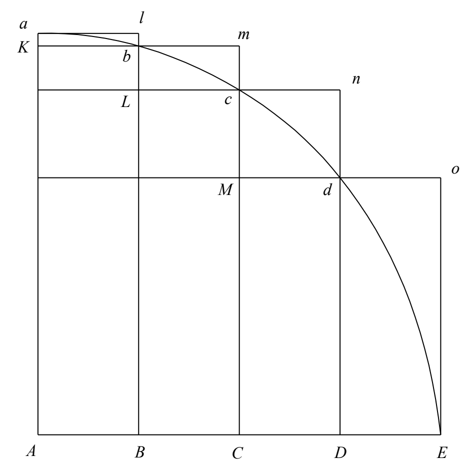


CLASS 18

LEMMA 2

Here is the text of Newton's next lemma, Lemma 2:

If in any figure whatever, AacE, comprehended by straight lines Aa, AE, and curve acE, there be inscribed however many parallelograms Ab, Bc, Cd, etc., contained by equal bases AB, BC, CD, etc., and sides Bb, Cc, Dd, etc. parallel to side Aa of the figure, and the parallelograms aKbl, bLcm, cMdn, etc. be completed. Then let the width of these parallelograms be diminished and the number be augmented into infinity: I say that the ultimate ratios that the inscribed figure, AKbLcMdD, the circumscribed figure, AalbmcndoE, and the curvilinear figure, AabcdE, have to one another are ratios of equality.



Newton's explanation is plain. In his figure, the complete inner figure and the complete outer one (which are both **rectilinear**) differ by the rectangle ABla. But as we proceed, that rectangle (or parallelogram) shrinks as small as you like, meaning that the difference between the outer and inner figures becomes less than any given difference. Hence the first ratio they do not reach is that of equality—that ratio is the limit of their successive ratios. So they are "ultimately equal."

And since the **curvilinear** figure is always between the inner and outer figures in its area, it is always even more equal to the inner than the outer is, and to the outer than the inner is. Hence its ratio to either the inner or outer also approaches that of equality, as the number of parallelograms increases. So the inner (or outer) figure is ultimately equal to the curvilinear figure.

Q.E.D.

NOTE: The "independent variable" is the number of the bases of the parallelograms, *n*. The "dependent variable" is the area of the inscribed figure, or the area of the circumscribed one.

QUESTION 1: Why does Newton divide the base into *equal* segments?

In order to get the difference between the inner and outer figure incarnated in a single rectangle, ABla. If the rectangles had different widths, the sum of their differences would not be equal to that first rectangle, which has its own width. And is it clear that no step-figure is actually equal to the curvilinear one?

QUESTION 2: Do the segments of the base have to be equal for the Lemma to be true? (Note: This is answered in Lemma 3) No—as long as all are shrinking to nothing. But it is *not* enough if we say "the difference is always shrinking." If some area is reserved, and will not be "invaded" by rectangles, then the figures being so constructed are not ultimately equal.

Is it enough to say "as the number of segments of the base increases to infinity"? No. We have to add "and as they are all taken smaller and smaller." Otherwise, we could leave some one rectangle the same, and multiply and shrink only the others, and then the inner figure would always differ from the curved figure by at least the amount by which our unchanging rectangle differs from the area under the curve which stands on the same base. For example, if, in Newton's diagram, we left rectangle MdDC unchanged, then the inner figure would always be short of the area of the curve by more than the curved segment cdM.

QUESTION 3: Do the figures have to be rectangles? No. He himself mentions "parallelograms" in the course of the Lemma.

QUESTION 4: Do the lines aA and AE have to be parallel to the tangents at *a* and E? Not for every figure, perhaps, but in order for the argument to work in all cases we might need the two lines aA and AE to be parallel to the tangents at E and *a*, at least when the curve is all-convex, as he has it in his diagram. Otherwise, suppose Eo were not tangent, but a secant. Then it would cut off a segment of our figure which will never be "invaded" by inscribed parallelograms, so the argument will fail. It is interesting that Newton nowhere mentions this fact about the figure in his Lemma. Probably he wishes us to gather it from the figure and from the argument. Also, he probably wants us to consider a portion of the curve that is all-convex, or else all-concave, not going up and down (although that would not complicate things much).

QUESTION 5: Is this argument "quia" or "propter quid"? That is, does it merely establish the truth of the conclusion, or does it also give the cause or the reason for the truth of the conclusion?

There are really two conclusions:

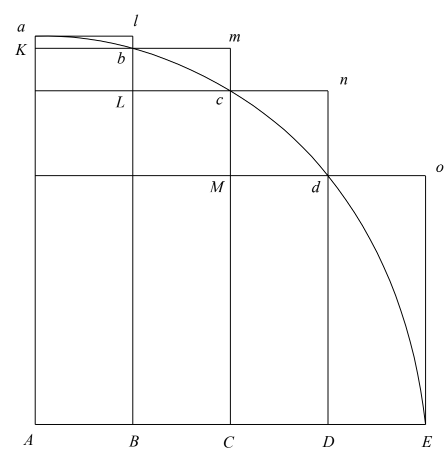
(1) The rectilineal areas are ultimately equal.

(2) Each rectilineal area is ultimately equal to the curvilinear one.

The cause of the two rectilineal areas being ultimately equal is that they are approaching the same area, i.e. the curvilineal one, as near as we please. So there is something a little backwards about showing first the ultimate equality of the rectilineal figures to each other, and then arguing to their ultimate equality to the curvilinear one by a "squeeze" argument. Still, that seems to proceed from what is better known to us—thus, a "quia" argument. (And I don't think that all arguments using ultimate equality or limits are by that very fact "quia.")

QUESTION 6: Will this Lemma be useful only for showing things are approximately equal, or as equal as we like, but never quite exactly equal? No—it will often be useful for showing that things are *exactly* equal, if we take it together with a few self-evident principles.

For example, if two variable quantities A and B are always equal, throughout their variation or succession, and they are "ultimately equal" respectively to C and D, it will follow that C and D are exactly equal—or else there would be no way for A and B to be always equal to each other and also to approach C and D by less than any given difference.



QUESTION 7: Is the process in this Lemma able to be completed in a "finite time"? We can imagine B going to A in a finite time, but not an infinity of rectangles coming into being. Can the process be completed at all? No, since if it could, there would have to be a last step-figure, which is impossible. Does that affect the conclusion of the Lemma? Well, if "ultimately equal" means "equal at the end of the process," and there IS no end of the process, then indeed it destroys the Lemma. But it remains true, and demonstrated, that the area of the step-figure approaches that of the curved figure by nearer than any given difference by this process. And that is all that Newton will really need in the upcoming Lemmas and Propositions, even if he himself prefers to speak in another way that is faster and more intuitive (though less intelligible).

CLASS 19

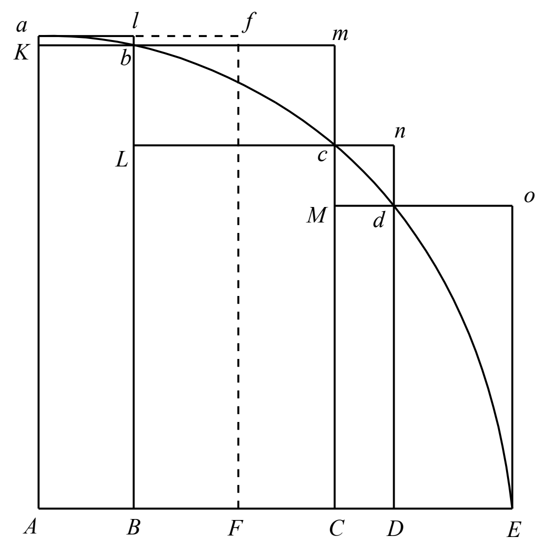
LEMMAS 3, 4, 5

LEMMA 3

"Unequal Bases Work, Too"

The same ultimate ratios are likewise ratios of equality, when the widths AB, BC, CD etc., of the parallelograms are unequal, and all are diminished to infinity.

For, let AF be equal to the maximum width, and let parallelogram FAaf be completed. This parallelogram will be greater than the difference of the inscribed figure and the circumscribed figure. But with its width, AF, being diminished to infinity, it will become less than any given rectangle. *Q.E.D.*



Q1. Is Newton assuming that the left-most rectangle is the widest?

No. He is simply constructing one there which is as tall as the tallest, and also as wide as the widest, and saying that even this constructed rectangle must shrink to nothing (since it stands on a base that shrinks to nothing), and yet it is greater than the difference between the inner step-figure and the outer one.

If the bases be unequal, and so the rectangles are of unequal width, take the greatest width among them and cut off AF equal to it from the left side of the base. Now complete rectangle FAaf. That is now equal to the tallest of the rectangles, and also as wide as any of them, and wider than many. So it is GREATER than the difference between the outer and inner rectangle-sums (since, if we shove all the other differences over to the left, they will **not** all be as wide as this tallest and widest rectangle, and so will **not** fill it up)—and yet FAaf, too, shrinks as close as we like to nothing, so that *a fortiori* the ACTUAL difference between the outer and inner figures, even when we take unequal bases, must also shrink as close as we like to nothing.

Q2. So if equal bases are not required, what IS required for the Lemma?

All that is required is that all the parallelograms shrink toward nothing—no single parallelogram is left, through the process, with a fixed base, and no base or part of a base gets left undivided, or "unshrunk," by the process. If we have even so much as a single, tiny, fixed parallelogram, then the step-figure will always differ from the curved one by at least as much as the amount by which this fixed parallelogram differs from the area of the curved figure sharing the same base.

COROLLARY 1

Hence the ***ultimate sum*** *of the vanishing parallelograms coincides in every part with the curvilinear figure.*

Q3. What does "ultimate sum" mean?

It means the first area which our sums of parallelograms cannot reach. But Newton might well mean the last one which we *do* reach, in which the parallelograms have "infinitely small" bases, as Galileo would say. The first way of speaking is correct, the second is more intuitive.

NOTE: This is as much a corollary to Lemma 2 as to Lemma 3; it is not about unequal bases in particular. What this Cor. is saying is that there will be no part of the area of the curved figure which the successive inner (or outer) figures do not eventually "**invade**."

QUESTION: What does this corollary add to Lemmas 2 and 3? If anything, we saw the truth of Lemma 1 because we already saw that the rectangle-sum would eventually coincide with all parts of the curved figure. So what is Newton's game? He seems to be bringing us back to the idea of coincidence in order to get us to think about the straight coinciding with the curved, as the next Corollaries seem to confirm.

Q4. Can we say, as a Newtonian extension of Euclid's 4th Common Notion, that "Things which ultimately coincide are ultimately equal"?

Provided the two things still have the same sort of magnitude at the end as they had throughout the process, the answer is "yes."

COROLLARY 2 (inscribed polygon)

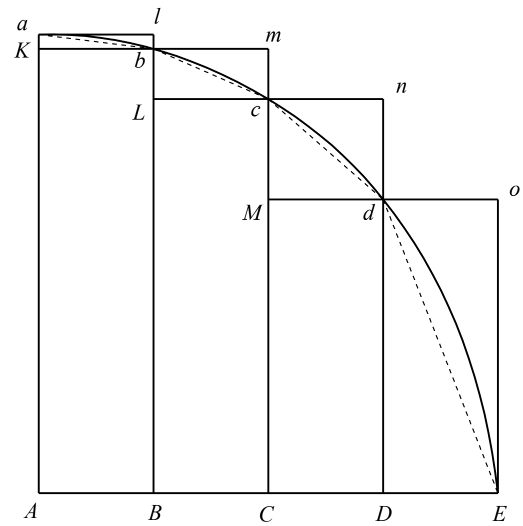
And much more does the rectilinear figure which is comprehended by the chords of the vanishing arcs ab, bc, cd, etc., ultimately coincide with the curvilinear figure.

That is, if we join ab, bc, cd, dE, we form a polygon contained by those chords and the original angle aAE. But such a polygon is even *closer* to the area of the curve than the "inner figure," i.e. the sum of inscribed rectangles. Hence such polygons also approach the area of the curved figure by nearer than any given difference.

Q5. What does "ultimately coincide" mean?

It means the first figure with which our inscribed polygons cannot entirely coincide. But Newton might well mean the last one with which they *do* coincide, in which the polygon has "infinitely many and infinitely small" sides, as Galileo would say. The first way of speaking is correct, the second is more intuitive.

So either we mean that there is a final polygon in our series, which polygon must have an infinity of sides which coincide with the curve (Galileo's way of thinking), or we mean that our polygons come as close as we like to coinciding with the whole curved figure (thinking now of areas coinciding rather than lines), or (and Newton seems already to be hinting at this) we could mean that the perimeter of the figure comes as close as we like to coinciding with the curved line.

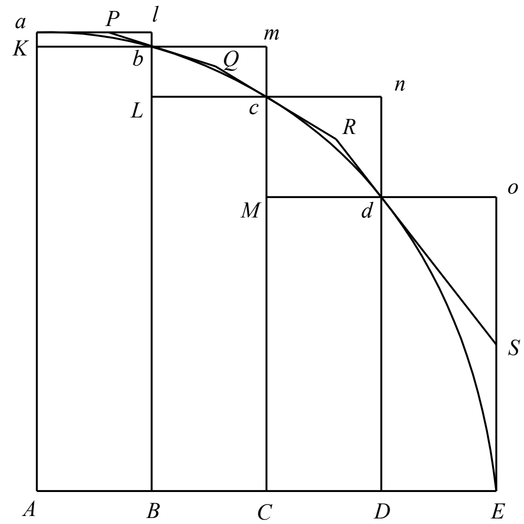


COROLLARY 3 (circumscribed polygon)

As also the circumscribed rectilinear figure which is comprehended by the tangents to the same arcs.

That is, if we draw tangents at a, b, c, d, E, we form an "outer polygon" which falls inside the "outer figure" composed of the rectangles, and hence this outer polygon is closer in area to the curved figure than that "outer step-figure" is, and therefore such outer polygons, too, approach the area of the curved figure by nearer than any given difference, as the segments of the base increase in number and decrease in width.

NOTE: Students often have trouble picturing this figure, and so it is good to have someone draw it on the board. The inscribed figure is easier, since the vertices of the perimeter are actually in the diagram already, a, b, c, d, E. So draw the figure as aPQRSE.



Q6. Why has Newton moved on from Step-figures to Polygons? What happens to the perimeter of the INNER POLYGONS as the process goes on? What happens to the perimeter of the OUTER POLYGONS? What happens to the perimeter of the INNER STEP-FIGURES? What happens to the perimeter of the OUTER STEP-FIGURES?

Newton began with the "stepped" figures, the rectangle-sums, because it is quick and easy to prove that they are ultimately equal to the curved figure and each other. But from that he can easily show that the inscribed and circumscribed polygons are ultimately equal to the curved figure and to each other, as he has done in the corollaries. And why does he go to them? Because he is also interested in getting perimeters to coincide with arcs—whereas the "perimeter" of the "stepped" figures do not tend toward equality with curves, but rather remain constant as one increases the number of rectangles.

Newton will need to approach CURVED LENGTHS with PERIMETERS as early as the very first Proposition in Book 1. But the perimeter of the STEP-FIGURE does not approach the length of the curve as we proceed; in fact, it does not change at all, but is always equal to Aa + AE, as the "CITY BLOCK THEOREM" shows. But the perimeter of the POLYGONS is always changing: ever-growing, for the inscribed polygons, and ever-shrinking, for the circumscribed ones. These can never be equal, but neither is there any limit to how close they can get to being equal—so there is a single straight line between all the lengths which can be inscribed and all those which can be circumscribed, and this is the straight line which Newton will say is equal to the curve. By contrast, the stepped perimeter does not approach the length of the curve.

COROLLARY 4

And on this account these ultimate figures (as to the perimeters acE) are not rectilinear, but curvilinear limits of rectilinear ones.

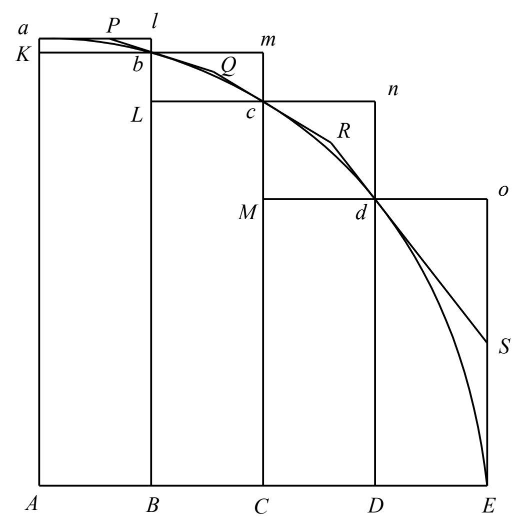
- Here Newton for the first time explicitly mentions LIMITS.
- He also uses the expression “ULTIMATE FIGURES.”

Q7. What does “ultimate figures” mean?

It means the first figure or shape which our inscribed polygons cannot assume. But Newton might well mean it is the last shape which they *do* assume, in which the polygon has “infinitely many and infinitely small” sides, as Galileo would say. The first way of speaking is correct, the second is more intuitive.

Q8. What does he mean by “as to their perimeters *acE*” ?

He wants to say that the “ultimate figures” are “not rectilinear,” i.e. that the perimeters such as acE approach the length of the curve as nearly as we please during the process. This can be proved if we allow that the curve joining the ends of a straight line is longer than that straight line. Anyway, he deliberately draws our attention to “acE,” i.e. the perimeter of the inscribed polygon rather than that of the inscribed step-figure, since that of the inscribed step-figure is obviously NOT approaching the length of the curve, but the sum of aA + AE, which is dependent on the angle at which we chose to draw them, and is in no way dependent on the length of the curve.

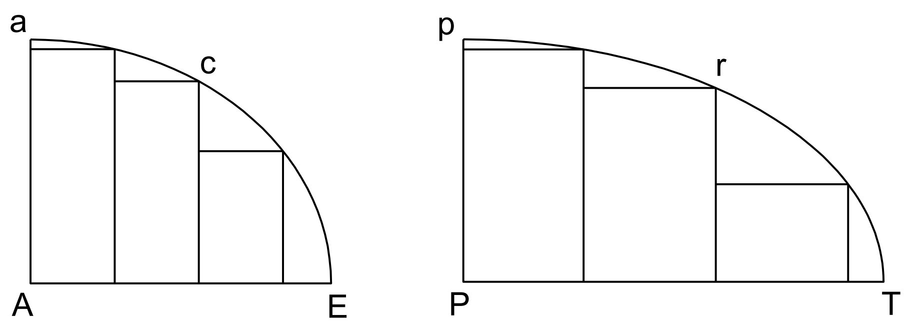


aP + Pb > ab

(1) Any tangent-perimeter is greater than any chord-perimeter.
(2) Any tangent-perimeter with more sides is LESS than any tangent-perimeter with fewer sides.
(3) Any chord-perimeter with more sides is MORE than any chord-perimeter with fewer sides.
(4) There is no minimum difference between the tangent-perimeters and the chord-perimeters.

LEMMA 4

If in two figures AacE, PprT, there are inscribed (as before) two series of parallelograms, an equal number in each series, and, their breadths being diminished in infinitum*, if the ultimate ratios of the parallelograms in one figure to those in the other, each to each respectively, are the same: I say that those two figures, AacE, PprT, are to each other in that same ratio.*



We have three new things in this Lemma:

(1) We are comparing two curvilinear figures.

(2) We are not proving that they are equal, in particular, but that they have the same ratio as that to which their inner rectangle-sums tend, say $a : b$.

(3) We are not assuming that the corresponding parallelograms *always* have the ratio $a : b$, or even that they *ever* have the ratio $a : b$, or even that they always or ever have the same ratio as each other, but only that this is the ultimate ratio to which each pair of corresponding parallelograms tends.

PROOF. We have two curvilinear areas like before. We draw first two parallelograms inside each, then three, then four, then five, etc., *ad infinitum*. If the ratio approached by the ratio of the first two parallelograms (i.e. taking the first one in each figure) is “R,” and again the ratio approached by the ratio of the second two parallelograms (i.e. taking the second one in each figure, and comparing them) is also R, and so on, each to each, then “by composition” the ratio of the sum of all in one to the sum of all in the other also approaches R nearer than any given difference. But the sum of all in one approaches the area of the curved figure in which it is inscribed, and the sum of all in the other approaches the area of the other. Hence the ratio of the two figures must actually be R. (We are using the “YOU CAN’T SERVE TWO MASTERS” principle, here. You must either overstep the nearer of the two limits, in order to get as near as possible to the further; or you must never go beyond the nearer of the two, thereby always remaining a minimum distance from the further.)

NOTE. The easiest case of this, of course, is when the two figures being compared are SIMILAR (e.g. two circular quadrants taken from unequal circles), and when the parallelograms are similarly drawn. Then the corresponding parallelograms, and also their sums, *always* have the ratio R, throughout the "growth" of the stepped figure.

Q1. Just to be clear: What is the ratio of the first rectangle to the second one in the figure on the left?

We don't know, or at least we don't have to know. What ratio are they approaching? Again, we don't know, or we don't have to know, in order for the Lemma to hold true. The ratios we are concerned with are those between corresponding parallelograms taken in the TWO figures.

Q2. Do the parallelograms have to be drawn on equal bases?

No. As long as the *number* of parallelograms is always the same in each figure, and as long as all corresponding parallelograms are approaching some fixed ratio R, the Lemma holds.

Q3. Do we have to use parallelograms?

No. We can also say that if the ratio of the areas of the inscribed polygons is approaching the ratio R, then the ratio of the two curved figures must be R. And we can divide the polygons into equal numbers of triangles with their common vertices at A and P.

Q4. Do the two curved figures have to be similar? (No.) How can they not be, if the corresponding rectangles are all approaching the same ratio?

Well, the heights of those on the left don't have to equal those of those on the right, and the bases of those on the left don't have to equal those of those on the right. We can have tall-and-skinny rectangles on the left, and fat-and-short ones on the right—and this is what happens when we consider two non-congruent quadrants of the same ellipse.

COMPARISON: This is very similar to *Elements* 12.2, where Euclid proves that circles are to each other as the squares on their diameters. As you inscribe similar polygons in them, the polygons are always in the ratio of the two squares, and the polygons get as close to the areas of the circles as you like.

There are some key differences, though.

(1) Euclid's proof is limited to circles. Newton's is much more universal.

(2) Euclid's proof uses inscribed figures which actually have the ratio R at any given step in our series; Newton's proof requires only that the inscribed figures approach the ratio R by nearer than any given difference.

COROLLARY: The same is true for two volumes, or two lengths, or two times, or whatever two continuous quantities you please. We know that the theorem still holds for these other kinds of quantities because we can always construct our curvilinear figures and parallelograms in the same ratios as the given quantities and their parts. So if the parts of our quantities approach R, then let the parallelograms in our figures approach R, and what follows for the parallelograms will have to follow for the parts, since the parts of the given figures are always as the parallelograms under our curves.

Really, there is no special reason to use curves and parallelograms except to make all this easier to imagine in a concrete case. Newton is asking us here to grow up, and realize this is not some special truth about curvilinear figures and rectangles. It doesn't matter whether the parts we are taking are angles, lengths, areas, volumes, times, speeds, weights, or whatever! This generality begins to suggest the generality of modern calculus.

EXAMPLE OF APPLICATION:

Two non-symmetrical "quadrants" of an ellipse that has been divided into four sectors (by non-axial conjugate diameters) may be proven perfectly equal in area by this technique. Ask the students to try to discover this proof in class.

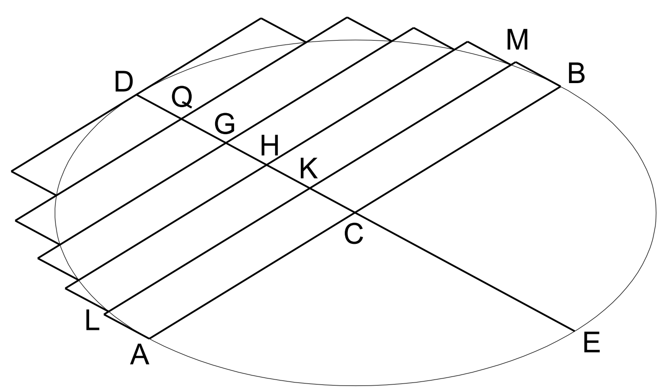
Let there be an ellipse of center C, with any non-axial conjugate diameters AB and DE. Obviously the quadrant DCB is equal to the quadrant ACE, since they are congruent, and again quadrant DCA is equal to quadrant BCE.

But what about quadrants DCA and DCB? Are they equal?

They are, and it is easy to prove using Lemma 2 and Lemma 4.

Divide DC into any number of equal segments (say at Q, G, H, K). Since AB and DE are conjugate diameters, therefore the lines through Q, G, H, K which are parallel to AB will be ordinatewise to diameter DE, and hence they will be bisected by DC and the ellipse. But that means KL = KM, and so the parallelogram HKL is equal to parallelogram HKM. And so the sum of parallelograms on one side of DC will be equal to the sum on the other. But such sums can approximate the areas of the two quadrants as nearly as we please, i.e. they are ultimately equal to those quadrants. And yet those sums are always equal. Hence the things they approach as nearly as we please must be exactly equal—that is, quadrant DCA is equal to quadrant DCB. *Q.E.D.*

Porism: Since every quadrant is ¼ the area of the ellipse, all quadrants of an ellipse, regardless of the conjugate diameters which form them, are equal.



LEMMA 5

All corresponding sides (whether curvilinear or rectilinear) of similar figures are proportional; and the areas are in the duplicate ratio of the corresponding sides.

No proof offered! This is laid down as though it were a kind of Axiom. And it is not about limits or ultimate equality, but is seen through such things (e.g. through Cor. 4 to Lemma 3).

Everything in it was proven in Euclid as to the straight lines and rectilinear figures. The curves are the only curve-ball here.

To see it in all its generality, one must see the following things:

(1) *That curved lines are comparable to similar curved lines* (i.e. have ratios to them, can be longer, shorter, equal, etc.). This is not immediately evident from Euclid's definition of "things which have a ratio," i.e. that they can be multiplied so as to exceed each other. Suppose A is a circle and B is a square. By Euclid's definition, we can see that these "have a ratio." Why? Because we can multiply the square into an area that is greater than the circle, which we can see because it contains and surrounds the circle. Since this multiple is greater than the circle, it has a ratio to it; and since it is a multiple of the square, it has a ratio to that; so the square also has a ratio to the circle.

Now we can easily multiply the circumference of a smaller circle, i.e. take it a repeated number of times (draw it three times over, for example). But how do we verify that the sum of these three lengths is longer than some other circumference of a larger circle? We cannot lay them off inside the greater circumference. Newton's way of thinking about this is implied in this Lemma: Do the circumferences of unequal circles differ? Yes. But how? In shape? No. Then how? In size! But they each have only one kind of size: length. So the circumference of the bigger circle is LONGER. This is how we see that they have a ratio. To see what ratio, we need the next step:

(2) *That similar curved lines are to each other in the same ratio as corresponding straight lines drawn in each of them*. Here is why. Since the curves are similar, they can each be approached by similar "bent lines" (series of chords) as near as you please. But therefore, by the Cor. to Lemma 4, the two curves must have the same ratio as that approached by the corresponding chords—but this is in fact always the same ratio. Hence the lengths of the similar curves are to each other as the corresponding chords. But Newton also says:

(3) *That all similar plane figures, whether polygons or curvilinear figures or mixed, are in the duplicate ratio of their corresponding sides*. *Elements* 6.19 and 6.20 prove it about similar triangles and similar polygons, and so it follows for all similar plane figures, since similar curvilinear figures are limits of similar polygons. Euclid also proves it about circles in Elements 12. One more thing is implied in Lemma 5:

(4*) That all similar surface figures, even if they be warped and not in a single plane, are also to each other in the duplicate ratio of corresponding chords*, since such figures are limits of similar "bent planes," e.g. bunches of little triangles.

"As the squares of the corresponding sides" or "in the duplicate ratio of the corresponding sides." What does this mean for curved sides? (We might have a figure with no rectilineal sides, after all.) It means "as the squares on the straight lines equal to those curves," OR it means as those straight lengths squared, i.e. multiplied by themselves Cartesian-style.

Consider, for example, two unequal circular quadrants, AGE and *age*.
Call their areas Q and q.

Then $Q : q = AG^2 : ag^2$ [Euc. 12.2 or Newton Lemma 4]

But if we cut arcs AE and ae at corresponding points,

then AC : ac = AG : ag
and CE : ce = AG : ag
so AC + CE : ac + ce = AG : ag

Similarly

(AB + BC + CD + DE) : (ab + bc + cd + de) = AG : ag

These corresponding perimeters are always in a fixed ratio. But as we increase the number of chords in them, they become as close to equaling the arcs as we please, which means the arcs themselves must actually BE in that fixed ratio (cf. Lemma 4). Hence

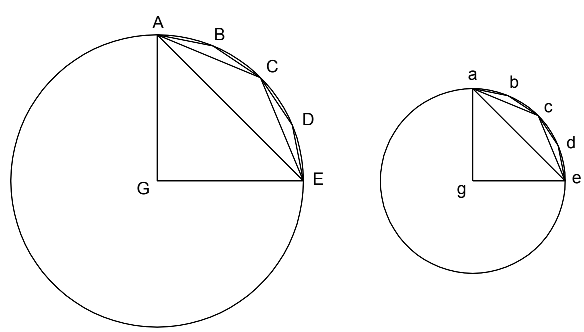
arc AE : arc ae = AG : ag

So, if we take the lengths of those arcs as straight lines, we can say:

$(\text{arc AE})^2 : (\text{arc ae})^2 = AG^2 : ag^2$

so $(\text{arc AE})^2 : (\text{arc ae})^2 = Q : q$

Q.E.D.



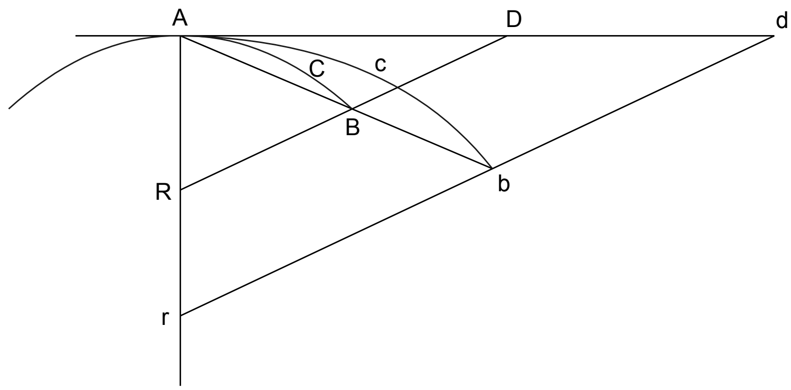
The same argument works for any similar figures.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 20

LEMMA 6

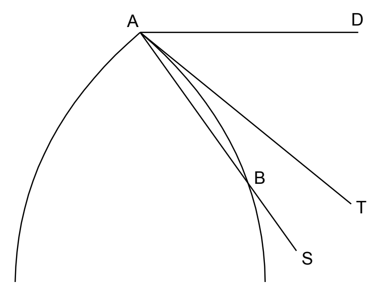
If any arc ACB, given in position, be subtended by chord AB, and at some point A in the middle of continuous curvature be touched by a straight line AD extended both ways, then the points A, B approaching towards each other and meeting: I say that the angle BAD, contained by the chord and the tangent, will be diminished in infinitum and ultimately will vanish.



PARAPHRASE: If you have a curve ACB (with A being in a place of "continuous curvature") and a chord AB and a tangent AD, then as points B and A get as close as we please, the angle BAD gets as close as we please to nothing.

QUESTION 1: What does Newton mean by "continuous curvature"?

He means there is only one tangent there. We can draw "pointy" curves, like a gothic arch, where at a certain point it is possible to draw two tangents. That is not a continuous curve. (And so, really, it is not one curve, but two that intersect at a point.) Newton defines the continuity of a curve (and hence, in a way, its unity) by the continuity in the movement of a unique tangent along the curve. If the tangent "jumps" (or stops and completely reverses direction) at any point, i.e. if in order to continue you must suddenly shift the tangent through an angle at a point, then you have a gap in continuity of curvature.



QUESTION 2: Is this a proof?

Since he does not say Q.E.D. at the end, it seems similar to Lemma 1, which is closer to being a definition than a proof. Here, the claim is that the limiting position of a chord rotating about point A in a curve must be the tangent at A, provided there is only one unique tangent to the curve at A.

QUESTION 3: Euclid proves a circle can have only one tangent at a point, and Apollonius proves the same for conics. Is Newton just assuming the same is true of all curves?

Well, no. He is saying that if you DO have two tangents, then the curve is discontinuous, i.e. you make the tangent have to sweep through an angle at one point in order to continue tracing along the curve. Conversely, if you have a curve that is continuous, where you don't make the tangent sweep through an angle, there can be only one tangent at any point, and so the tangent will always be the limit-position of the chord shrinking to the point of tangency.

CAN A STRAIGHT LINE BE COMPARED TO A CURVED ONE?

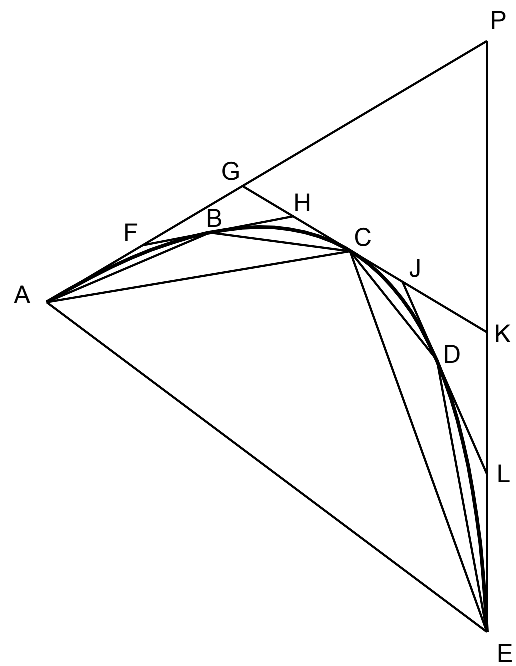
Can a straight line be less than, or greater than, or even equal to a curved one? This is not the sort of comparison one would run into in Euclid. It is not possible to take a straight line and lay it off in a curved one some number of times, so as to measure the curved line by the straight one, and get a numerical ratio between them that way. A straight line and a curved one can never be made to coincide except in points, and never in any amount of their lengths.

Nonetheless, there is another way to compare them, namely through *non-coincidence*. The straight line is the shortest length joining two points, and any curve joining those two points will be *longer*. Similarly, if a curve is all-concave on one side and all-convex on the other (as opposed to one that squiggles back and forth), then the two tangents to it at its endpoints (and intersecting each other) will be longer, taken together, than the curve itself. These ideas are intuitive. Given the choice between a straight path between two points, and a curvy one, we automatically choose the straight one. And given the choice between a smooth, gentle curve between two points that does not wiggle back and forth, and the jagged path of the two tangents to its endpoints, we automatically choose the curved path, if we are trying to save time.

But once we admit these ideas, there is also a way to say what it means for a straight line to be equal to a curved one, as follows.

DEFINING THE STRAIGHT LINE
EQUAL TO A CURVE

Given: A finite, continuous curve, AE.
Prove: There is a unique straight line which is neither greater nor less



Pick any number of points on the curve, A, B, C, D, E, and join the chords and draw tangents to each as well.

Let the tangent-perimeter and the chord-perimeter which are defined by the same number of points be called perimeters of the SAME ORDER.

For example:

tangent-sum AP + PE and chord-sum AE, since they are defined by points A and E, and so these are both of the *first order*.

tangent-sum AG + GCK + KE and chord-sum AC + CE, since they are defined by points A, C, E, and so these are both of the *second order*.

tangent-sum AF + FBH + HCJ + JDL + LE and chord-sum AB + BC + CD + DE, since they are defined by points A, B, C, D, E, and so these are both of the *third order*.

And so on.

STEP 1: *Any tangent-sum is greater than any chord-sum, going from A to E.*

Plainly	AP + PE > AE	[△APE]
so	1^{st} tangent-sum > 1^{st} chord-sum	
Again	AG + GC > AC	[rAGC]
and	<u>CK + KE > CE</u>	[△CKE]
so	AG + GK + KE > AC + CE	
so	2^{nd} tangent-sum > 2^{nd} chord-sum	

and in general

n^{th} tangent-sum > n^{th} chord-sum.

i.e. *any tangent-sum is greater than the chord-sum of the same order.*

Furthermore, since the tangent-sum is decreasing with the increase in its order, we have to say that

$(n + m)^{th}$ tangent-sum < $(n)^{th}$ tangent-sum

but $(n + m)^{th}$ tangent-sum > $(n + m)^{th}$ chord-sum [same order]

so $(n)^{th}$ tangent-sum > $(n + m)^{th}$ chord-sum

Might it be possible, though, to take a small enough tangent-sum, and a large enough chord-sum, so that

$(n)^{th}$ tangent-sum $\leq$ $(k)^{th}$ chord-sum ?

After all, the tangent-sums get shorter as n grows, and the chord-sums get longer as k grows, and therefore, if k and n are taken large enough (but not of the same order!), might we not be able to get this order of inequality?

Let's assume it and see what happens:

$(n)^{th}$ tangent-sum $\leq$ $(k)^{th}$ chord-sum [ASSUMED]

Well, we know that n and k cannot be equal, since, if they were, the tangent-sum would be greater than the chord-sum, being of the same order. Therefore

either $k > n$
or $n > k$

First let $k > n$, i.e. $k = n + m$

thus $(n)^{th}$ tangent-sum $\leq$ $(n + m)^{th}$ chord-sum

But we showed above that

$(n)^{th}$ tangent-sum > $(n + m)^{th}$ chord-sum

which is absurd. Hence k is not greater than n.

Then let $n > k$, i.e. $n = k + m$

thus $(k + m)^{th}$ tangent-sum $\leq$ $(k)^{th}$ chord-sum

but $\underline{(k + m)^{th} \text{ tangent-sum} > (k + m)^{th} \text{ chord-sum}}$ [same order]
so $(k)^{th}$ chord-sum > $(k + m)^{th}$ chord-sum

which is absurd, because chord-sums *increase* as their order increases.
So n is not greater than k.

So n and k can be neither equal, nor unequal. Hence there are no values of n and k for which it is true that

$$(n)^{th} \text{ tangent-sum} \leq (k)^{th} \text{ chord-sum}$$

which means this is never true.

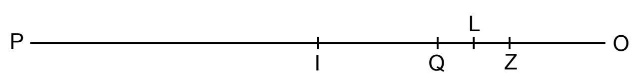
Therefore $(n)^{th}$ tangent-sum > $(m)^{th}$ chord-sum

regardless of the values of n and m.

Q.E.D.

STEP 2: *It is impossible that TWO lengths between a chord-sum and a tangent-sum be neither a chord-sum nor a tangent-sum.*

Let the straight line PI be a chord-sum, capable of being inscribed in the given curve ACE as a series of chords.



Let the straight line PO be a tangent-sum, capable of being circumscribed about the Outside of the given curve ACE as a series of tangents.

Hence PO > PI [by Step 1 above]

If possible, let it be that TWO lengths falling between PO and PI (such as PQ and PZ) are such that neither can be inscribed in nor circumscribed about ACE, and of these two let PZ > PQ.

Since PQ cannot be inscribed (given), it is longer than any chord-sum. For any line equal to a chord-sum is obviously able to be cut up and inscribed as a chord-sum; so PQ is not equal to any chord-sum. And any line shorter than a chord-sum is also able to be inscribed in ACE, although it might not reach all the way to E (if it is shorter than AE). But it is given that PQ cannot be inscribed in ACE. Therefore it is not shorter than any chord-sum, either. Therefore it is longer than every chord-sum.

Again, since PQ cannot be circumscribed (given), it is shorter than any tangent-sum. For any line equal to a tangent-sum is obviously able to be cut up and circumscribed as a tangent-sum; so PQ is not equal to any tangent-sum. And any line longer than a tangent-sum is also able to be circumscribed about ACE, although it might shoot past E (if it is longer than AP + PE). But it is given that PQ cannot be circumscribed about ACE. Therefore it is not longer than any tangent-sum, either. Therefore it is shorter than every tangent-sum.

Similarly PZ is both longer than any chord-sum, and shorter than any tangent-sum.

But *a fortiori* all the lengths between PQ and PZ will be both longer than any chord-sum, and shorter than any tangent-sum.

Hence QZ is a MINIMUM DIFFERENCE between any chord-sum and any tangent-sum.

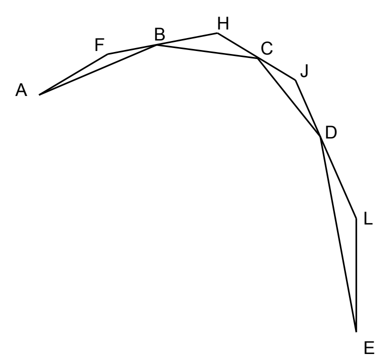
But there is no minimum difference between chord-sums and tangent-sums (see below).

Therefore it is impossible for there to be two lengths, each of which is neither able to be inscribed in nor able to be circumscribed about the given curve ACE.

Q.E.D.

PROOF of the premise that there can be no minimum difference between any chord-sum and any tangent-sum:

Obviously, the difference between tangent-sums and chord-sums of the same order decreases as the order increases, since the tangent-sums are all greater than the chord-sums, but the tangent-sums grow shorter, and the chord-sums grow longer, as the order increases.



But might there be a minimum difference between chord-sums and tangent-sums, such as QZ?

Well, any same-order tangent-sum and chord-sum will form a connected series of triangles. And in any one triangle, such as AFB, as the order increases, the peak angle AFB increases, and the base angles (where the chord AB is the "base") decrease. But if the difference (AF + FB) – AB approached a minimum value, this would imply that (∠AFB + ∠FBA) could not shrink below a given value, or that ∠AFB must always differ from 180° by a minimum value, which is false for a continuous curve. Hence (AF + FB) – AB, in all our little triangles, gets less than any assigned value as the order increases. But then it follows that the difference between the sum of all the tangents (AF + FB + BH + HC etc.) and the sum of all the chords (AB + BC + etc.), gets less than any assigned value. Therefore there can be no minimum difference between chord-sums and tangent-sums.

Q.E.D.

STEP 3: *There is one and only one straight line which is not greater than, and not less than, the curve ACE, and hence is equal to it.*

Let the straight line PO be a tangent-sum, and the part PI a chord-sum.

Since every length less than PI is a chord-sum, too, but every length greater than PO is greater than every chord-sum (by Step 1), there is some limit to the lengths which can be chord-sums: therefore it is either the case that there is a LAST length which can be a chord-sum, i.e. a longest chord-sum, or there is a FIRST length which cannot be a chord-sum.

But there is no last length which can be a chord-sum. For any length which can be a chord-sum can be broken up into segments which can be inscribed in the curve, and either stop at E or fall short of it. And any length we can do this with, will make it possible for us to find a greater one, just by taking more points on the curve. So there is no last, and hence longest, length which can be a chord-sum.

Hence there is a first length which cannot be a chord-sum.

Similarly there must be a limit to the lengths which can be tangent-sums, but there is no last length which can be a tangent-sum, and therefore there is a first length which cannot be a tangent-sum.

These cannot be different lengths.

If possible, let PQ be the first length which is too long to be a chord-sum, and PZ the first length which is too short to be a tangent-sum. Then, picking any point between Q and Z, such as L, it follows that L is both too long to be a chord-sum (since it is longer than PQ) and also too short to be a tangent-sum (since it is shorter than PZ), and therefore there are many lengths which are neither tangent-sums nor chord-sums, which is impossible (by Step 2).

Therefore there is a single length which is the limit of both, i.e. it is both the first which is too long to be a chord-sum, and the first which is too short to be a tangent-sum. Call this PL.

I say that PL is equal to the curve.

For, employing the postulates of Archimedes (ratified by Aristotle and Thomas), the curve is longer than every chord-sum, but shorter than every tangent-sum. But every length less than PL is a chord-sum. And every length greater than PL is a tangent-sum. Therefore the curve is greater than every length less than PL, and less than every length greater than PL—which can happen only if the curve is equal to PL.

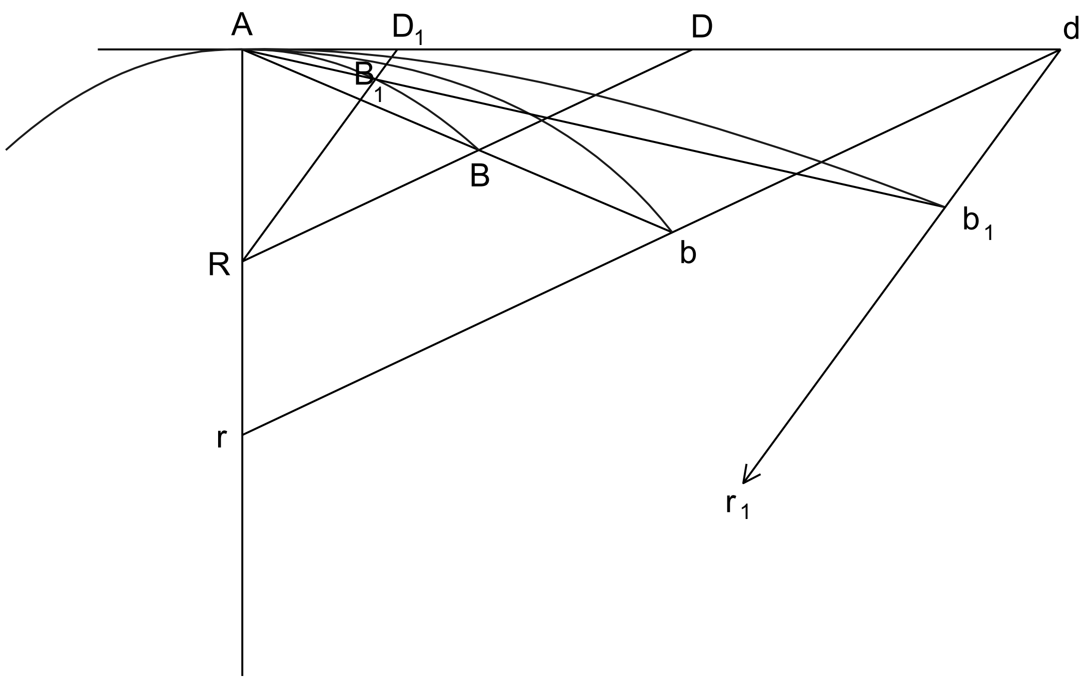
Q.E.D.

CLASS 21

LEMMA 7

"Newton's Microscope"

The same things being supposed, I say that the ultimate ratio of the arc, the chord, and the tangent to each other, is the ratio of equality.



Let the points B and D be produced to the further points b and d, such that bd is always parallel to BD.

Then	AB : AD = Ab : Ad	[always]
but	<u>Ab = Ad</u>	[ultimately]
so	AB = AD	[ultimately]

The point of the "microscope," i.e. the magnification of the whole figure ABD into the similar, larger figure abd, is to give us a **FIXED LINE** Ad, which will always be in our argument. The lines AB and AD are both shrinking to nothing, and at different rates, so it is hard to see what we can say about them in themselves. But the line Ad always stays the same, and so it is clear that Ab gets as close as we like to Ad in length.

NOTE: The points R, r are not important for this Lemma.

Q1: How do we construct the new point b each time?

Newton does not specify how far to go out, but if we are to keep Ad the same (and keeping some such line the same is the point of the construction), then the construction is decided for us, as follows: Taking the new location of B, namely B_1, join A to it. Keeping R the same, too, if we like, we then join RB_1 through to D_1 on the given tangent, and then draw dr_1 parallel to RD_1, and extend AB_1 until it intersects dr_1. Call this point of intersection our new b_1.

Q2: Is this more a tool for the imagination than a demonstration?

The magnification allows us to see the lines Ad and Ab and curve Ab (which are always as AD and AB and curve AB) tending toward coincidence in the permanent line Ad, instead of vanishing into points. Probably this magnification is not an argument from the real reason why. It is not because the magnified versions are ultimately equal that the original lines are ultimately equal.

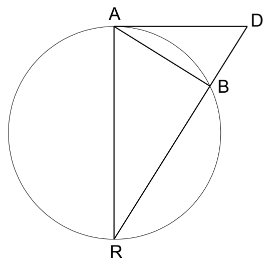
Q3: How do we see that the CURVE ACB is ultimately equal to the chord AB and the cut-off tangent AD ?

First we see that the similar curve Acb is ultimately equal to Ab and Ad; then we argue that ACB is ultimately equal to AB and AD from their similarity to the magnified versions.

And how do we see that curve Acb is ultimately equal to Ab and Ad?

Newton is not very explicit about that! But he refers to this arc as “the intermediate arc Acb,” implying the “sandwich principle.” We see that Ab and Ad are ultimately equal, since b gets as close as we please to d, and Ad is a nice permanent line. But arc Acb is always BETWEEN those two straight lines which ultimately coincide. So all the more does it ultimately coincide with either of them.

NOTE: Newton seems to be assuming that we have a finite arc Acb, with smooth and simple curvature. If it is assumed to be one of those horrid “fractal” things, then it has infinite length, and it is in no way “straightening” as we go, and the argument here would not work. Then again, those “fractals” cannot really exist all at once, but are more like processes themselves; and we are interested in paths of motion, whereas nothing can move through one of those curves.

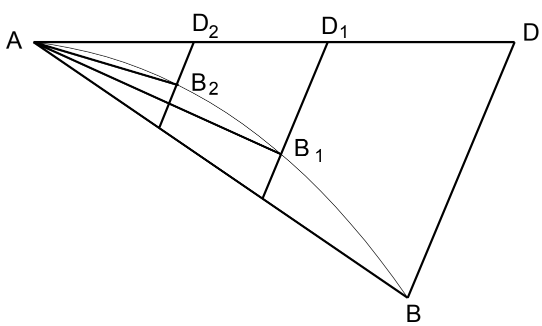


NOTE: An easy, but specific, case would be that of the circle, with RA as diameter. In that case, ∠RAD is 90° (since AD is tangent), and ∠ABR is 90° (since AR is the diameter). Hence △DAB is similar to △DRA, and therefore DA : AB = RD : RA always. But RD is ultimately equal to RA (as B goes to A). Therefore also DA is ultimately equal to AB. *Q.E.D.*

Q4: Does this argument prove that chord and arc are comparable? No. It assumes this, and then shows that they tend toward equality as B goes to A.

NOTE: We can also see the ultimate equality of chord and tangent (the ultimate equality of chord and arc is really an easy matter, and would not require this Lemma) in a different case, if we draw the secants BD always parallel to the original position. One can see (by using the original AD as our fixed line) that as D_1B_1 becomes D_2B_2 and again D_3B_3 etc., the ratio of AD and AB approaches that of equality. Just extend AB_1 till it meets BD, and we will have our magnification, in which the magnified version of AB_n is approaching the fixed length AD.

NOTE: This way of cutting off our tangents is actually USED IN COROLLARY 1.



COROLLARY 1. If we draw any fixed straight line AF through the point A (see Newton's figure), and continually draw BF to it, parallel to the tangent, as B goes to A, then BF is also ultimately equal to the vanishing arc AB.

Proof: Complete parallelogram AFBD (thus determining how we cut off our tangents!):

BF = AD (always)
<u>AD is ultimately equal to arcACB (by the Lemma itself)</u>
so BF is ultimately equal to arcACB

NOTE: Although he doesn't mention these names yet, AF is the **SAGITTA**, which will represent forces, and FB is the semi-chord or **SINE** of the arc, and BD is the **SUBTENSE** of the arc.

COROLLARY 2. Newton draws more lines, cutting off equal portions AE and BG from AD and BF, and says these abscissas AE and BG are also ultimately in the ratio of equality to AB and arc ACB.

Why? Presumably because AE is just a new tangent and AG a new diameter, and Lemma 7 applies to them just as well—in other words, there was nothing special about the diameter AR and tangent AD that we used in Lemma 7.

This means that regardless of the *original* ratio of the secant to the tangent (e.g. whether it be BD : AD, or BE : AE), the *ultimate* ratio is one of equality.

COROLLARY 3. So we can use "ALL THESE LINES" indifferently in our reasoning about ultimate ratios, i.e. ratios which are limits approached by ratios among such lines. This statement is important for the later application of Lemma 7.

But WHICH LINES? He means those whose ultimate ratio he has just proved to be that of equality, i.e. the ABSCISSAS AD, AE, BF, BG, and AB (the chord itself) and ACB (the arc).

NOTE: But AF, AG, BE, BD are a different story!!!

He does not mention these lines as being ultimately equal to AD, and there is good reason for that.

Consider the case where the subtense BD is always taken parallel to itself.

Then what is the ultimate ratio of BD to AD?

We see it by moving b up toward d while keeping Ad fixed: bd shrinks to nothing while Ab grows to Ad. But then the ratio Ad : bd increases as much as you like, and can be made as large as you please, and so does not tend to any finite ratio at all, much less the ratio of equality.

Also, in Lemma 11 he will show that the "subtenses," like BD, *have to each other* ULTIMATELY the same ratio as the SQUARES of the corresponding chords like AB.

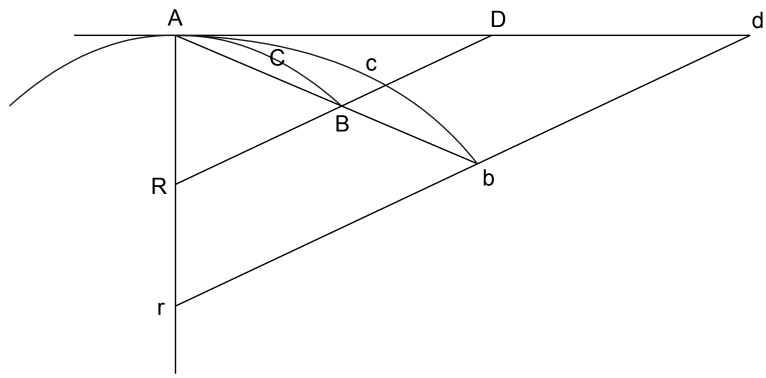
THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 22

LEMMA 8

If the given straight lines AR, BR, with the arc ACB, the chord AB, and the tangent AD, constitute three triangles RAB, RACB, RAD, and the points A and B approach one another: I say that the ultimate form of these vanishing triangles is one of similitude, and their ultimate ratio is that of equality.

Newton argues again as in Lemma 7, introducing a magnification of the original arc ACB for the sake of giving us a PERMANENT LINE Ad, which gives us a fixed target through which to understand what something is tending to.



PROOF. As B goes to A, ∠BAD goes to nothing (by Lemma 5).

Therefore also ∠bAd goes to nothing, since this is just the same angle with longer legs.

Therefore b goes to d, since Ad remains fixed, and bd is shrinking to nothing.

Therefore Ab ultimately coincides with Ad.

Therefore △rAb and △rAd ultimately coincide, and so are ultimately equal and ultimately similar.

But also the curvilinear triangle △rAcb ultimately coincides with those two triangles, since it is always caught between them, and so it is ultimately equal and similar to them also.

But then △RAB, △RAD, △RACB, always similar (in shape) and proportional (in area) to the magnified versions, must also be ultimately equal and similar to each other.

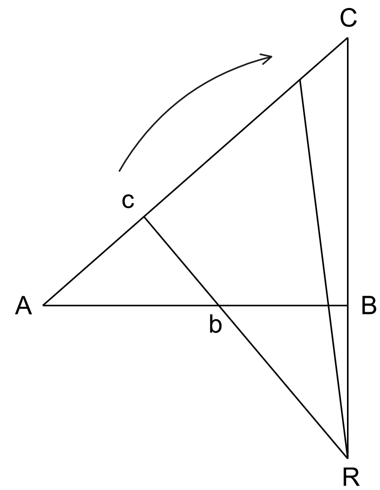
Q.E.D.

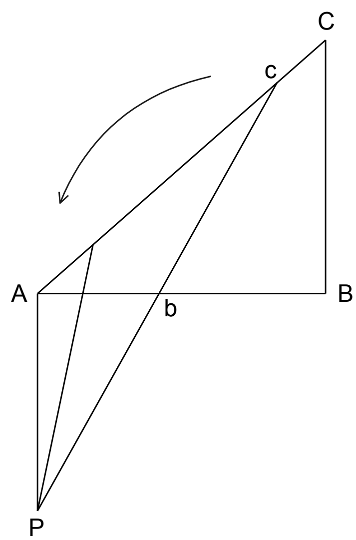
COROLLARY: So, again, in reasoning about ultimate ratios of things, we can use any of these three triangles that we like, indifferently.

Q1. Here Newton speaks of the “uItimate form” of the vanishing triangles. In general, what does that mean? Is it possible for there to be a last shape which some changing shape will attain? Or a first shape which it does not attain?

Yes, both are possible.

Let ABC be some triangle, and extend CB down to any point R. Now draw from R a secant through the triangle, Rbc. This forms the triangle Abc. Plainly Abc is not similar to ABC. But if we now rotate Rbc toward RBC, we see that the LAST SHAPE Abc WILL attain is ABC.





Now draw AP down parallel to CB, and draw a secant through the triangle, Pbc. This forms the triangle Abc. Plainly Abc is not similar to ABC. But if we now rotate Pbc toward AP, we see that the FIRST SHAPE Abc WILL NOT attain is that of similarity to $\triangle$ABC.

Q2. In the case of Lemma 8, is there an ultimate shape toward which the three triangles are tending? Is there a last shape they will attain, or a first triangular shape that they will not attain?

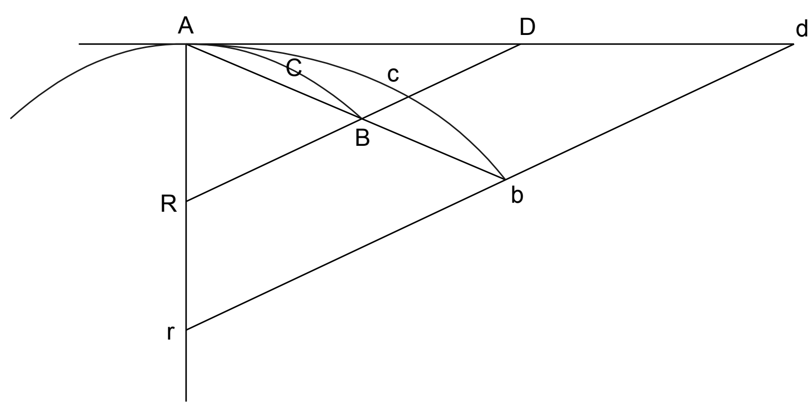
This might depend on how we define the process.

Is R a fixed point? Newton speaks of AR as a "given" line, which makes it sound like it is given in both position and length. Then R would be a fixed point.

If that is so, then ∠ARB (or ∠ARD) is closing up to nothing.

But then △ARB (or △ARD) is becoming longer and skinnier, *in infinitum*. But there is no "skinniest triangle" (or triangle that is most like a straight line), which could be either the first shape not reached in the process, or the last one reached. So in that case, there is no ultimate form.

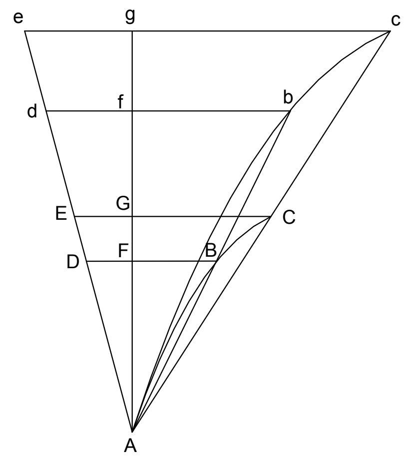
On the other hand, if R is not fixed, but instead RBD is a fixed orientation, so that RBD is always drawn parallel to its last position, through each new B, then rbd, always parallel to this, and always passing through the fixed point d, remains fixed through the whole process. And in that case it is evident that the final form of △rAb is that of △rAd, a fixed triangle. Hence the final form of △RAB, always similar to △rAb, is also that of △rAd.



THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 23

LEMMAS 9 AND 10



LEMMA 9

PARAPHRASE: *Consider a simple curve ABC forming an angle with straight line ADE, where DB and EC are always parallel, and B and C both move to A. Then the ratio of the areas of the curvilinear triangles ABD and ACE will approach the ratio of the squares of their corresponding sides, as B and C approach A.*

CONSTRUCTION:

Let AFG be the tangent to ABC, cutting DB and EC at F and G.
Extend AE to e, and let Ae be our PERMANENT LINE, magnifying AE.
So we cut Ae at d such that

Ad : Ae = AD : AE [always]

Draw the parallels to DB and EC through d and e, and let these parallels meet AB and AC produced at points b and c. And let ec and db cut tangent AFG produced at f and g.

So we have magnified our original rectilinear triangles into those with the lower case letters, which are similar. Now add in a curve through A,b,c similar to the given curve ABC. And we will always do this as B and C go to A (at whatever rates), always beginning our construction with Ae, a fixed line.

ARGUMENT:

What does c do? It moves along cg until it coincides with g. And ge sits still.
What does b do? It moves along bf until it coincies with f. And bfd might move up or down.
But as points b and c go over to f and g, $\angle$cAg vanishes (Lemma 6).
But since $\angle$cAg vanishes, or goes down to nothing, therefore everything that is always inside it also vanishes or goes down to nothing.
But that means all the angles and areas contained in it go to nothing, and specifically

curvilinear $\triangle$Abf goes to nothing
curvilinear $\triangle$Acg goes to nothing

Hence curvilinear $\triangle$Abd, which is (curvilinear $\triangle$Abf + $\triangle$Afd), ultimately becomes just $\triangle$Afd.
So too curvilinear $\triangle$Ace, being (curvilinear $\triangle$Acg + $\triangle$Age), ultimately becomes just $\triangle$Age.

Therefore,

	curv.$\triangle$Abd : curv.$\triangle$Ace = $\triangle$Afd : $\triangle$Age	[ultimately]
but	$\underline{\triangle\text{Afd} : \triangle\text{Age} = \text{Ad}^2 : \text{Ae}^2}$	[always]
so	curv.$\triangle$Abd : curv.$\triangle$Ace = Ad^2 : Ae^2	[ultimately]

but	curv.△Abd : curv.△Ace = curv.△ABD : curv.△ACE	[always]
and	$Ad^2 : Ae^2 = AD^2 : AE^2$	[always]
so	curv.△ABD : curv.△ACE = $AD^2 : AE^2$	[ultimately]

Q.E.D.

Q1. What is left at the end of the whole process? And where is fd at the end?
Nothing but △Age is left, and whatever happens to be in it.
df could be fixed the whole time, if EA : DA is a fixed ratio. Or it can go up and down.
df could coincide with eg at the end, if EA and DA are ultimately equal.
Could df disappear down at A by the end of the process?

Q2. Are the curved triangles ABD and ACE tending to some ultimate form?
Yes, and the same form, that of △Age.

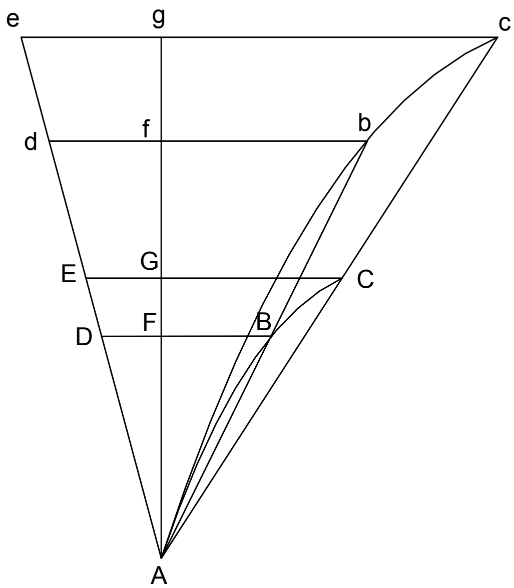
Q3. Do the points B and C have to be heading to A at some special rate, e.g. uniformly?
Nope. Just as long as they get there *together* at the end of a finite time.
If they do not vanish together, is the Lemma true?
(Then one will have vanished when the other is still there, and we cannot magnify the vanished one; and even when they vanish together, their ultimate ratios change depending on the two rates at which they drop. So long as they vanish together, though, their ratio will be ultimately the same as the squares of EA and DA)
Note: CG and BF form all the same triangles on their way down to A; therefore when they are at various points is the only thing giving us definite triangles to compare.

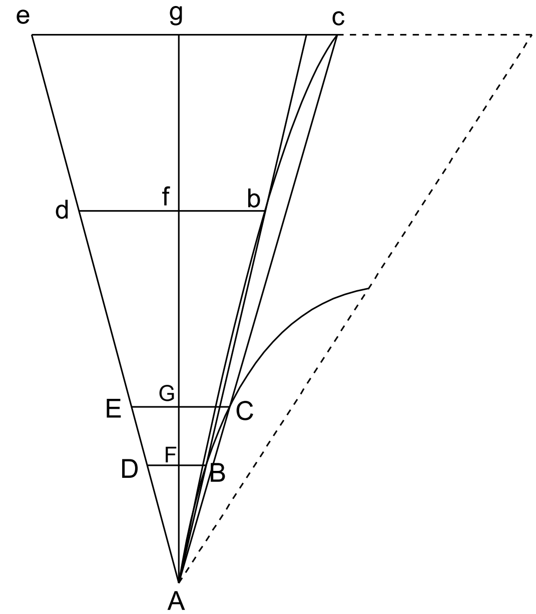
Q4. Is Lemma 5 really necessary for Lemma 9?
Newton cites it, but he is really talking more about the ultimate ratios and figures toward which the curved figures are tending, not so much any ratio they ever in fact have. They are in fact always dissimilar.

Q5. Are the curved triangles ABF and ACG ultimately similar? (yes)
Are they ever in fact similar? (no, unless B and C coincide at some point in the process)
Do they approach a fixed ultimate shape? (no)

Q6. Can curved △ABD be ultimately equal (or congruent) to curved △ACE?
(Yes, if AE is ultimately equal to AD.)

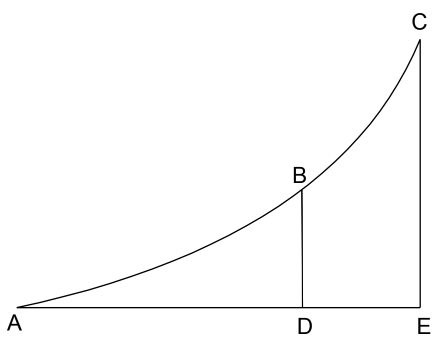
FOR CLARITY it might be good to illustrate the construction at an earlier stage (as in the left accompanying figure below) and again at a later stage (as in the right figure):





LEMMA 10

The distances which a body travels [by any finite force urging it, whether that force is definite and unchanging, or is continually increased or continually diminished] are, in the very beginning of the motion, to each other in the duplicate ratio of the times.



1. Whoa! Not just geometry anymore. Now we are talking about motion and forces and times. And FORCE means "acceleration" here, or at least that is its measure (and for a long time to come)!

2. Newton is using the diagram for Lemma 9 again, and it is just a physical application of Lemma 9. We don't need the "magnification" part of the figure for this Lemma.

3. We let the times be represented by such lines as AD, AE, and we let the velocities generated in those times be represented by the corresponding ordinates DB, EC. When he says "in the very beginning of the motion," he means that the motion begins from rest; the force might continually diminish, which means that although the velocity will increase (it must, as long as there is force), the rate at which it increases will diminish, i.e. the slope of the tangent will diminish as we progress over the curve, which is to say it will (in such a case) be a convex curve. Anyway, since the speed at A is zero, we are talking about some kind of *acceleration from rest.*

4. So the endpoints of the ordinates representing the acquired speeds will trace out some line ABC over the time ADE, and this will enclose an area. QUESTION: Does ADE have to be tangent to the curve? No. Lemma 9 did not require that. (In fact, it is a further question whether the velocity-curve CAN BE TANGENT TO THE TIME-AXIS, since if that were so, the initial acceleration would be zero!)

5. Newton assumes we see that the distances travelled by the body in any two given times (starting from A) will be as the AREAS standing on those times (under line ABC and cut off by the corresponding ordinates). For example:

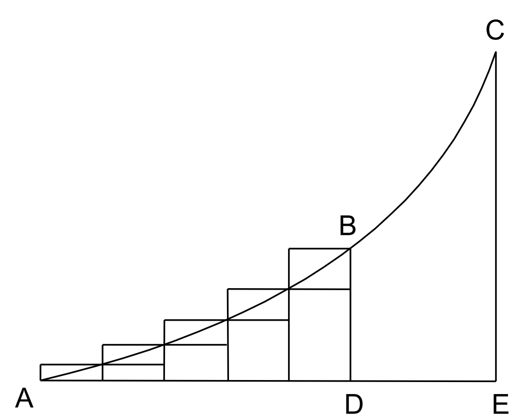
Distance in AD : Distance in AE = Curved area ADB : Curved area AEC

6. **Galileo** proved this (after a fashion) for the case of uniform acceleration from rest, where ABC is a straight line, and the whole figure is a triangle. But the general method he used to argue that the distances are as the areas standing on the lines representing the times did not depend on the uniformity of the acceleration.

7. So let's supply a refresher argument that in a velocity-over-time diagram, AREA REPRESENTS DISTANCE. Divide AD into as many segments as you like, and on them describe circumscribed and inscribed rectangles. Each of these represents a uniform speed maintained for the time on which it stands—but we saw in Galileo that for any two uniform speeds at which bodies move for any two times, the distances covered will be in the ratio compounded of the speeds and the times, i.e.

$d_1 : d_2 = (s_1 : s_2)\ c\ (t_1 : t_2)$

Now we represent the ratio of the speeds and that of the times by ratios among straight lines. And the ratio compounded of two ratios among straight lines is the same as the ratio of the rectangles they contain, so that the ratio of the distances covered at two uniform speeds over two times is the same as that of the "representative rectangles" we drew. So the total distance covered by a sum of such motions is represented by the total area of all the little rectangles. The total distance covered by the accelerated body, of course, is always *between* (a) the total distance covered by the uniform motions which the *outside* rectangles represent, and (b) the total distance covered by the uniform motions which the *inside* rectangles represent. So the area representing the distance covered by the accelerated body must be an area that is *always between the rectangle-sums*. But the only area which is always between these (less than the one, greater than the other) is the area of the curve (by Lemmas 2 and 3). Therefore the distance covered by the accelerated body is as the curved area.



8. Now back to Lemma 10. Since any two distances travelled by a body beginning

from rest are as the areas described on the times (and contained by speed-representing ordinates), i.e. as curved figures ADB, AEC, and since those areas in turn *are ultimately in the duplicate ratio of the corresponding sides AD, AE as these are taken smaller and smaller, i.e. closer and closer to A* (by Lemma 9), and since this comes to saying "closer and closer to the beginning of the motion," it now follows that two distances travelled by a body accelerated from rest are ultimately in the duplicate ratio of the times, as we take the times closer and closer to the beginning of the motion. *Q.E.D.*

9. Now we see that for EVERY CONTINUOUSLY ACCELERATED MOTION, the closer we get to the beginning of it, the more it behaves like UNIFORMLY accelerated motion. For in the case of uniformly accelerated motion, any two distances covered (from rest) are actually in the duplicate ratio of the times.

10. We have to say every *continuously* accelerated motion, since Newton said "continually augmented or continually diminished," which will guarantee the continuity of the curvature, and hence the applicability of Lemma 6 and hence of Lemma 9.

11. GALILEO. Galileo would plot our acceleration (from rest) due to gravity by a straight line. But Newton (who has already said the force increases as we get closer to the center of the Earth) would plot it as a curve. But here Newton is explaining, as it were, to what extent Galileo is right. Since we are always near the beginning of an accelerated motion to Earth, it is very nearly true that the distances covered (from rest) are as the squares of the times.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 24

LEMMA 10 COROLLARIES AND SCHOLIUM

COROLLARY 1

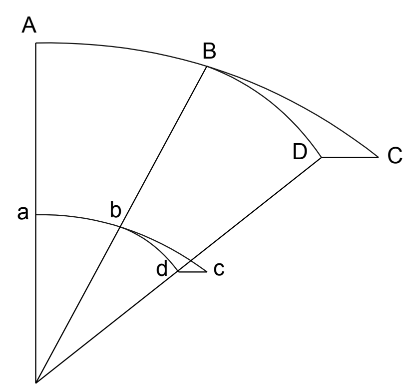
And hence it may be easily gathered that the deviations [of bodies describing similar parts of similar figures in proportional times] which are generated by any equal forces whatever similarly applied to the bodies and are measured by the distances of the bodies from those places of the similar figures to which the bodies in those proportional times would have arrived without those forces, are NEARLY as the squares of the times in which they are generated.

Newton gives no argument. We have to "gather" it for ourselves.

The main thing we need to see is *that the SLOPE OF A TANGENT to a curve in a velocity-over-time diagram corresponds to instantaneous acceleration.* That should be evident by now. The vertical lines in the diagram represent speeds, and the horizontal location of those lines represent the time those speeds are had by the mobile. But then at any point in the velocity-curve, what is the instantaneous rate of change of speed, i.e. the instantaneous acceleration? It is nothing else than the limit of the change of speed over short intervals of time shrinking down to that instant. But since those changes of speed are changes of heights (which represent the speeds) of the curve, and the horizontal locations of those heights represent the times, we are actually finding the limit of the "rise over run" ratio over short intervals of the curve, which intervals are shrinking down to the point in question. But that is the same thing as the slope of the tangent at that point. Hence the slope of the tangent to a velocity-curve at a given point is the same as the acceleration of the mobile at that instant.

With that principle in hand, we can now argue for Newton's first corollary.

Let ABD and abd be similar arcs which two bodies *would* describe in times proportional to the arcs (that's what Cor. 1 takes as given). Note that since the linear distances are always as the times to go through them, the bodies are NOT always at corresponding points, but their linear VELOCITIES ARE ALWAYS EQUAL to each other at the corresponding points (that is, until the interfering forces come along and ruin things!).



But let it be that the bodies are at corresponding points at some one moment, say when they are at B, b and let them there be acted on by equal interfering "forces" (i.e. accelerations) and in the same direction, and suppose at the end of the proportional times in which they would have been at d and D the bodies, due to the always-equal forces, are instead at c and C, so that the resulting "deviations" are dc and DC.

When the bodies are at B and b (which is at the same time, by our supposition), start plotting a velocity-over-time diagram for each body, as in the figure for Lemma 10, beginning from the moment the bodies are at B and b, and plotting only the additional velocities taking them away from their original orbits (which additional velocities are due to the always-parallel and always-equal-at-corresponding-points forces).

Now, since arcs BD and bd are *similar*, the times for them are not equal, but proportional to those similar arcs (given). [NOTE: This is why the deviations are not simply equal: the always-equal forces have more time to produce DC than dc.] Hence the time for deviation DC and the time for deviation dc are also not equal, but are as those arcs (since they are accomplished in the same times in which arcs BD and bd *would have been* accomplished).

This means the velocity-graphs will be different for our two deviations, and will not coincide.

But now let TdD be our time-axis, and let ed and ED be the final velocities among those producing deviations dc and DC.

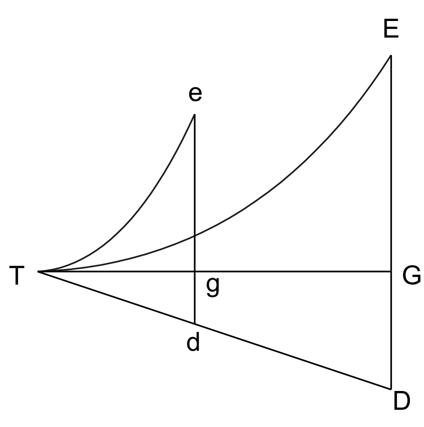
As it happens, the two velocity-curves will share a common tangent TgG, since we are given that at the original corresponding points B, b the accelerations are equal (so the slopes of the tangents to the velocity-curves must be equal, there). But if the initial accelerations are equal, so that TgG is a common tangent to both curves, what happens to the curved triangles Ted and TED as d and D go to T? They become as similar as we please to the always-similar rectilinear triangles Tgd and TGD. In other words, the triangles Ted and TED are themselves ultimately similar, and therefore

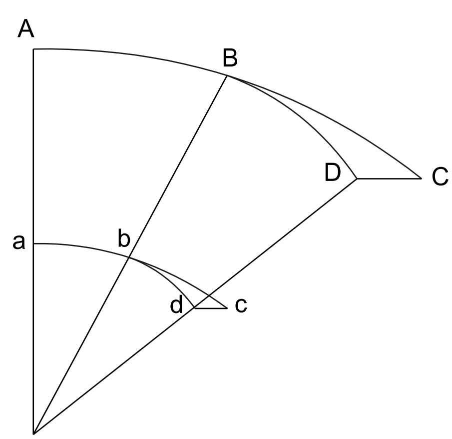
$$\triangle Ted : \triangle TED = Td^2 : TD^2 \quad \text{[ultimately]}$$

so the distances always represented by those triangular areas, namely DC and dc, are also ultimately as the squares of the times represented by TD and Td,

i.e. $DC : dc = t_{DC}^2 : t_{dc}^2$ [ultimately]

Q.E.D.

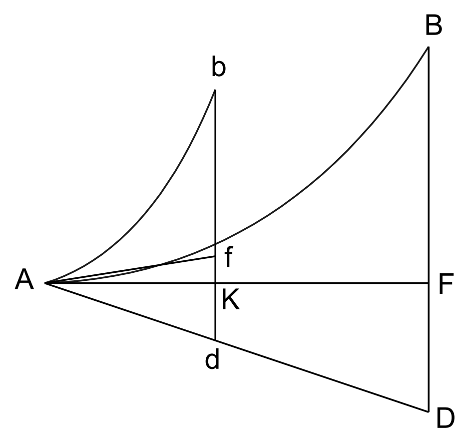




NOTE: The usefulness of this corollary seems to be for showing, much later, how the Sun affects the orbits of the moons of Jupiter (for example). The Sun is so far away that it accelerates the moons almost equally, and along almost parallel lines.

COROLLARY 2

The deviations, however, which are generated by proportional forces similarly applied to similar parts of similar figures, are as the forces and the squares of the times conjointly.



Now instead of equal forces applied at corresponding points, we have "proportional" forces, i.e. the two forces (i.e. accelerations) acting at any corresponding points are always in a fixed ratio.

So lets draw velocity-graphs again, and let A represent the moment when our two bodies are at corresponding points in their original orbits and the proportional forces begin to act. Let AB be one velocity-curve, ab the other, with time-axis AdD; and bd, BD the velocities at the proportional times Ad, AD.

Now since the initial accelerations are not equal, but similar (i.e. proportional to all other pairs of accelerations at corresponding points), therefore the tangents Af and AF will not coincide, but will form an angle. Using the time-axis AD as a coordinate-axis, the slopes of Af and AF are

fd : Ad   and   FD : AD

i.e.   fd : Ad   and   Kd : Ad   [FD : AD = Kd : Ad]

and therefore the two slopes (of corresponding tangents) are always to each other as fd
Kd.

Hence fd : Kd is the ratio of the proportional forces at time A (expressed in our diagram as the slopes of the tangents to point A).

Now at the beginning of the motions, at A, the two accelerations are instantaneous, and thus NEAR the beginning they are as close as you please to being two uniformly accelerated motions whose accelerations are as fd : Kd; and therefore their *distances* (our "deviations") are in a ratio as near as you please to the ratio of the *areas* of △Afd and △AFD, i.e. Deviation$_1$: Deviation$_2$ = $\triangle$Afd : $\triangle$AFD [ultimately]

but $\Delta Afd \ : \ \Delta AKd \ = \ fd \ : \ Kd$

so $\Delta Afd \ = \ \dfrac{fd}{Kd} \Delta AKd$

so $Deviation_1 : Deviation_2 \ = \ \dfrac{fd}{Kd} \Delta AKd \ : \ \Delta AFD$ [ultimately]

but $\Delta AKd \ : \ \Delta AFD \ = \ AK^2 \ : \ AF^2$

so $Deviation_1 : Deviation_2 \ = \ \dfrac{fd}{Kd} AK^2 \ : \ AF^2$ [ultimately]

so $Deviation_1 : Deviation_2 \ = \ fd \cdot AK^2 \ : \ Kd \cdot AF^2$ [ultimately]

i.e. $Deviation_1 : Deviation_2 \ = \ (fd : Kd) \ c \ (AK^2 : AF^2)$ [ultimately]

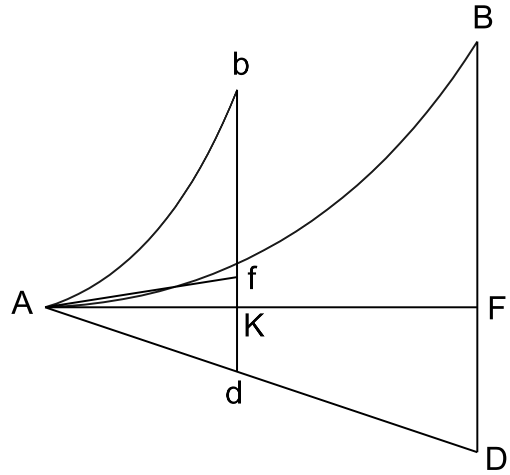
i.e. $Deviation_1 : Deviation_2 \ = \ (fd : Kd) \ c \ (Ad^2 : AD^2)$ [ultimately]

i.e. $Deviation_1 : Deviation_2 \ = \ (f_1 : f_2) \ c \ (t_1^2 : t_2^2)$ [ultimately]

And so if we take actual deviations after short proportional times, it is approximately true that

Deviation$_1$: Deviation$_2$ = (f_1 : f_2) c (t_1^2 : t_2^2)

Q.E.D.



COROLLARY 3

The same thing is to be understood of any distances whatever traversed by bodies urged with different forces, all which, in the very beginning of the motion, are as the product of (the forces) and (the squares of the times).

The argument of Cor. 2 does not require that we be speaking of "deviations" in particular, but we can be speaking of any two distances accomplished by two accelerated motions; also it does not require that we have proportional forces throughout, so long as we are content with speaking ultimately (i.e. about what is going on at the very beginning of the motion). Whatever the forces at the beginning, since any two accelerations in teensy-weensy times are as near to uniform accelerations as we like, it will be true that

$d_1 : d_2 = (f_1 : f_2) \text{ c } (t_1^2 : t_2^2)$ [ultimately, as t_1 and t_2 go to zero]

where "f" means acceleration.

COROLLARY 4

This is just a mathematical manipulation of Cor. 3:

$f_1 : f_2 = (d_1 : d_2) \text{ c } (t_2^2 : t_1^2)$ [ultimately, as t_1 and t_2 go to zero]

COROLLARY 5

This is just a mathematical manipulation of Cors. 3 and 4:

$t_1^2 : t_2^2 = (d_1 : d_2) \text{ c } (f_2 : f_1)$ [ultimately, as t_1 and t_2 go to zero]

SCHOLIUM

Here Newton defines some terms he has already been using:

"**A is directly as B**" means $A_1 : A_2 = B_1 : B_2$

So if $3A_1 = A_2$

Then $3B_1 = B_2$

"**A is inversely as B**" means $A_1 : A_2 = \frac{1}{B_1} : \frac{1}{B_2}$

So if $3A_1 = A_2$

Then $\frac{3}{B_1} = \frac{1}{B_2}$

"**A is directly as B and C**" means $A_1 : A_2 = B_1 C_1 : B_2 C_2$

So if $3A_1 = A_2$

Then $3B_1 C_1 = B_2 C_2$

"**A is inversely as B and C**" means $A_1 : A_2 = \frac{1}{B_1 C_1} : \frac{1}{B_2 C_2}$

So if $3A_1 = A_2$

Then $\frac{3}{B_1 C_1} = \frac{1}{B_2 C_2}$

"A is directly as B and C, and inversely as D" means $A_1 : A_2 = \frac{B_1 C_1}{D_1} : \frac{B_2 C_2}{D_2}$

So if $5A_1 = 7A_2$

Then $\frac{5B_1 C_1}{D_1} = \frac{7B_2 C_2}{D_2}$

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 25

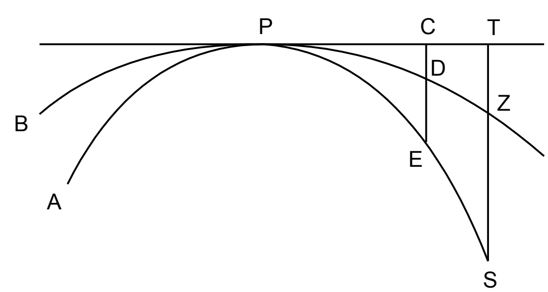
PRELIMINARY TO LEMMA 11:
EXPLANATION OF CURVATURE

EXPLANATION OF CURVATURE

We naturally speak of one curve as being "sharper" or "tighter" than another which is "blunter" or "gentler" than it. There is a quantitative more and less in the curvature of curves—and this is obviously of interest in the science of motion, as any driver will attest.

But how do we define the amount of curvature in a curve? We might say that the more a curve departs from a straight line, the more curved it is. Consider curve A, and compare it to one of its tangents, PT. If we draw TS at some angle, TS is a measure of the departure of curve APS from the straight line PT. The longer TS is, the greater the curvature, right? This is a good start, but is not good enough, as we'll see.

Suppose, for instance, we had another curve, B, to which PT was also tangent at P, but B cut TS at Z. Then curve B departs from the tangent less than curve A does, and so it would seem fair to say that curve B is less curved than A is. (And certainly there is some truth to that.)

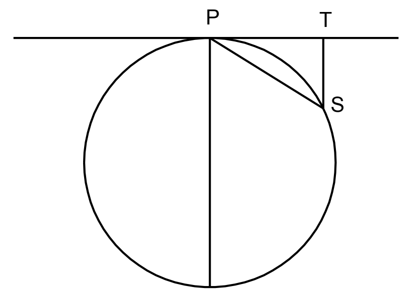


But now take C along PT somewhere, and draw CDE parallel to TZS. Is it the case that

CE : CD = TS : TZ ?

Not necessarily! And that can't be true continuously without making one curve a blow-up of the other (a special case). But this prompts us to wonder what is happening to those ratios—in particular, what if they are changing toward the same ultimate ratio? Then we would be inclined to say that the curvature of the two curves becomes as equal as we like as we approach point P!

The problem we are bumping into is that curvature can vary continuously in a curve (and indeed must do so if the curve is not a circle), and therefore curvature is a little bit like “instantaneous velocity.” When curvature changes continuously, we have to talk about curvature at a point—where there is no curve! When velocity changes continuously, we cannot identify a single fixed speed except at an instant, when there is no speed! In the case of instantaneous velocity, we escape the paradox by defining the “speed at an instant” not as speed which the mobile actually possesses, but as one which its average speed (defined by the distance actually traversed in the short time interval) approaches as nearly as we please (but does not attain) during the time bounded by the instant, as we shrink the time to nothing. Somewhat similarly we define “curvature at a point” by the ultimate comparison of the subtense-tangent ratio at one point in a curve to the subtense-tangent ratio at another point.

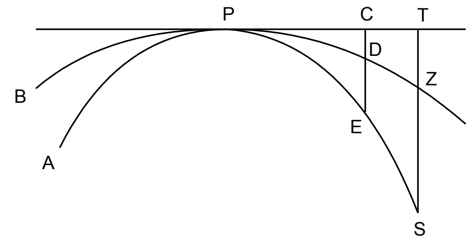


At first, we might be tempted to quantify the curvature at a point simply by the ultimate ratio approached by the subtense-tangent ratio as we shrink the tangent to that one point. But we cannot do that, since generally there IS no such ultimate ratio! Consider a circle, for example, with tangent PT and subtense TS. As we shrink PT, what is the ultimate ratio to which TS : TP will go? There is none. Suppose there were some finite ratio which limited this process. Then construct PT and TS in that ratio and join PS. Then the angle TPS is the ultimate angle as we shrink PT (or PS) to nothing. But that is false! The ultimate angle is zero. Put the other way: if we are given a ratio of TS : TP, and told that is the limit ratio, and we cannot in the process find a LESSER ratio than that, then we just construct the corresponding triangle, and cut the ∠TPS with any intercepting line, and the resulting subtense-tangent ratio will be still less.

So, at least as we will consider it, curvature is a RELATIVE thing only. We do not look at one point on a curve and try to find the limit of the ratio TS : TP as T goes to P. Rather, we compare TWO POINTS in a curve, or in two curves, and ask whether the CORRESPONDING SUBTENSE-TANGENT ratios are ultimately equal, or if one is ultimately greater than the other. Those subtense-tangent ratios are “corresponding” which are taken from the same point on the tangent (or with tangents of equal lengths) and with the same subtense-angle.

[NOTE: Will this ultimate ratio be different if our subtenses have different angles from each other? Newton’s Case 3 of Lemma 11 suggests that this is so. He seems to think it is necessary for the subtenses in his own lemma at least to be ultimately parallel, if they are not so throughout.]

If two curves share a point P, the two curvatures there are in the same ratio as the limit, as T goes to P, of the ratio



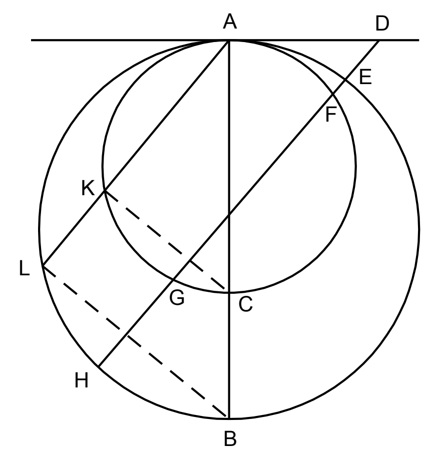
$$\left(\frac{TZ}{TP} : \frac{TS}{TP}\right)$$

$$\lim_{T \to P}\left(\frac{TZ}{TP} : \frac{TS}{TP}\right)$$

$$\lim_{T \to P}(TZ : TS)$$

Now, by this definition of "curvature" it is clear that every point on a CIRCLE will have the same curvature, and any two equal circles will have equal curvature. But what about UNEQUAL CIRCLES? Instinctively, we would say that the SMALLER circle has GREATER curvature, and indeed this definition harmonizes with that thought. According to our definition, it follows that *THE CURVATURES OF TWO CIRCLES ARE TO EACH OTHER INVERSELY AS THEIR DIAMETERS* or radii.

Consider two circles with a common tangent at A, and let diameter AB be greater than diameter AC. Let DEF be a common subtense, drawn at whatever angle (which we will keep constant). Produce DEF through to G and H to exit the circles. Draw AKL at the subtense angle.



I say that	DE/DA : DF/DA = AC : AB [ultimately, as D goes to A]	
For	AD^2 = DE.DH	[Euc.3.36]
and	$\underline{AD^2 = DF.DG}$	[Euc.3.36]
so	DE.DH = DF.DG	
so	DE : DF = DG : DH	[always]
but	$\underline{DG : DH = AK : AL}$	[ultimately, as D goes to A]
so	DE : DF = AK : AL	[ultimately]
thus	DE : DF = AC : AB	[ultimately]
so	DE/DA : DF/DA = AC : AB	[ultimately]

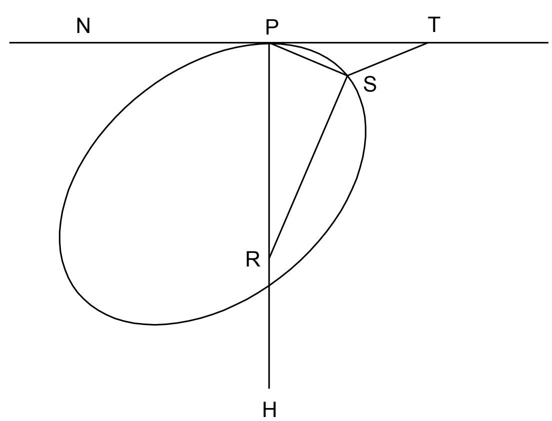
Q.E.D.

This makes it apparent that we can have circular curvatures as large and as small as we like, and in any ratio.

Given this, and given the uniformity of curvature in circles, and given that curvature is defined by the relation of ultimate ratios (and not by overall shapes), it seems reasonable to try to measure curvature in **other** curves by the circles which have the same curvature as them at a given point. Such a circle is called a CIRCLE OF CURVATURE.

To illustrate, let NPT be tangent to some curve, such as an ellipse. Draw PH at right angles to it (this is called the "normal" to the curve at P). Hence any circle also tangent to NPT at P will have its diameter along PH. Cut off the tangent at any start-length PT, and draw TS at some angle to PT, which angle we will always use as we move T to P. Also, join the chord PS, and draw SR at right angles to it, forming right triangle PSR. Plainly a circle can be drawn through P, S, R, having diameter PR, and tangent to NPT at P. As T goes to P, we repeat the construction continuously, and if it should happen that, when S and P and T all coincide, there is a final length PR, then we will have a circle whose TS : TP ratio is ultimately equal to that of the given curve (although these do not converge on a finite ratio, as explained above—they are like triangles which get as similar as you please without converging on a fixed triangular shape).

NOTE: How do we know the limit circle has the same curvature as the curve at P? Because ST : TP is the same ratio at S *always* for the curve and for the changing circle, and in the case of our limit-circle, it is the *ultimate* ST : TP that is the same for both it and the curve, since S is at P.

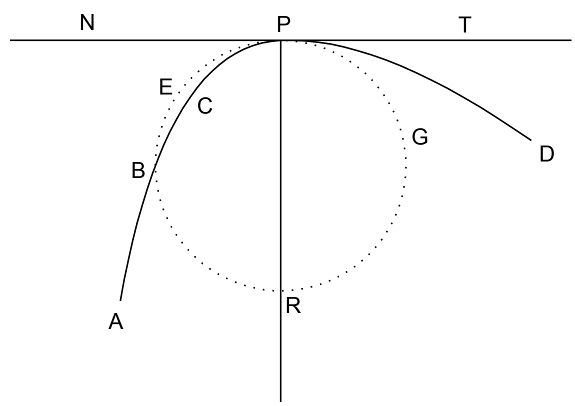


Very well, but how do we know there will BE such a final PR? Well, we need to know the specifics of the curve. In some cases, there will not be, as we will learn in the SCHOLIUM following Lemma 11.

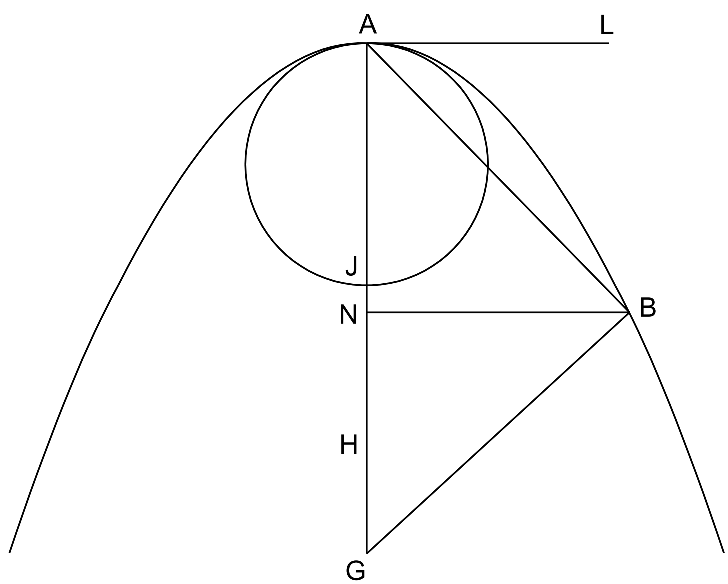
For now, we will just say this: by "FINITE CURVATURE" we will mean curvature which is equal to the curvature of some circle. Non-finite curvature comes in two flavors: (1) sometimes a curve is of such a nature that, at a certain point P, if we shrink the chord to P, there is **no greatest** circle sharing that chord and tangent, so that the "circle of curvature," the limit-circle, is infinite, and degenerates into the tangent itself, (2) other times a curve is of such a nature that, at a certain point P, if we shrink the chord to P, there is **no smallest** circle sharing that chord and tangent, so that the "circle of curvature," the limit-circle, is a point. In other words, compared to circles, some curves have infinitely small curvature, and are like straight lines, while others have infinitely great curvature, and are like points, or like rectilineal angles.

The curves which are of principal concern to Newton are the conic sections, and these, being of second degree (like the circle), cannot have non-finite curvature compared to the circle (like certain third-degree curves, etc., can; more on this in the SCHOLIUM following Lemma 11).

NOTE: Since a circle has uniform curvature, and no other curve has uniform curvature, it follows that a circle of curvature will have the same curvature as the given curve ONLY AT THE GIVEN POINT (or a few others like it). Since the non-circular curve is continuously changing curvature, it is (for instance) more curved than the circle before P, and less curved than the circle after P. Consider curve ABCPD, with circle of curvature PR at point P, and common tangent PT. Going clockwise from R, prior to P the circle falls between the curve and the tangent; but after P, the curve falls between the circle and the tangent.



This means that A CIRCLE OF CURVATURE OFTEN CUTS THE CURVE at the point where they have the same curvature, as our circle cuts the curve at P. Exceptions occur—for example, when the curve is at a point of maximum curvature or minimum curvature, so that it is symmetrical on both sides of the given point.



Also, a circle of curvature is UNIQUE at a given point in a curve. This comes from the fact that there is only one tangent there (assuming continuous curvature), and we get the circle of curvature by drawing a circle with its diameter along the normal and which cuts the curve at the other end of the chord—then we shrink the chord to the limit. At any stage, we have only one circle. And one such process can have only one limit. You can't get a second

circle of curvature by now shrinking chord BP to P, since you obviously cannot have a smallest circle that way.
For the sake of clarity, let’s find the CIRCLE OF CURVATURE FOR THE PRINCIPAL VERTEX OF A PARABOLA.

Let AB be a parabola, axis AH, latus rectum AL (drawn tangent), B a random point on the curve.

Join chord AB, draw BG at right angles to it, and drop BN at right angles to the axis.
Thus a circle on diameter AG passes through B and is tangent to AL at A.
If there is a limiting-circle among such circles as we shrink AB, we have our circle of curvature for point A.

Well, note that

	$BN^2 = AN.NG$	[circle]
but	$BN^2 = AN.AL$	[parabola]
so	$NG = AL$	

always, throughout the shrinking-process.
But AN shrinks to nothing, as AB does.
So the limit of the process is when AN = 0, and we have nothing left but NG.

So draw AJ = NG = AL, the upright side.

The circle on diameter AJ is our circle of curvature.

To prove it, we need to verify that our definition applies, i.e. we need to show that, if we draw a common subtense TZS (e.g. at right angles to tangent AT), and shrink AT to nothing, then

$$TS : TA = TZ : TA \qquad \text{[ultimately, as T goes to A]}$$

So, draw CSE as a common ordinate, and join BZ.
Now, as T goes to A, the point C also goes to A. Therefore, as T goes to A

$$CJ = AJ \qquad \text{[ultimately]}$$

so $$\frac{CS^2}{CJ} = \frac{CS^2}{AJ} \qquad \text{[ultimately]}$$

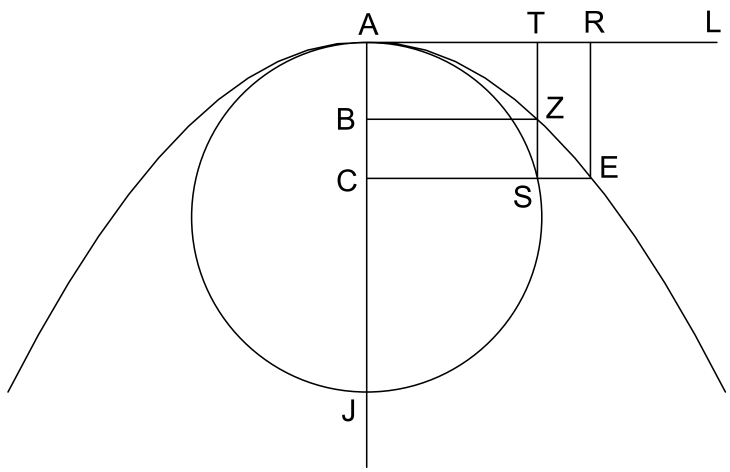
so $$\frac{CS^2}{CJ} = \frac{BZ^2}{AL} \qquad \text{[ultimately, since CS = BZ, and AJ = AL]}$$

so $AC = AB$ [ultimately, since $CS^2 = AC.CJ$ in circle, $BZ^2 = AB.AL$ in parab]

so $TS = TZ$ [ultimately]

so $\frac{TS}{TA} = \frac{TZ}{TA}$ [ultimately]

Q.E.D.

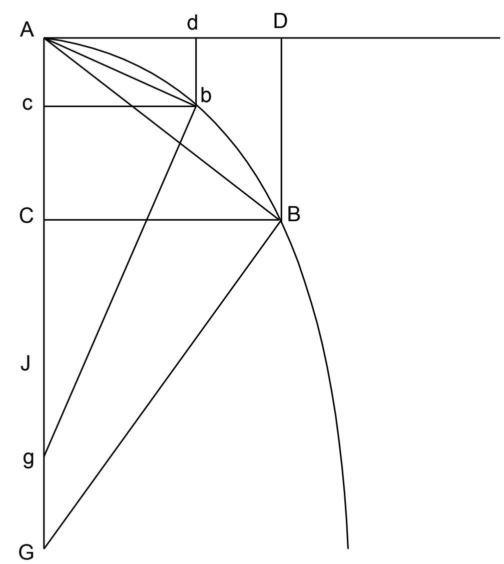


THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 26

LEMMA 11

The evanescent subtense of the angle of contact, in all curves which at the point of contact have a finite curvature, is ultimately as the square of the subtense of the coterminous arc.



A “subtense” is what subtends something else, i.e. what stands under it.

So the “subtense of the angle of contact,” where the angle of contact is BAD, is the straight line BD, since it stands under the angle BAD.

And the “subtense of the arc,” where the arc is AB, is the chord AB, since it stands under that arc.

“Coterminous” just means “has the same endpoint.” So chord AB and arc AB are “coterminous,” since they share endpoints A and B.

Newton aims to prove that as we shrink down two subtenses BD and bd, their ratio approaches that of the squares on AB and Ab, i.e. the lines subtending the corresponding arcs.

This Lemma is divided into three cases.

CASE 1. (BD is perpendicular to AD)

Draw AG perpendicular to tangent AD, and draw BG perpendicular to AB, and bg perpendicular to Ab. Now since we are assuming finite curvature, there is a circle of curvature that can be drawn there, whose diameter is the limit of AG as B goes to A. (Since $\angle ABG = 90°$, therefore a circle can always be drawn through A, B, G, and AG is the diameter, and to assume finite curvature is to assume a final finite value of AG as B goes to A. Call the ultimate value of AG "AJ".)

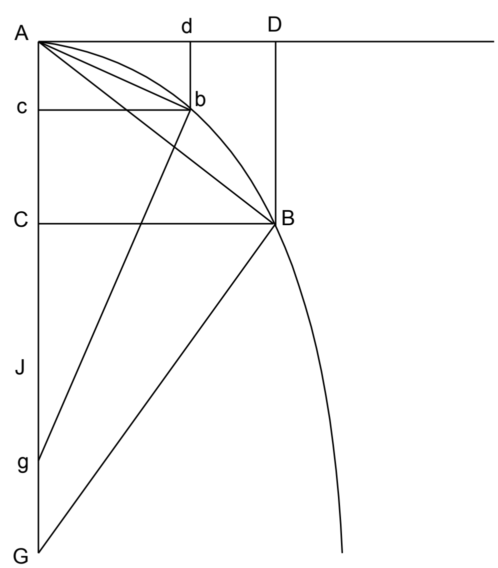
Now, Newton brings up the circles that might be drawn through points Abg and ABG, and says:

	$AG : AB = AB : BD$	[since $\triangle ABG$ is similar to $\triangle ABD$]
so	$AB^2 = AG \cdot BD$	
and	$\underline{Ab^2 = Ag \cdot bd}$	[by the same reasoning]
so	$AB^2 : Ab^2 = AG{\cdot}BD : Ag{\cdot}bd$	

But, since AJ is the limit of both AG and ag, therefore

	$AG : Ag = 1 : 1$	[ultimately]
so	$AB^2 : Ab^2 = BD : bd$	[ultimately]

Q.E.D.



CASE 2. (BD is given at some non-right angle to AD)

Not a very interesting difference. But it is reminiscent of Lemma 7, in which we learn that the angle at which certain lines are drawn makes no difference to the ultimate proportion or equality under consideration.

Here we can reason like this. Suppose we do not draw BD and bd at right angles to AD, but at some other fixed angle, e.g. 60°. Then let BZ and bz be drawn at right angles to AD. Then, by Case 1,

$$AB^2 : Ab^2 = BZ : bz \quad \text{(ultimately)}$$

But BZD and bzd will always be similar triangles, so that

$$BZ : bz = BD : bd$$

Hence it remains true that

$$AB^2 : Ab^2 = BD : bd \quad \text{(ultimately)}$$

QUESTION

We might think that according to Lemma 7, Ab is ultimately equal to Ad, and AB is ultimately equal to AD. So why can't we say that they are ultimately proportional? i.e. why not this:

$$AB : BD = Ab : bd \quad \text{(ultimately, by Lemma 7, each ratio is an equality)}$$

so $$AB : Ab = BD : bd \quad \text{(ultimately, by alternating)}$$

But that is not compatible with saying

$$AB^2 : Ab^2 = BD : bd \quad \text{(ultimately)}$$

which is the conclusion of Lemma 11!

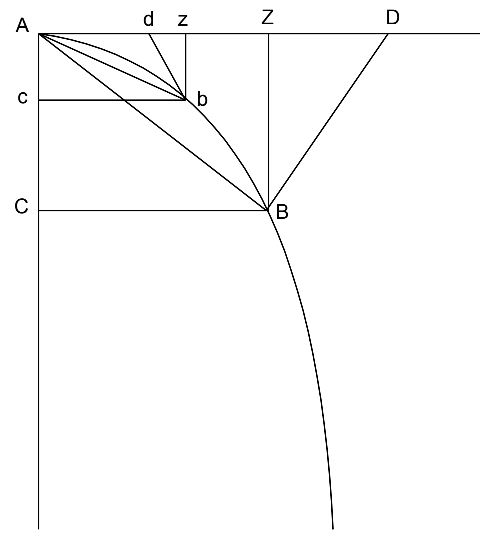
So what is going on here?

Actually, Lemma 7 never says anything about BD and bd, the "subtenses." He mentions the "secant" BD in Lemma 7, but never draws any conclusion about it, precisely because he cannot. So Lemma 11 is getting at those pesky lines BD, bd, which Lemma 7 did not allow us to say anything about.

QUESTION: What if BD is always drawn at one given angle, e.g. 60°, and bd is always drawn at another given angle, e.g. 20°?

If we draw bz and BZ perpendicular to AD, we now have two dissimilar triangles bzd and BZD, each of which maintains its shape as B and b go to A. In this case, bd and BD are not ultimately parallel, so it seems unlikely that they are ultimately in the same ratio as bz and BZ, and hence it seems unlikely that they are ultimately in the same ratio as ab^2 and AB^2.

This fits with what Newton says in Case 3, namely that we must at least be given that ∠Adb and ∠ADB are ultimately equal, i.e. that BD and bd are ultimately parallel.



CASE 3.

Here the angle of BD to AD is not given, but fluctuates according to some rule as D goes to A, but it is given that bd and BD are ultimately parallel, i.e. tend toward it as near as you like, as B and b go to A. This does not change the ultimate ratio of BD : bd.

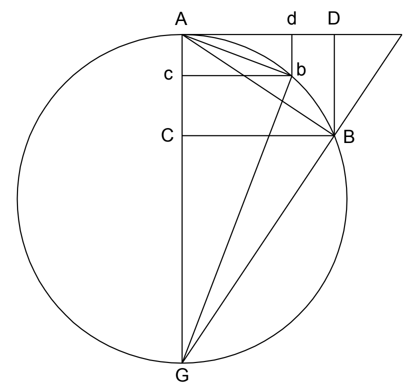
Example: Let the curve be a circle, and let BD be taken by being the extension of radius CB, and let bd be always taken at right angles to AD. Then BD and bd will never be parallel, but they are ultimately parallel as B and b go to A.

QUESTION

What if the curve in Lemma 11 [Case 1] were a circle?
Then, since the angle inscribed in a semicircle is right, G, g, J would all coincide, so AJ = AG, the diameter. And also it would follow that $AB^2 : ab^2 = BD : bd$ *actually and always*, not just ultimately (provided we are thinking of Case 1 or 2, not 3):

For	$AB^2 = BD \cdot AG$	[as before]
and	$ab^2 = bd \cdot AG$	[since Ag and AG are the same now]
so	$AB^2 : ab^2 = BD \cdot AG : bd \cdot AG$	
so	$AB^2 : ab^2 = BD : bd$	

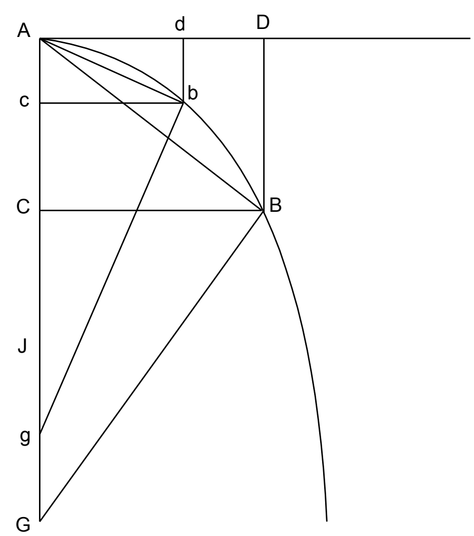
Q.E.D.



This is reminiscent of Galileo's circle theorems, in which the "ramps" are AB, Ab, and they are in the duplicate ratios of the heights, BD, bd, and therefore the times down them are equal.

QUESTION

What if the curve in Lemma 11 were a parabola? What would the value of AJ be then? (Let R be the upright side, AG the axis.)



Since $BC^2 = AC \cdot R$
and $BC^2 = AC \cdot CG$
thus CG = R (actually and always).
Again cg = R (for the same reasons).
And since AC (as well as BC^2) is shrinking to nothing, therefore AJ is nothing else than the final CG, which must therefore be R.
So AJ = R.

In the case of a parabola, then, the circle of curvature at its principal vertex, A, has a diameter equal to the upright side. That is the first circle small enough not to fall between the parabola and the common tangent AD.

QUESTION

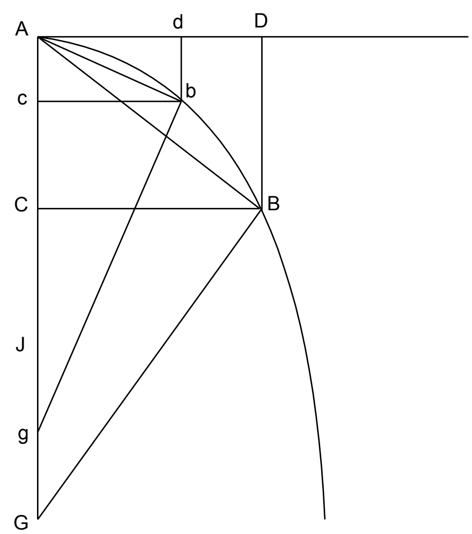
What if the curve in Lemma 11 were an ellipse? What would the value of AJ be then? (Let R be the upright side, AG part of the major axis, the whole of which we'll call M.)

In an ellipse,	$BC^2 = R \cdot AC - R/M\ (AC^2)$	[Conclusion from nature of ellipse]
so	$BC^2 = AC\ [R - R/M\ (AC)]$	
but	$BC^2 = AC \cdot CG$	[from the property of the circle]
so	$CG = R - R/M\ (AC)$	

But R and M are constants, and AC vanishes into nothing, so that

	$CG = R$	(ultimately)
Hence	$AJ = R$	

So once again, AJ, the diameter of the circle of curvature for point A, is the upright side.



THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 27

LEMMA 11 COROLLARIES AND SCHOLIUM

Newton next develops some corollaries to Lemma 11:

COROLLARY 1

> *Whence, since the tangents AD, Ad, the arcs AB, Ab, and their sines BC, bc become ultimately equal to the chords AB, Ab, likewise, the squares of all these ultimately will be as the subtenses BD, bd.*

This follows from Lemma 7.

Lemma 7 says that

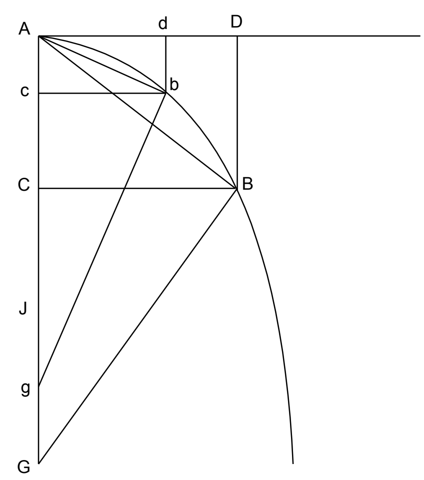
	$AD : Ad = AB : Ab$	[ultimately]
so	$AD^2 : Ad^2 = AB^2 : Ab^2$	[ultimately]
but	$\underline{BD : bd = AB^2 : Ab^2}$	[ultimately, by Lemma 11]
so	$BD : bd = AD^2 : Ad^2$	[ultimately]

Thus for the "tangents" AD, Ad.
Likewise for the arcs AB, ab

$$BD : bd = \text{arc } AB^2 : \text{arc } ab^2$$

and again for the sines BC, bc

$$BD : bd = BC^2 : bc^2$$



COROLLARY 2

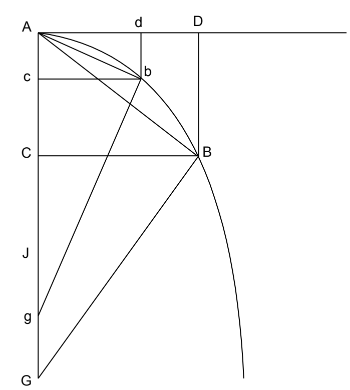
All their squares are likewise ultimately as are the sagittae (of the arcs) which bisect the chords and converge to a given point. For, those sagittae are as the subtenses BD, bd.

SAGITTA is often translated as "versed sine," which is wrong. A "versed sine" is a particular kind of sagitta, specific to the unit circle of trigonometry—given an arc in the unit circle, θ, the "versed sine" is the leftover portion of the radius which bisects the chord of θ. In other words, the "versed sine" = $1 - \cos\theta$.

But the idea of a *sagitta* is much more general than that. It is not found only in unit circles, but in all curves. And it is not defined by a center, but by any fixed point. And it need not bisect the arc, but only the chord. *A "**sagitta**" is a straight line bisecting the chord of an arc (which does NOT necessarily bisect the arc itself).*

Now, what are the "sagittae" ("arrows") that Newton is talking about?

Easy instances would be AC, Ac (the "bow" is aimed straight up, ready to fire those "arrows"), if bc and BC happened to be chords of the curve and bisected at c and C. If that were true, then Newton is here saying:

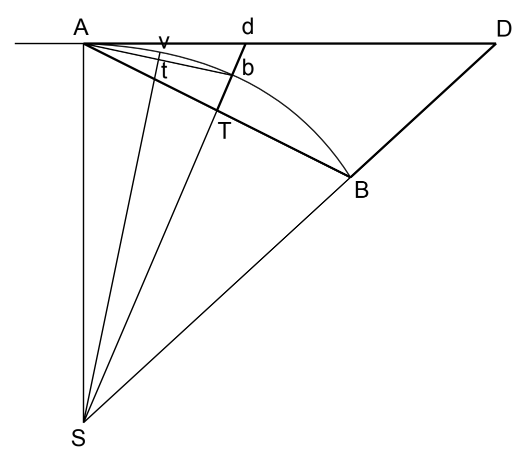


	AC : Ac = BD : bd	[actually and always]
but	$\underline{\text{BD : bd} = \text{AB}^2 : \text{Ab}^2}$	[ultimately, by Lemma 11]
so	AC : Ac = $\text{AB}^2 : \text{Ab}^2$	[ultimately]

and so the sagittae AC, Ac are also ultimately as the squares of the tangents, arcs, and sines, according to Cor. 1.

Easy indeed, but that cannot be quite what Newton means, since bc and BC are always dropped at right angles to AG, and that is a very specific situation. Also, in Cor. 1 he calls BC the "sine," and so the angle he is talking about is AGB, and so the "sagitta" of that must bisect chord AB.

So he means the *undrawn* sagittae which bisect the chords Ab and AB, and drawn from whatever fixed point we please. If we pick a point S on the concave side of the curve, and then join S to the midpoints of AB, Ab, (call these T, t), and draw SBD through to the tangent, and STbd also, and join St through to v on the arc, then Newton is saying that



Tb : tv = $\text{AB}^2 : \text{Ab}^2$ (ultimately)

Proof:

I say first that

$Tb = \frac{1}{4} BD$ (ultimately)

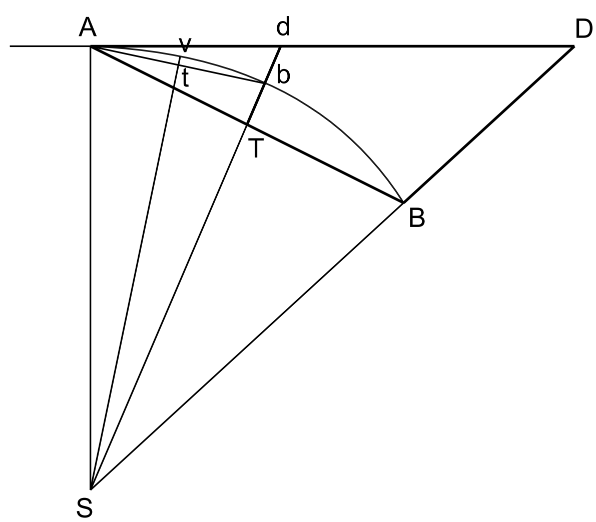
For, as B goes to A, BD and Tbd become ultimately *parallel*,
so that △AdT and △ADB become ultimately *similar*,

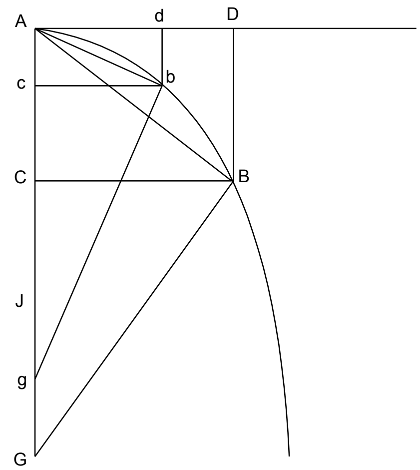
thus	$Td : BD = AT : AB$	[ultimately]
i.e.	$Td = \frac{1}{2} BD$	[ultimately]
But	$bd : BD = Ab^2 : AB^2$	[ultimately, Lemma 11, Case 3]
so	$bd : BD = Ad^2 : AD^2$	[ultimately, Lemma 11, Cor. 1]
so	$bd : BD = AT^2 : AB^2$	[ultimately, since Ad : AD = AT : AB ultimately]
so	$bd : BD = 1 : 4$	[ultimately]
i.e.	$bd = \frac{1}{4} BD$	[ultimately]
So	$Td - bd = \frac{1}{2} BD - \frac{1}{4} BD$	[ultimately]
i.e.	$Tb = \frac{1}{4} BD$	[ultimately]

Likewise, if we bisect chord Ab at t always, and form sagitta tv,

then	$tv = \frac{1}{4} bd$	[ultimately]
Thus	$Tb : tv = \frac{1}{4} BD : \frac{1}{4} bd$	[ultimately]
so	$Tb : tv = BD : bd$	[ultimately]
so	$Tb : tv = AB^2 : Ab^2$	[ultimately, by Lemma 11 case 3]

Q.E.D.





COROLLARY 3

And so the sagitta is in the duplicate ratio of the times in which a body describes the arc with a given velocity.

Since the velocity is given, we are talking about a uniform motion, here, at constant (linear) speed V. So the body describes portions of the arc in times proportional to their lengths.

Hence	arc Ab : arc AB = time thru arc Ab : time thru arc AB	(at speed V)
But	arc Ab : arc AB = Ab : AB	(ultimately, Lem.7)
so	Ab : AB = time thru arc Ab : time thru arc AB	(at speed V; ultimately)
so	$Ab^2 : AB^2 = (\text{time thru arc Ab})^2 : (\text{time thru arc AB})^2$	(squaring all)
but	$Ab^2 : AB^2 = tv : Tb$	(ultimately, by Cor. 2)
so	$tv : Tb = (\text{time thru arc Ab})^2 : (\text{time thru arc AB})^2$	(ultimately, for speed V)

Note: Newton does not say "ultimately" in this Corollary, but I think he has to, since all the ratios we are considering are fluid.

COROLLARY 4

The rectilinear triangles ADB, Adb are ultimately in the triplicate ratio of the sides AD, Ad, and in the sesquiplicate ratio of the sides DB, db, inasmuch as being in a ratio compounded of the sides AD and DB, Ad and db. So also triangles ABC, Abc are ultimately in the triplicate ratio of the sides BC, bc. Verily, the subduplicate ratio of the triplicate ratio I call the sesquiplicate ratio, which of course is compounded of the simple ratio and the subduplicate ratio.

Here Newton makes two claims, and then offers a little definition.

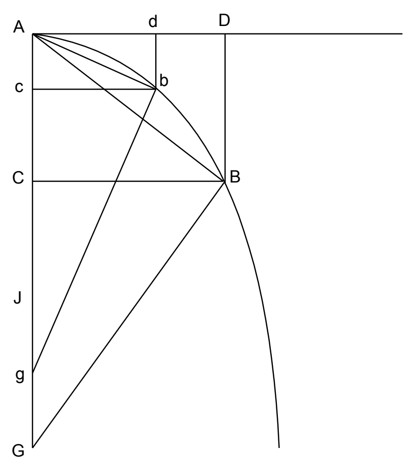
(CLAIM 1)	$ADB : Adb = AD^3 : Ad^3$	(ultimately)
(CLAIM 2)	$ADB : Adb = \sqrt{DB^3} : \sqrt{db^3}$	(ultimately)
(DEF.)	"sesquiplicate ratio"	

(CLAIM 1)

Plainly	ADB : Adb = AD·DB : Ad·db	(at least ultimately, since DB, db ult. parallel)
so	ADB : Adb = (AD : Ad) c (DB : db)	(ultimately)
but	(AD : Ad) = AB : Ab	(ultimately, by Lemma 7)
and	(DB : db) = AB^2 : Ab^2	(ultimately, by Lemma 11)
so	ADB : Adb = (AB : Ab) c (AB^2 : Ab^2)	(ultimately)
i.e.	ADB : Adb = AB^3 : Ab^3	(ultimately)
thus	ADB : Adb = AD^3 : Ad^3	

(CLAIM 2)

We said	ADB : Adb = (AD : Ad) c (DB : db)	(just above)
but	(AD : Ad) = (AB : Ab)	(ultimately, by Lemma 7)
and	(AB : Ab) = ($\sqrt{DB}$: $\sqrt{db}$)	(ultimately, by Lemma 11)
so	ADB : Adb = ($\sqrt{DB}$: $\sqrt{db}$) c (DB : db)	(ultimately)
i.e.	ADB : Adb = $\sqrt{DB^3}$: $\sqrt{db^3}$	(ultimately)



(DEFINITION)

By "sesquiplicate ratio" Newton means the subduplicate ratio of the triplicate ratio, i.e. the ratio between the square roots of the cubes. For example, if X : Y is the simple, original ratio, then the "sesquiplicate ratio" of X : Y is $\sqrt{X^3}$: $\sqrt{Y^3}$.

And we have just seen that the sesquiplicate ratio is also the ratio compounded of the simple ratio and the subduplicate ratio (i.e. the ratio of the square roots):

(X : Y) c ($\sqrt{X}$: $\sqrt{Y}$) = X $\sqrt{X}$: Y $\sqrt{Y}$ = $\sqrt{X^3}$: $\sqrt{Y^3}$.

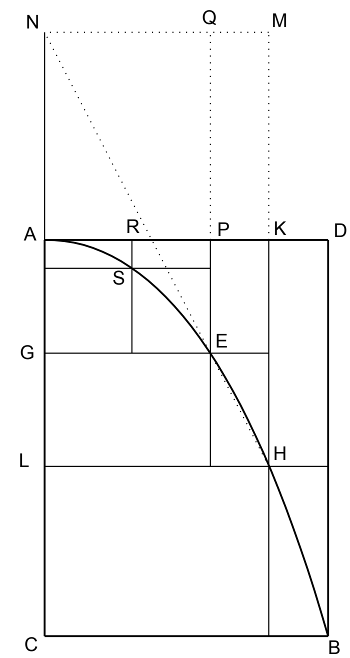
A funny word using the same prefix is "sesquipedalian," which describes a word which is more polysyllabic than it has to be (it's "three half-feet," rather than just a "foot"). The word itself seems to be sesquipedalian.

COROLLARY 5

And since DB, db are ultimately parallel and [ultimately] in the duplicate ratio of AD, Ad, the ultimate curvilinear areas ADB, Adb, will be (from the nature of the parabola) two third-parts of the rectilinear triangles ADB, Adb; and the segments AB, Ab third-parts of the same triangles. And so these areas and these segments will be in the triplicate ratio of the tangents AD, Ad; as well as of the chords and arcs AB, Ab.

Here we need a little preliminary, namely the Quadrature of the Parabola, which we will accomplish in true Newtonian fashion.

Given: Parabola AB, diameter AC, CB ordinatewise, AD tangent, BD parallel to AC.
Prove: The convex parabolic segment is double the concave triangle ABD.



Pick as many points as you like S, E, H along the curve, and draw straight lines through each of these parallel to AC and AD, forming parallelograms inside the convex figure ABC (such as GS, LE, CH) and also inside the concave figure ABD (such as SP, EK, HD). Plainly we can exhaust the two figures this way, with such corresponding parallelograms, and therefore if the ratio of the sums of their parallelograms should approach a ratio, that ratio must be the ratio of ABC : ABD (by Lemma 4).

But each pair approaches the ratio 2 : 1 as the number of parallelograms increases. Consider, for example, the pair LE and EK, as H approaches E.

Draw HE, extend it to N on CA produced. Complete parallelogram HLNM, and extend EP to Q.

As H approaches E, HE approaches tangency at E, which means

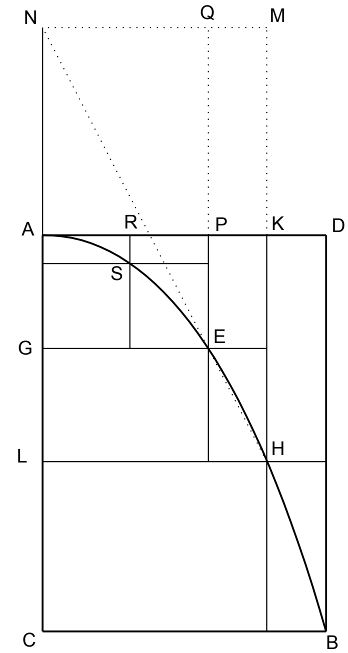
	GA = AN	(ultimately, by the parabolic tangent property)
i.e.	GN = 2AG	(ultimately)
or	EQ = 2PE	(ultimately)
so	EM = 2EK	(meaning the areas, here) (ultimately)
but	LE = EM	(always; complements)
so	LE = 2EK	(ultimately)

And the same goes for every such pair of parallelograms; those inside ABC approach being double the corresponding ones inside ABD.

Therefore, *componendo*, the ratio of the "ABC parallelograms" (as a total) to the "ABD parallelograms" (as a total) approaches the ratio 2 : 1 as near as you like. But they also approach the ratio of the curved areas ABC and ABD as near as you like.

Hence ABC : ABD = 2 : 1.

Q.E.D.



Porism 1: Since convexABC : concaveABD = 2 : 1
but their sum is parallelogram ACBD
hence convex ABC is 2/3 parallelogram ACBD
and concave ABD is 1/3 of ACBD.

Porism 2: Since concaveADB : ACBD = 1 : 3
thus concaveADB : ½ ACBD = 2 : 3
so concaveADB : $\triangle$ACB = 2 : 3

Porism 3: Join AB, forming parabolic segment AB.
Since $\triangle$ADB : ACBD = 1 : 2
and concaveADB : ACBD = 1 : 3
thus concaveADB : $\triangle$ADB = 2 : 3
so $\triangle$ADB – concaveADB : $\triangle$ADB = 3 – 2 : 3
i.e. parabolic segment AB : $\triangle$ADB = 1 : 3
and so too parabolic segment AB = 1/3 of $\triangle$ACB.

That'll do for the preliminaries. Now back to Newton's Corollary 5.

Newton is saying:

Since	$BC^2 : bc^2 = BD : bd$	(ultimately, by Cor. 1 above)
and	$AC : Ac = BD : bd$	(always and truly, or at least ultimately)
thus	$BC^2 : bc^2 = AC : Ac$	(ultimately)

But that is the property of the parabola. In other words, the curve becomes as close to being a parabola as we please, the closer we get to A. (Consider the parabolic path of a projectile; maybe it is really just the tiniest portion of an ellipse!)

But that means the curve will approach having the other properties of a parabola as nearly as we please, as we approach A. So, even in Newton's figure, in keeping with Porism 2 about the parabola, just above:

concaveADB = 2/3 $\triangle$ACB = 2/3 $\triangle$ADB (ultimately)

And again, in keeping with Porism 3 about the parabola, just above:

segmentAB = 1/3 $\triangle$ACB = 1/3 $\triangle$ADB (ultimately)

so	concaveADB : concaveAdb = $\triangle$ADB : $\triangle$Adb	(ultimately)
so	concaveADB : concaveAdb = $AD^3 : Ad^3$	(by Cor.4)
also	segmentAB : segment Ab = $AD^3 : Ad^3$	(same reasoning)

QUESTION: In Cor. 5, Newton says that the curve AbB, whatever conic it may be (e.g. a circle), is ultimately parabolic as we approach A, since

$AD^2 : Ad^2 = BD : bd$ (ultimately; and that is the parabola's property).

But we also see that arc AbB becomes as straight as you please as we shrink it. So that means curved figures ABD and Abd become as nearly similar as we please, right? (cf. Lemma 9) Hence

AD : Ad = BD : bd (ultimately).

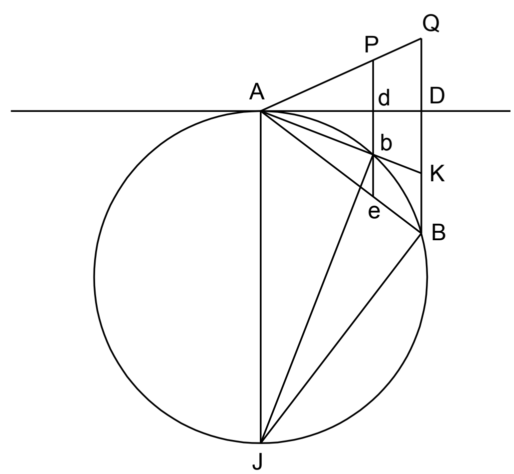
But how can BD : bd approach two different ratios as their limits? Of course, if it happens that BD = bd (ultimately), then so too AD = Ad (ultimately) and $AD^2 = Ad^2$ (ultimately), and everything is fine. But there is no guarantee that is the case, and it is in fact only a special case.

ANSWER: Only the first ultimate proportion, drawn from Lemma 11, is correct. The second one is false! Take the case of a circle (although any curve in which there is finite curvature at A will do), and also draw the subtenses at right angles to tangent AdD, or at least actually parallel, so that it is actually and always true that de : BD = Ad : AD. Now draw in a secant APQ, with P and Q lying along de and BD extended. To say that AbB is ultimately straight (and coincides with the tangent) is enough to guarantee that △AbP is ultimately similar to △AdP (as Newton says in Lem. 9), and △AJB is ultimately equal to the curvilinear triangle AJB. But it is **not** enough to say that △Abd is ultimately similar to △ABD, **nor** is it enough to say that db = de (ultimately), because all that depends not just on AB shrinking, or on arc AbB straightening, but depends also on *where b is along AB, and on the nature of the curve*! In fact, we could specify that for every new location of B, the way we will choose the new location of b is such that db = be; or such that db : be = 3 : 5, or whatever ratio you like, since the segments of dbe can have any ratio at all as we shift dbe right or left. Just cut DB at K in the desired ratio, join KA, and this must cut the curve; call that point b, draw dbe parallel to DB, and—*presto!*—we have the ratio sought. Therefore it is false and illegitimate to conclude that db = be (ultimately). But then, since

de : BD = Ad : AD [actually and always]

thus we cannot say that

db : BD = Ad : AD [ultimately]



And accordingly we never catch Newton saying such a thing, not even back in Lemma 9, where he only said the areas of the triangles (curved and straight) are ultimately equal, and hence those areas are ultimately in the same ratio as the squares on the corresponding rectilinear sides.

SCHOLIUM

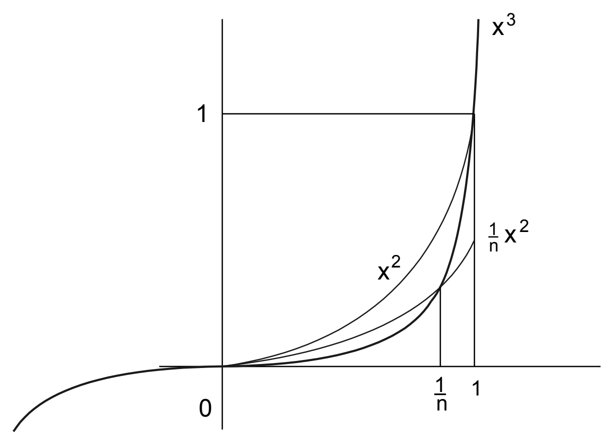
Newton does three things in this Scholium:

(1) He explains his assumption, throughout the Lemmas, of “finite curvature.”
(2) He explains his motivation for developing these Lemmas.
(3) He defends himself from an objection.

(1) THE ASSUMPTION OF FINITE CURVATURE.

A straight line has no curvature, zero curvature, the least possible curvature, and so less curvature than any curve, including any circle, no matter how large. And a circle has more curvature the smaller it is, and the limit of this is the point (infinite curvature). But we can even say that one curve has “infinitely more” curvature than another, if a curve of the one kind, no matter how small it is, must always fall between the other curve and a common tangent. To illustrate, notice in Lemma 11, Newton assumes that we can draw circles with their diameters along AG, through A, so that AD is tangent to them, but also they cut the curve at points such as b and B. Hence these circles fall *between* the given curve and the tangents ad, AD until they cut the curve. But this is not possible with all curves.

Consider two functions, $f = x^3$ and $y = x^2$ (a parabola). Which of these, immediately after tangency to the x-axis at the origin, falls below the other, and closer to the x-axis? That will be the “straighter” of the two. At $x = 1$, it happens that $f = 1$ and $y = 1$, too, so the curves intersect there. What happens *before* $x = 1$, for values of x between 0 and 1 ? Take ½ ; for that value of x, $y = ¼$ while $f = 1/8$. So the curve x^3 is closer to the x-axis than the parabola x^2, and falls between x^3 and the x-axis. But what if we take a bigger and bigger parabola, thus diminishing its curvature? Can we eventually find one that falls between x^3 and the x-axis?



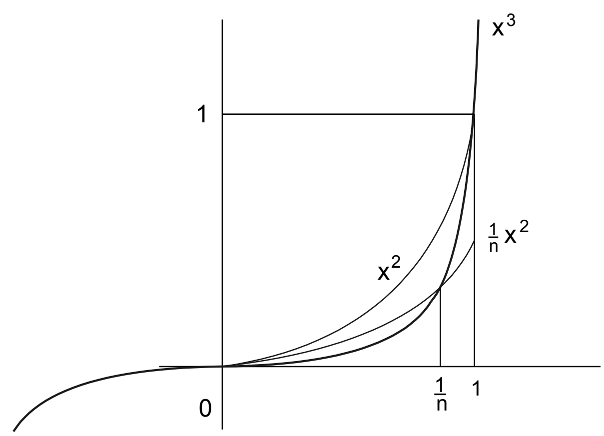
Consider the curve $y = 1/n(x^2)$. That is simply a “magnified” parabola (it is still the case that the ordinate-squares are as the abscissas, i.e. that $x^2_1 : x^2_2 = y_1 : y_2$, since the common factor $1/n$ does not affect the proportion), and it is bigger depending on how large the denominator n is. What n does is to pull the parabola down toward the x-axis, and “flatten” it out, which is the same as to magnify it.

Well, if n is big enough, shouldn’t we be able to get $y = 1/n(x^2)$ to fall *between* x^3 and the x-axis?

We cannot, no matter how big n is!

To see it, notice that our parabola $y = (1/n)x^2$ and our curve $f = x^3$ will intersect where $(1/n)x^2 = x^3$, i.e. where $(1/n) = x$. But what will happen *before* that, for values of x that are between 0 and $1/n$?

Such values, where x is greater than 0, but less than $1/n$, are values of the type $1/(n + z)$. What happens with those values?



Well, if $x = 1/(n + z)$, then:

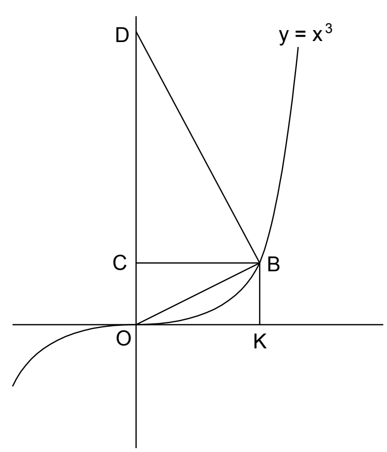
$$y = \left(\frac{1}{n}\right)\left(\frac{1}{n+z}\right)\left(\frac{1}{n+z}\right)$$

$$f = \left(\frac{1}{n+z}\right)\left(\frac{1}{n+z}\right)\left(\frac{1}{n+z}\right)$$

Plainly the value of f is less than that of y for such values of x. The denominator for f will be greater than that for y, since the first factor has a greater denominator $(n + z)$, and the remaining two are the same for both functions. But the greater denominator makes a smaller final value. Hence x^3 still falls between $(1/n)x^2$ and the x-axis, regardless of how large x is.

Another way to see that there can be no circle of curvature at the origin for the curve $y = x^3$ is simply to start with a random chord, draw the circle about it which is also tangent to the x-axis at the origin, and start shrinking the chord. Do we end up with a finite circle? No. Take chord OB, and drop BK at right angles to the x-axis, BC at right angles to the y-axis, and draw BD at right angles to chord OB and intersecting the y-axis at D. Then

$$\frac{CD}{CB} = \frac{CB}{CO} = \frac{OK}{KB} = \frac{x}{y} = \frac{x}{x^3}$$



So $$\frac{CD}{CB} = \frac{x}{x^3}$$

so $$\frac{CD}{x} = \frac{x}{x^3}$$

so $$CD = \frac{1}{x}$$

Which means the smaller x becomes (and it gets as small as you like as B goes to O), the larger CD becomes, being the inverse. For example, if x = 1/1000000, then CD = 1000000. So CD does not approach a finite value. So although segment CB is going to nothing, segment CD is going to infinity, so that the "circle of curvature" has a diameter of infinite length, and so degenerates into the x-axis.

Note the similarity to Euc. 3.16, in which Euclid shows that it is not possible to fit a straight line between the circle and its tangent.

So in Lemma 11 Newton was assuming there is a circle of curvature at the given point, which is not something true of all curves, but only some (or only at some points on certain curves). Curves of second degree, however, will have x's that cancel out, if we try to reproduce the above reasoning, and so they will not have an infinite CD, and so will have a circle of curvature at every point. And all conics (which are of greatest interest in what is to come) are of second degree.

He then goes a bit wild, talking about the different ways of generating curves of incomparable curvature.

(2) HIS MOTIVATION BEHIND THE LEMMAS. Newton says he introduces these Lemmas to avoid the "tedious" methods of the ancients who reduced things *ad absurdum*. He is thinking of Euclid's "Method of exhaustion", e.g. in *Elements* 12. (One gets tired of looking for a new kind of proof for each kind of figure.) And he excuses himself for using shortcut talk, like speaking of continuous quantities as if they were composed of indivisibles, or of infinitely small curves and treating them as straight lines. He says the real force of demonstration is to be taken from the foregoing and (supposedly) more careful way of speaking. He is correcting Galileo to please the contradictious.

(3) AN OBJECTION. Now he considers an objection: "there is no ultimate proportion of evanescent quantities, because the proportion, before the quantities have vanished, is not the ultimate, and when they are vanished, is none."

His answer is remarkably poor after all this work, or so it would seem. He says, for instance, that the ultimate velocity is not that before the motion is ended, nor that after it is ended, but that "with which it ends," which seems plenty open to objection. What is the ratio "**with which things vanish**"? Does this mean the last ratio they ever get? Or the one they have in the very instant in which they go out of existence? This is all just as impossible as Galileo. Possibly he means the ratio they would have had in the moment of vanishing, i.e. the first they never got. But he is unclear.

He raises another objection: "if the ultimate ratios of evanescent quantities are given, their ultimate magnitudes will also be given, and so all quantities will consist of indivisibles, which is contrary to what *Euclid* has demonstrated concerning incommensurables."

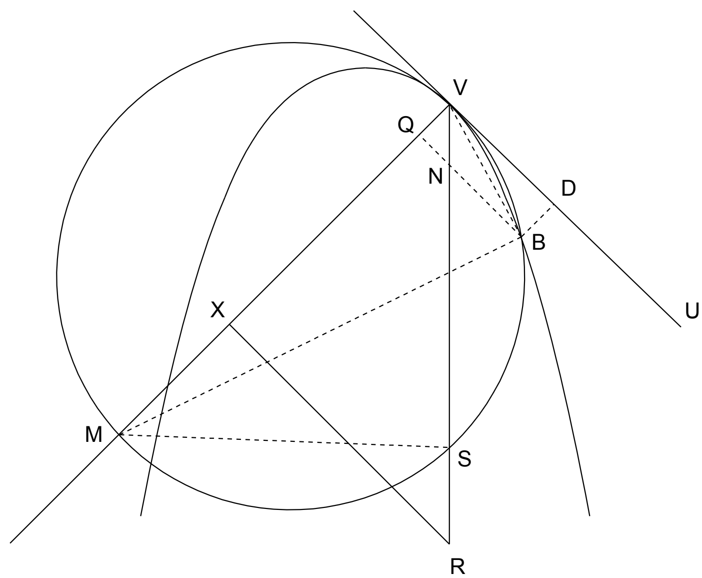
He means that if there are ultimate ratios, there would seem to be ultimate magnitudes, too, and hence indivisible magnitudes, and so magnitudes should all be commensurable, like numbers which are composed of indivisible units.

His response is better than before: "Those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limits towards which the ratios of quantities decreasing without limit do always converge, and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to"—perfect answer, until he adds "till the quantities are diminished *in infinitum*"!

CLASS 28

CIRCLES OF CURVATURE IN CONIC SECTIONS

Since Newton will from time to time mention circles of curvature at various points along a conic section, we will here develop the method for constructing such circles. This is not strictly necessary for the continued reading of Newton, however, since Newton generally uses circles of curvature as an alternative to some other way of reaching his main conclusions. For the interested reader, however ...



Given a PARABOLA with any point V on it, VR the diameter, VU the upright side drawn tangent at V, how do we find the circle of curvature for point V?

Draw VM at right angles to VU.
Draw RX at right angles to VX. (Hence VR : VX is a fixed ratio.)
Take any point B on the parabola on the side of VR making an acute angle with VU.
Draw BD at right angles to VU.
Draw BM at right angles to VB.
Draw BNQ at right angles to VM.

If we can find the ultimate value of VM as B goes to V, that will be the diameter of the circle of curvature at V.

Here's how to find it:

VM : VB = VB : BD [similar triangles]

So $$VM = \frac{VB^2}{BD}$$

But BD = VQ always, and VQ : VN = VX : VR always,

So $$BD = \frac{VX}{VR}VN$$

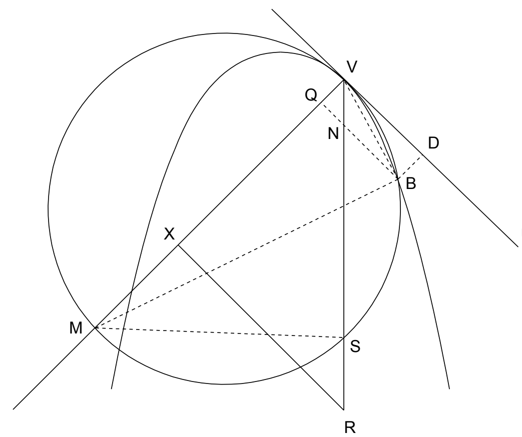
So $$VM = \frac{VB^2}{\frac{VX}{VR}VN}$$

Now $$VB^2 = VN^2 + NB^2 + 2QN \cdot NB$$

And VN : QN = VR : RX

so $$QN = \frac{VN}{VR} RX$$

So $$VB^2 = VN^2 + NB^2 + 2\left[\frac{VN}{VR} RX\right] NB$$



So $$VM = \frac{VN^2 + NB^2 + 2\left[\frac{VN}{VR} RX\right] NB}{\frac{VX}{VR} VN}$$

Now multiply the right side of the equation by $$\frac{\dfrac{VR}{VX \cdot VN}}{\dfrac{VR}{VX \cdot VN}}$$

that is, by 1, and we have $$VM = VN\frac{VR}{VX} + \frac{NB^2}{VN} \cdot \frac{VR}{VX} + 2\left[VR \cdot RX\right] NB \cdot \frac{VR}{VX}$$

Now VN occurs in the denominator in the second term on the right side of this equation, and we can replace it with

$$\frac{NB^2}{VU}$$

thanks to the property of the parabola. This gives us

$$VM = VN\frac{VR}{VX} + \frac{NB^2}{NB^2 / VU} \cdot \frac{VR}{VX} + 2\left[VR \cdot RX\right] NB \cdot \frac{VR}{VX}$$

or more simply

$$VM = VN\frac{VR}{VX} + VU \cdot \frac{VR}{VX} + 2\left[VR \cdot RX\right] NB \cdot \frac{VR}{VX}$$

So the value of VM is always exactly that, throughout the process of moving B to V.

But as B goes to V (or, what is the same thing, as N goes to V), VN goes to nothing, and therefore the first term on the right side of the equation, which is the product of VN and a fixed ratio, goes to nothing.

Also, as B goes to V (and hence as N goes to V), NB goes to nothing, and therefore the third term on the right side of the equation, which is the product of NB times a constant coefficient and a constant ratio, also goes to nothing.

Hence the value of VM, as B goes to V, gets as near as you please to being just

$$VU\frac{VR}{VX}$$

Since that is the limit of the value of VM as B goes to V, it is the value of the diameter of the circle of curvature at V. So it is just the upright side at V, adjusted by the fixed ratio of VR : VX.

So $$VM = VU\frac{VR}{VX}$$ [ult. as B goes to V]

But $$VM = VS\frac{VR}{VX}$$ [since VM : VS = VR : VX]

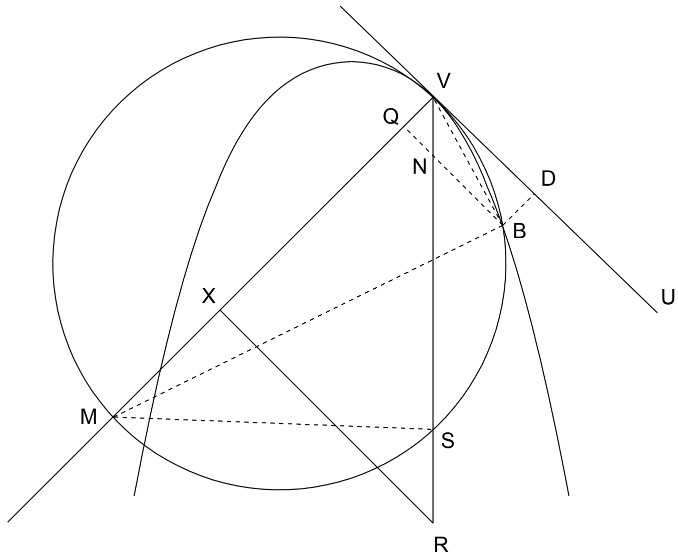
Hence $$VS\frac{VR}{VX} = VU\frac{VR}{VX}$$ [ult. as B goes to V]

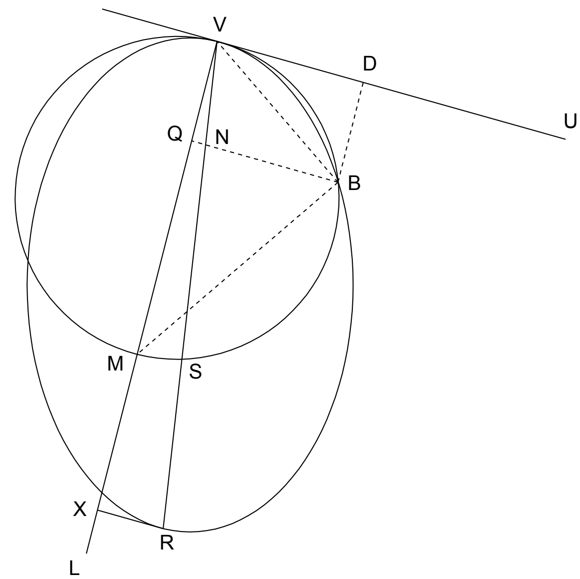
i.e. $$VS = VU$$ [ult. as B goes to V]

Hence the ultimate value of the chord VS inside our shrinking circle is just VU, the upright side.

So an easy way to construct the circle of curvature at any point V on a parabola is to cut off VS equal to the upright side at V, draw SM at right angles to diameter VR, cutting VM (the normal at V) at point X, and then draw the circle on VM as diameter.

Q.E.I.





Given an ELLIPSE with any point V on it, VR the diameter, VU the upright side drawn tangent at V, how do we find the circle of curvature for point V?

Draw VL at right angles to VU.
Draw RX at right angles to VX. (Hence VR : VX is a fixed ratio.)
Take any point B on the ellipse on the side of VR making an acute angle with VU.
Draw BD at right angles to VU.
Draw BM at right angles to VB.
Draw BNQ at right angles to VM.

If we can find the ultimate value of VM as B goes to V, that will be the diameter of the circle of curvature at V.

Here's how to find it:

VM : VB = VB : BD [similar triangles]

So $$VM = \frac{VB^2}{BD}$$

But BD = VQ always, and VQ : VN = VX : VR always,

So $$BD = \frac{VX}{VR} VN$$

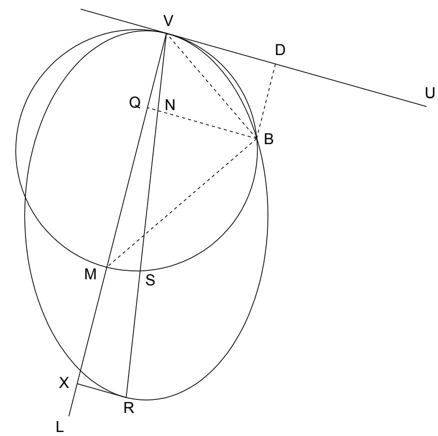
So $$VM = \frac{VB^2}{\frac{VX}{VR}VN}$$

Now $$VB^2 = VN^2 + NB^2 + 2QN \cdot NB$$

And VN : QN = VR : RX

so $$QN = \frac{VN}{VR}RX$$

So $$VB^2 = VN^2 + NB^2 + 2\left[\frac{VN}{VR}RX\right]NB$$



So $$VM = \frac{VN^2 + NB^2 + 2\left[\frac{VN}{VR}RX\right]NB}{\frac{VX}{VR}VN}$$

Now multiply the right side of the equation by $$\frac{\frac{VR}{VX \cdot VN}}{\frac{VR}{VX \cdot VN}}$$

that is, by 1, and we have $$VM = VN\frac{VR}{VX} + \frac{NB^2}{VN} \cdot \frac{VR}{VX} + 2[VR \cdot RX]NB \cdot \frac{VR}{VX}$$

Now NB^2 occurs in the numerator in the second term on the right side of this equation, and we can replace it with

$$VU \cdot VN - \frac{VU}{VR}VN^2$$

thanks to the property of the ellipse. This gives us

$$VM = VN\frac{VR}{VX} + \left[\frac{VU \cdot VN - \dfrac{VU}{VR}VN^2}{VN}\right] \cdot \frac{VR}{VX} + 2\left[VR \cdot RX\right]NB \cdot \frac{VR}{VX}$$

or more simply

$$VM = VN\frac{VR}{VX} + VU\frac{VR}{VX} - \frac{VU}{VR} \cdot \frac{VR}{VX}VN + 2\left[VR \cdot RX\right]NB \cdot \frac{VR}{VX}$$

So the value of VM is always exactly that, throughout the process of moving B to V.

But as B goes to V (or, what is the same thing, as N goes to V), VN goes to nothing, and therefore the first term on the right side of the equation, which is the product of VN and a fixed ratio, goes to nothing.

For the same reason, as B (or N) goes to V, and so VN goes to nothing, the third term on the right side of the equation, which is the product of VN and a fixed ratio, goes to nothing.

Also, as B goes to V (and hence as N goes to V), NB goes to nothing, and therefore the fourth term on the right side of the equation, which is the product of NB times a constant coefficient and a constant ratio, also goes to nothing.

Hence the value of VM, as B goes to V, gets as near as you please to being just

$$VU\frac{VR}{VX}$$

Since that is the limit of the value of VM as B goes to V, it is the value of the diameter of the circle of curvature at V. So it is just the upright side at V, adjusted by the fixed ratio of VR : VX.

So $$VM = VU\frac{VR}{VX}$$ [ult. as B goes to V]

But $$VM = VS\frac{VR}{VX}$$ [since VM : VS = VR : VX]

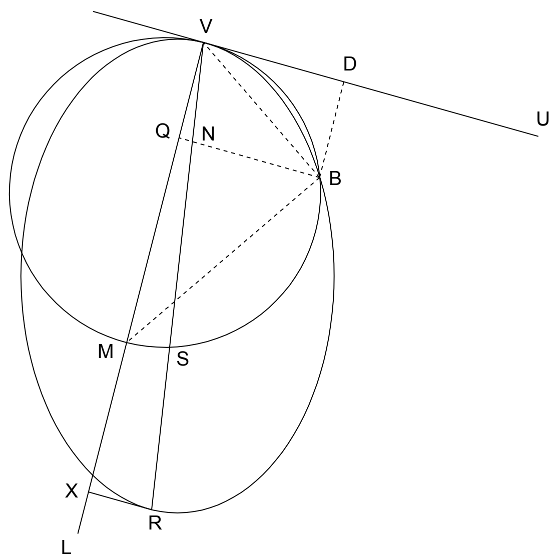
Hence $$VS\frac{VR}{VX} = VU\frac{VR}{VX}$$ [ult. as B goes to V]

i.e. $$VS = VU$$ [ult. as B goes to V]

Hence the ultimate value of the chord VS inside our shrinking circle is just VU, the upright side.

So an easy way to construct the circle of curvature at any point V on a parabola is to cut off VS equal to the upright side at V, draw SM at right angles to diameter VR, cutting VL (the normal at V) at point M, and then draw the circle on VM as diameter.

Q.E.I.



THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 29

PROPOSITION 1

PRELIMINARY NOTES:

• Section 1 was the Lemmas, the calculus. We are now transitioning into the *Principia* proper, the part of the treatise analogous to the Theorems in Euclid's *Elements*.

• Section 2 is entitled "*On the Finding of Centripetal Forces*."

• So we are dealing specifically with forces to a center (and forces whose rule of variation is a function of distance from that center). Section 3 will be even more specific, and focus on conic sections.

• We are given motions, and finding the force-rules producing them, e.g. in Prop. 10 (but also finding the properties of those motions, as in Prop. 1; later, in Section 3, we will also be given force-rules, and from them deduce the paths of motion, as in Prop. 17).

• Prop. 1 is foundational to almost everything in the *Principia*.

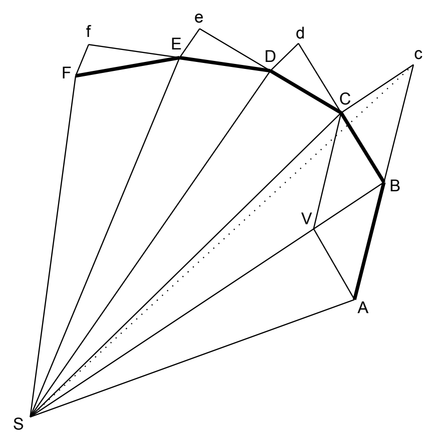
• Prop. 1 uses the Laws and the Lemmas, as one would expect, so we are beginning to see their utility.

• Prop. 1 is a proof of a generalization of Kepler's 2^{nd} Law. It is more general because it is not about planets in particular, nor about ellipses or even conics in particular. It is one of Newton's purposes in the *Principia* to derive Kepler's Three Laws of Planetary Motion from his own Three Laws of Motion.

• KEPLER'S THREE LAWS OF PLANETARY MOTION:
(1) The planets, including Earth, move in elliptical orbits with the Sun at one focus of the ellipses.
(2) The line drawn from the Sun to a planet sweeps out equal areas in equal times.
(3) The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of their mean distances from the Sun. That is, if r_1 is the mean distance for Planet 1, and p_1 is the period for Planet 1, then $(r_1)^3 : (r_2)^3 = (p_1)^2 : (p_2)^2$

PROPOSITION 1

The areas which bodies moving in orbits describe (by radii drawn to an immobile center of forces) remain in immobile planes and are proportional to the times.



The proof:

(A) SETTING OUT. Let a body move from A to B by its own inertia in some unit of time, say one hour. Then, at point B, let it suddenly be impressed with a force (acceleration) toward fixed point S, so that instead of arriving at c after the 2nd hour, it is at C. Again, at C, let it be acted on by a force toward S, so that at the end of the 3rd hour, instead of being at d by its inertia, it is at D. Thus we have the body at A, B, C, D etc., with equal times between.

(B) ALL IN ONE PLANE. Now, the triangles described by such motions are all in one plane. For, AB and Bc are in a straight line, and are equal, by Law 1 (inertia). Since the body is found at C (vs. c) at the end of the first hour, therefore cC represents the change of motion, and therefore by Corollary 1 to the Laws (the parallelogram of forces), we know that cC is parallel to the direction of the sudden force at B, i.e. cC is parallel to BS. Therefore cC (and hence BC) is in the same plane as △ASB, since cC cuts ABc and it is parallel to BS. And so △BSC is in the same plane as △ASB, and the same goes for all the subsequent triangles.

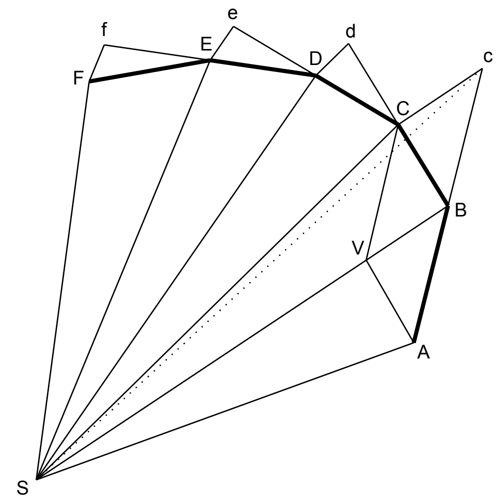
(C) TRIANGLES EQUAL IN AREA. And all these triangles are equal in area.

For,	△SBc = △SAB	[since AB = Bc]
and	△SBc = △SBC	[since Cc is parallel to SB]
so	△SBC = △SAB	

and the same goes for all the subsequent triangles.

(D) COMPONENDO ON THE TRIANGLES. Hence, if we take any number of these triangles which the body describes in equal times, the total area will be as the time.

(E) GETTING TO THE CURVE. Now, if we divide the time more finely into smaller and smaller equal time increments (having a sudden centripetal force occur at the end of each), and hence increase the number of triangles toward infinity, and diminish their bases (AB, BC, CD, etc.) toward points, the total area will be approaching some curved area, and the rectilinear perimeter will be approaching a curvilinear one (Cor. 4 to Lemma 3). And since any two triangle-sums are swept out in times proportional to their areas, and this law is always true as we get as near as we please to the curve, therefore the same law will hold for the ultimate curve itself, i.e. that as the body moves along the curve (by its inertial movement together with some continuously acting, but possibly growing and shrinking, force toward S), it will sweep out areas about S that are proportional to the times.



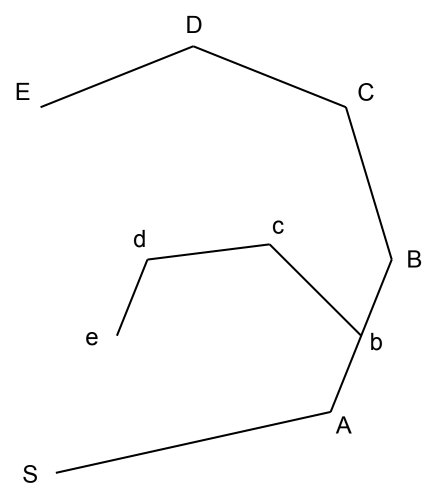
Q.E.D.

QUESTION 1: *Is the force uniform?*

No, except in special cases (e.g. uniform speed around a circle). So the acceleration BV might not be equal to the next acceleration (i.e. Cc and Dd and Ee etc. need not be equal).

QUESTION 2: *Are we approaching a definite curve?*

Only if lines like BV and cC are given in length, i.e. we must be given the rule which says how strong the force is at given distances from S. But once the force-rule is given, or all the forces are definite, then we are approaching a definite curve—or so Newton expects us to believe. But we seem to be approaching it by "curling down" to it, not by drawing inscribed polygons of increasing numbers of sides in the fixed orbit, as the next question brings out.



QUESTION 3: *Are A, B, C, D, etc. points on the final curve?*

Not necessarily! Suppose our first impulse-path results from *dividing the time into hours*, and we get the path ABCD etc. Now divide the time into half-hours—this means the body does not get to B, but instead at the midpoint, b, the force toward S acts again and pulls the body off of AB. So now we get a new impulse-path, Abcd etc. So although we might be approaching a curve, it is not necessarily a curve on which the vertices of our impulse-paths lie! It is significant that Newton does not draw a curved path in his diagram.

QUESTION 4: *CAN the points A, B, C, D, E etc. be on the curve?*

But we could divert the body at A along a different line from AB, and have it bend back to B, by allowing an impulse to happen in between. So maybe, doing things that way, we could get all our impulse-points to be on the final curve.

Suppose A, B, C, D, etc. are all on the final curve resulting from the force continuously acting, and it is also true that the body is at A, B, C, D etc. after equal increments of time. Suppose, further, that *all the new points* introduced by dividing the time more finely are also on the final curve. Then

$\triangle$SAB = $\triangle$SBC etc.

as Newton shows. But it is also true (if Prop. 1 is correct) that

curvilinear SAB = curvilinear SBC etc.

And so in our curve, equal pie-pieces contain equal rectilineal triangles—which is true only of a circle (with geometrical center S). So the assumption that all the points in the impulse-paths are on the final curve over-specifies things, or will work only in that specific case.

If we start with a definite orbit, we can certainly divide the time as finely as we please, and draw chords between successive locations of the body, and then contrive uniform motions which would carry a body along the chords to those successive locations in the same times as the given body goes through those points in its curved orbit. But then the speeds along the chords are determined by the lengths and shapes of the arcs covered by the given body in the equal times, so it might not be the case that the uniform speeds along the chords are simply as the chords—as Newton's way of doing things requires. In other words, it is not safe to assume that the straight lines in Newton's figure are chords of the final orbit, and it is usually not possible for them to be so.

QUESTION 5: *Is it legitimate to assume that what happens with the impulse-paths must also happen with the curved path?*

Newton argues geometrically for the impulse-path, but he simply assumes that what is true of the impulse-path must also be true for the curve which the impulse-path is approaching. But isn't that assuming that what is true in the approach is also true in the limit? And that is not always true! For example, the areas cut off inside two equal and adjacent squares by rotating the diagonal of their rectangle are always in the ratio 1:3, but it does not follow that the final lines are in that ratio. Isn't this a Galilean error?

And isn't it even doubtful that we are approaching a curve? We go from a finite number of impulses to a CONTINUOUSLY ACTING and changing acceleration—that seems quite a leap!

Newton seems to be thinking SYNTHETICALLY: If you have inertial motion and a centripetal force, you can't get anything out of them except equal areas in equal times, try as you might. There simply is no cause present which is capable of doing that, even if you use them continuously or with infinite repetition. Hence the final result, even if they both act together continuously, must be equal areas in equal times.

Even modern mathematicians say that Newton is a bit lazy or fast-and-loose here—or at least they say he is leaving out some important considerations. But the result is accepted, and is demonstrated today ANALYTICALLY rather than synthetically, i.e. given the actual curved orbit and that it is the result of a continuously acting centripetal force and of the inertial tendency of the body, it is proven that the body sweeps out equal areas in equal times around S. Such an analytic proof will be supplied below.

QUESTION 6: *What if the centripetal force is zero? Does a body moving inertially sweep out equal areas in equal times?*

Yes! A body moving uniformly in a straight line sweeps out equal areas in equal times around EVERY point in space! (This is what one might call a "degenerate" case.) To see it, just draw the straight path, chop it into equal segments (accomplished in equal times by the uniform speed), and pick any point S not on that straight line, and join S to the endpoints of the equal segments. The triangles are all equal. So the body is sweeping out equal areas in equal times "about S."

ANALYTIC VERSION OF PROPOSITION 1
IN NEWTON'S *PRINCIPIA*

Given a path of motion resulting from (1) a body's innate force and (2) a continuously acting (and perhaps fluctuating) centripetal force toward a fixed point S, I say that the path is in one fixed plane and that the body sweeps out areas in that plane, about S, which are proportional to the times.

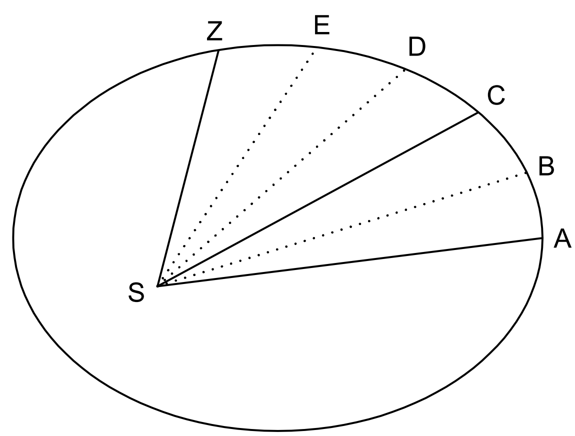
(Let the symbol $\underline{U}$ mean "is ultimately equal to.")

PART 1

At any instant during the motion, there are two straight lines along which the body is endeavoring to move: (1) by its innate force it is endeavoring to move uniformly along the straight line tangent to its path, and (2) by the centripetal force it is endeavoring to move along the straight line between itself and S. These two straight lines define a plane. If we now let time go forward, it is clear that the body must remain in that same plane. Its innate force will never draw it out of that plane unless it is drawn out of that plane by another force; and the centripetal force will never draw it out of that plane unless the body is drawn out of that plane by another force. So the two forces are incapable of moving the body out of that plane unless some other force is introduced. Hence the body remains in that one plane, and hence the given curve (which is the path of the body's motion) must be a plane curve.

That suffices to prove the first part of the enunciation.

PART 2



Now let the given body, G, by moving in its plane path, sweep out two consecutive areas about S, namely SAC and SCZ, and let these be swept out in commensurable times—and for the sake of concreteness, let us say

Time SAC : Time SCZ = 2 : 3.

I say that

Area SAC : Area SCZ = 2 : 3.

Let the total time of arc ACZ be divided into equal increments of time called Δt, as many as you please, and let the body be successively at A, B, C, D, E, etc., after each Δt. Let the instantaneous velocity of G at A be called V_A, and so on for the other points.

Join the chords.

Let a motion be contrived by which a body Q travels chord AB in Δt at uniform velocity, and then, by a sudden impulse, travels chord BC in the next Δt at a new uniform velocity, etc. Since the times are equal, these velocities are as the chords:

$$V_{AB} : V_{BC} = AB : BC$$

Now, I say that the motions of G over arcs AB and BC are ultimately the same as the motions of Q over chords AB and BC, as Δt goes to zero (or, what is the same thing, as B is taken closer to A, and C closer to B, *ad infinitum*).

First, for the sake of simplicity, let G always accelerate from A to C (any non-uniform motion can be broken up into motions in which the body is always acclerating or always decelerating). So G is slowest at A, fastest at C, increasing speed monotonically in between.

Plainly, as arc AB shrinks to nothing (with Δt), the instantaneous speeds of G at A and B (and throughout) differ by as little as you please, and hence are ultimately equal. And since G is speeding up throughout arc AB, this means that the uniform speed which would cover arc AB in Δt is *greater* than the speed of G at A, but *less* than the speed of G at B. Since this uniform speed is always *between* the speeds of G at A and B, and since the speeds of G at A and B are ultimately equal to each other, *a fortiori* they are ultimately equal to that uniform speed. But that uniform speed along arc AB is ultimately equal to the uniform speed of Q along chord AB, since these motions always take place in the same time (Δt), and since they begin and end at the same points (A and B), and since the two distances between those points (i.e. chord AB and arc AB) are ultimately equal (by Newton's Lemmas). Hence it follows further that the speed of G over arc AB is ultimately equal to the speed of Q over chord AB. But the times of these motions are always the same, and their paths are ultimately the same. Hence the motion of G over arc AB in Δt is ultimately the same as a uniform motion over chord AB in Δt.

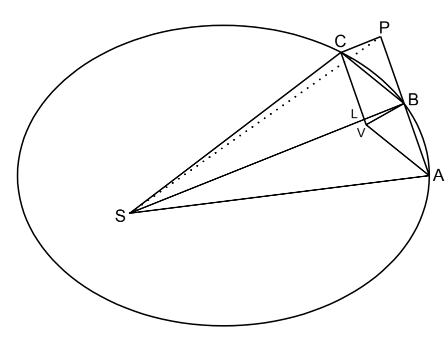
Similarly, the motion of G over arc BC in Δt is ultimately the same as a uniform motion over chord BC in Δt.

Hence the accelerated motions of G through arcs AB and BC are ultimately the same as uniform motions through chords AB and BC.

This means that as Δt goes to zero, the motion of G through arcs AB and BC becomes as similar as we please to the motion through the rectilinear angle ABC with uniform speeds in AB and BC which are to each other as AB and BC.

And that means that the action on G in the instant it is at B is as similar as we please to a sudden "knock" which alters Q from uniform speed AB in line AB to uniform speed BC in line BC. Or, putting it the other way: the instantaneous acceleration (or change of motion) from speed AB in AB to speed BC in BC is ultimately the same as the action on G when it is at B. But this is the centripetal force on G toward S. Hence the instantaneous acceleration from speed AB in AB to speed BC in BC is ultimately the same as the centripetal force on G at B toward S.

Complete the parallelogram ABCV. BV represents the change in velocity from AB to BC. This might not point along the line BS, but this is only because the motion through chord AB is not exactly the same as that through arc AB, and likewise the motion through chord BC is not exactly the same as that through arc BC. But, as we take Δt smaller and smaller, and the motions through the arcs differ less and less from the motions through the chords, BV must get as close as we please to pointing along the line BS. (Let BS and CV intersect at L.) So BV ultimately coincides with BS.



Now, while Δt goes to zero, draw CP always parallel to BS, and let P be the point on this parallel where AB extended intersects it. Since BV ultimately coincides with BS (as we showed), therefore it is ultimately parallel to CP. Also, since BV ultimately coincides with BS, V is ultimately on BS; so if we call "L" the place where CV intersects BS, and V itself is ultimately on BS, this means that V and L are ultimately the same point.

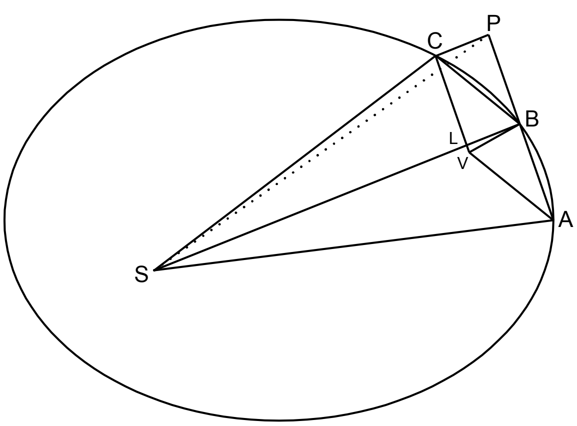
But then	CV $\underline{U}$ CL	
so	AB $\underline{U}$ BP	[since CV = AB, CL = BP]
so	$\triangle$SAB $\underline{U}$ $\triangle$SBP	
but	$\underline{\triangle\text{SBC} = \triangle\text{SBP}}$	[since CP parallel to BS]
so	$\triangle$SAB $\underline{U}$ $\triangle$SBC as Δt goes to zero	
But	$\triangle$SAB $\underline{U}$ area SAB	
and	$\triangle$SBC $\underline{U}$ area SBC	[Newton's Lemmas]
so	area SAB $\underline{U}$ area SBC	[actually an understatement]

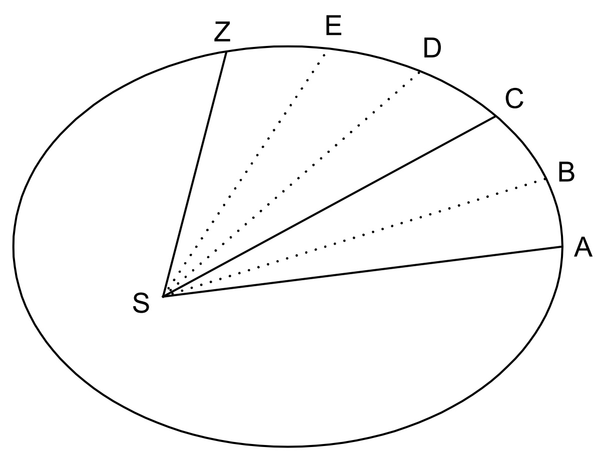
And since this is true of any two consecutive areas swept out in Δt, it is true of any two whatsoever in arc AZ (since things ultimately equal to the same are ultimately equal to each other, and two areas with one in between are each ultimately equal to the one in between, and hence are ultimately equal to each other; but then two areas with two in between will also be ultimately equal, etc.).

for example, area SAB $\underline{U}$ area SCD

Now if we take Δt smaller and smaller, the number of sectors in SAC and SCZ increases as much as we please, but the number in SAC is to the number in SCZ always as 2 : 3, and all of these little pie-slivers are ultimately equal to one another. Hence the ratio of SAC to SCZ is the limit of $(x_1 + x_2 + ... + x_n) : (y_1 + y_2 + ... + y_m)$, where each x or y is one of the little pie-slivers swept out in Δt, and n : m = 2 : 3. But since each x is ultimately equal to each other x, and also to each y, therefore the limit of this ratio is the ratio nx : mx, where x is anything (finite) you like. But that is just the ratio 2 : 3. Therefore

area SAC : area SCZ = 2 : 3





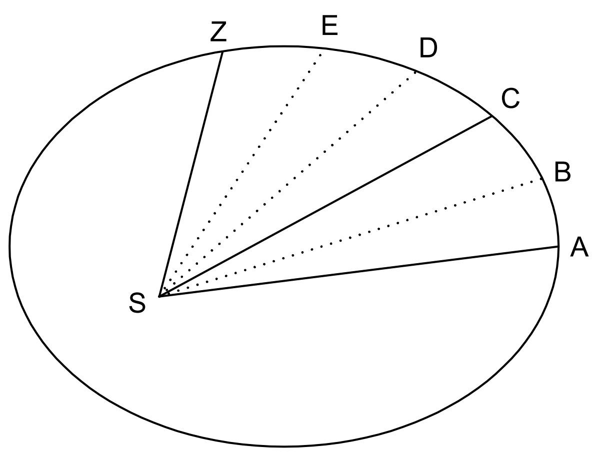
Porism: So it is evident that area SAB = area SBC throughout the process, and these are not just "ultimately equal."

Corollary 1: If we take any two consecutive areas swept out in *incommensurable* times, we can approach these areas and their times as nearly as we please with areas swept out in commensurable times, which areas will always be exactly as the times in which they are swept out (by the above). Therefore the areas swept out in the incommensurable times must also be as the times in which they are swept out. Hence, generally, consecutive areas will be as the times in which they are swept out.

Corollary 2: If we take any two areas swept out which are not consecutive, then each will be to the area in between as the time of each to the time of the area between (by Cor. 1). Hence the non-consecutive areas will also be as the times in which they are swept out.

Hence, generally, in such an orbit, any two areas whatever will be to one another as the times in which they are swept out.

Q.E.D.



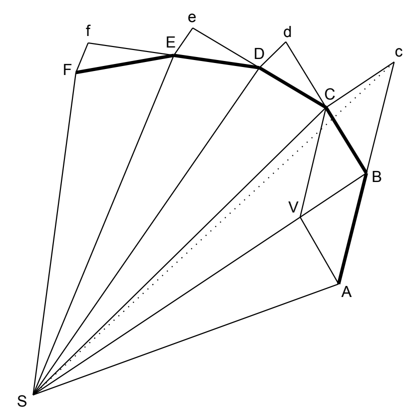
THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 30

PROPOSITION 1 COROLLARIES

Newton next develops six corollaries to Proposition 1. I here include the text of each corollary, with notes to accompany them.

COROLLARY 1



The velocity of a body attracted towards an immovable center (in non-resisting spaces) is reciprocally as the perpendicular let fall from that center onto the straight line tangent to the orbit. For, the velocity at those places A, B, C, D, E is as are the bases, AB, BC, CD, DE, EF, of the equal triangles, and these bases are reciprocally as the perpendiculars let fall on them.

1. In our impulse-path, the velocities at A, B, C, D, E are as AB, BC, CD, DE, EF, since the times are equal and those motions are uniform (inertial), and for uniform motions the speeds are as the distances accomplished in equal times.

2. And those bases AB, BC (etc.) are reciprocally as the perpendiculars dropped from S onto them. That is:

   AB : BC = (perp. from S to BC) : (perp. from S to AB)

   And this is because of the equality of the triangles:

   △SAB = △SBC

   thus ½ AB x (height of △SAB) = ½ BC x (height of △SBC)

   hence ½ AB : ½ BC = (height of △SBC) : (height of △SAB)

   so AB : BC = ht. of △SBC : ht. of △SAB

   and those heights are the perpendiculars from S to BC and AB respectively.

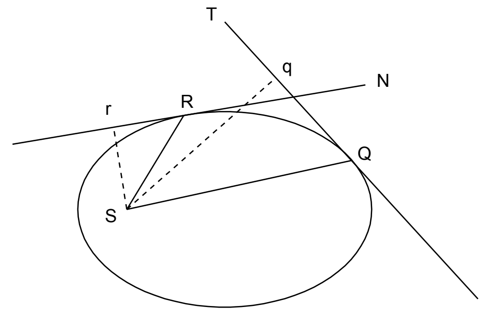
3. It follows that the velocities, which are as the bases of the triangles, are reciprocally as the heights of the triangles.

4. Of course, the ***bases*** of the "impulse-orbit" are, ultimately, ***points*** in our final, curved orbit. That is one motivation for finding *something else* that is as the speeds, something else that is still there once we are at the final orbit—namely the perpendiculars to the tangents. So the bases of the triangles become more and more like points, and their directions more and more like tangents, but the heights

from S (which are always inversely as those bases) do not go away, but remain as the perpendiculars from S to the tangents.

5. So pick any two points on an orbit (formed by a body moving through non-resisting space by its own inertia and by centripetal force toward a fixed point S), e.g. Q and R; draw tangents TQ, NR; from S drop perpendiculars to these Sq, Sr; and it will then be actually and exactly true that

$$V_Q : V_R = Sr : Sq$$

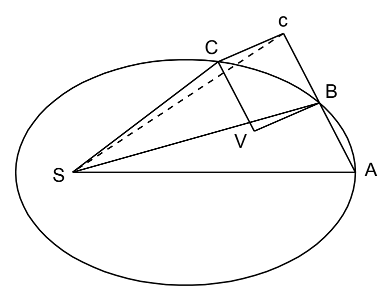


NOTE: Kepler, in *Astronomia Nova*, hypothesized that the velocities in a planetary orbit are to each other inversely as the solar distances at those points. The sun is the "S" point, since the planets sweep out equal areas in equal times around it. So he was nearly right, especially since the orbits are so circular. But the instantaneous speeds at two points are not to each other inversely as their distances from S, but inversely as the perpendiculars from S to the tangents at the points.

Of course, the ***bases*** of the "impulse-orbit" are, ultimately, ***points*** in our final, curved orbit. That is one motivation for finding *something else* that is as the speeds, something else that is still there.

QUESTION: Is Cor. 1 true "actually" or just "ultimately"? (Answer: Actually.)

COROLLARY 2

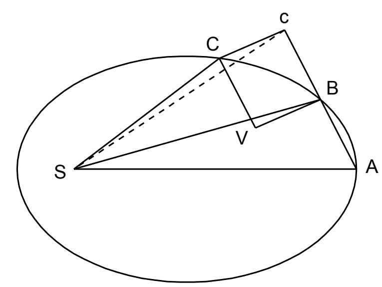


If the chords AB, BC of two arcs successively described, by the same body, in equal times, in non-resisting spaces, be completed into the parallelogram ABCV, and its diagonal, BV, in that position which it ultimately will have when those arcs are diminished in infinitum, is produced both ways, the same will pass through the center of forces.

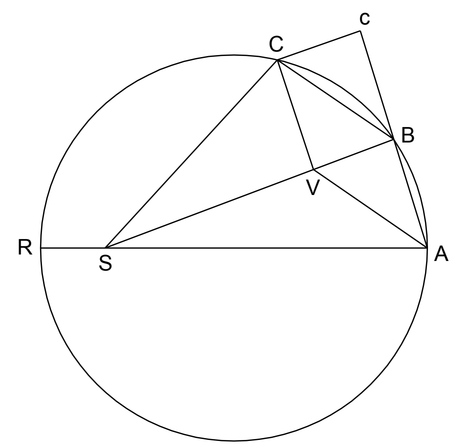
(1) Notice he calls AB and BC "chords." And he is not saying chords AB, BC are traversed in equal times, but arcs AB, BC are. That means (by Prop. 1 itself) that the curvilinear areas SAB, SBC are equal; but, except in special cases (like the circle, or if B is a point of symmetry in some other curve), the rectilinear triangles SAB, SBC will *not* be equal. They will only be ultimately equal, since they differ as little as you please from the curved figures as we shrink the arcs to B.

(2) WHY "ULTIMATELY"? Doesn't BV *always* point to S? In our impulse-paths, yes. But if we are talking about an actual curved orbit in which we draw chords to successive arcs traversed in equal times, we just have just seen (in (1) above) that the resulting rectilinear triangles are not (necessarily) equal, and hence if we draw ScB equal to SAB, ScB and SCB are not (necessarily) equal, and so Cc is not (necesarily) parallel to BS. So if we draw BV parallel to Cc, BV does not (necessarily) point to S.

But BV ultimately points to S, as we argued in our analytic version of Prop. 1. So we can see that BV ultimately points to S in two ways. (a) SYNTHETICALLY. This way begins from the construction of Prop. 1, i.e. from the impulse-path. In that case, BV always points to S throughout the approaching-process, i.e. as we divide the time more finely, but it also changes position. So it “ultimately” points to S, i.e. in the final, limit orbit, the curved one, since it always does throughout the approaching process. (b) ANALYTICALLY. This way, which seems to be the one Newton has in mind now (since he begins from arcs done in equal times), is to begin with the actual curved orbit, and to shrink arcs AB and BC in our curve, and note that the arcs ultimately do not differ from rectilinear bases AB, BC—that is, the more we shrink those arcs, the less our curved orbit differs from our polygonal impulse-path, and the more true the things we said about the impulse-path become, e.g. the closer BV comes to pointing to S.



If you are **not convinced** that BV can ever fail to point exactly at S, consider the following explanation. As long as we have an actual parallelogram ABCV (vs. an “infinitely-small” one), its sides merely approximate what the body actually does, and its diagonal therefore merely approximates the change in what it does, i.e. the acceleration it undergoes, and hence the force on it at B. So it will (or can) be a little off, in both magnitude and direction.



EXAMPLE. Let AR be a diameter of a circle, and S any point on the diameter other than the geometric center. Let arcs AB and BC be swept out in equal times around S as the center of forces. Therefore the pie-pieces SAB and SBC are equal, by Prop. 1.

Now extend AB to c so that AB = Bc, and join Cc.
Complete the parallelogram ABCV.
Now, IF POSSIBLE, let V lie on BS.

Since	CV and AB are equal and parallel, and Bc = AB,	
thus	CV and Bc are equal and parallel,	
so	Cc and BV are equal and parallel.	
Thus	$\triangle$SBC = $\triangle$SBc	
but	$\underline{\triangle\text{SAB} = \triangle\text{SBc}}$	
so	$\triangle$SAB = $\triangle$SBC	
but	$\underline{\text{sect.SAB} = \text{sect.SBC}}$	[Prop. 1]
so	seg.AB = seg.BC	[remainders]

That is, the circular segments cut off by chords AB, BC are equal.
Hence AB = BC.
Hence ABCV is a rhombus.
Hence $\angle$ABV = $\angle$CBV.
Hence BV is a diameter of the circle.
But AR is a diameter of the circle.
Hence where they intersect, S, is the geometric center of the circle.
Which is against the given.

COROLLARY 3

If the chords AB, BC and DE, EF, of arcs described in equal times, in non-resisting spaces, be completed into parallelograms ABCV, DEFZ, the forces at B and E are to each other in the ultimate ratio of the diagonals BV, EZ, when those arcs are diminished in infinitum.

1. Again, as in Cor. 2 above, he is talking now not about an approximating impulse-path, but about the actual curved path produced by the continuously acting centripetal force, and about the chords in it.

2. And now he is giving us a ratio of forces or accelerations as opposed to a ratio of velocities at any two points on the orbit.

3. BV and EZ do not necessarily point to S in fact, but (as Cor. 2 said) they ultimately do, as the chords are taken smaller and smaller. Likewise BV and EZ do not necessarily have the ratio of the centripetal force at B to that at E, but they do ultimately, i.e. the ratio they approach, as we shrink the arcs accomplished in equal times, is the ratio of the centripetal forces at B and E.

Why? Well, SYNTHETICALLY, the velocity along AB is to that along BC as AB : BC (done in equal times). And therefore BV is the change of motion due to the sudden centripetal force at B (Cor. 1 to the Laws), i.e. BV is the acceleration or "accelerative centripetal force." Likewise for EZ at E. So

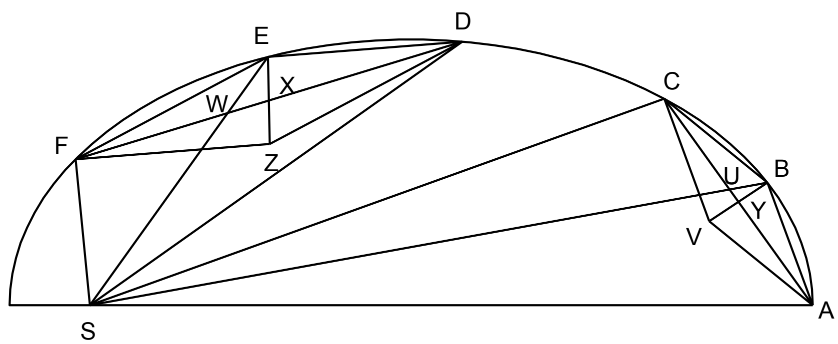
BV : EZ = Force at B : Force at E

always, for the impulse-path, and therefore the ultimate ratio of these is the ratio of the instantaneous accelerations at B and E in the curved orbit.

We could come at this ANALYTICALLY too: if arcs AB and BC in the actual curved orbit are done in equal times, then the ratio of the chords, AB : BC, is ultimately the same ratio as that of the average velocities in arcs AB and BC, and therefore the diagonal BV is ultimately equal to the change in motion from arc AB to arc BC, and hence it is ultimately as the force at B.

COROLLARY 4

The forces by which any bodies whatever (in non-resisting spaces) are drawn back from rectilinear motion and turned aside into curved orbits are to each other as those sagittae of arcs described in equal times which converge towards the center of the forces and bisect the chords when those arcs are diminished in infinitum. For, these sagittae are the halves of the diagonals we needed in Cor. 3.



Join AC and DF. These diagonals will bisect the other diagonals of our parallelograms, EZ and BV (which ultimately pass through S as we shrink the arcs FE and ED toward E, and CB and BA toward B, all four arcs always being done in equal times). Call the bisection points X and Y. Then EX and BY ultimately point to S, and are always half of EZ and BV, which are ultimately as the forces at E and B. Hence EX and BY (the sagittae) are also, ultimately, as the forces at E and B.

NOTE: EX and BY are *actually* sagittae, and *ultimately* point to S; conversely, WE and UB *actually* point to S, and *ultimately* are the sagittae. So it is also true that the forces at B and E are ultimately as BU : EW.

COROLLARY 5

And so, the same forces are to the force of heaviness {gravity}, as these sagittae are to the sagittae perpendicular to the horizon of the parabolic arcs which projectiles describe in the same time.

1. Woah! Heaviness now.

2. Newton is now comparing different forces not on the same path, but on different paths. He is considering our original curve ABCDEF, from the past few corollaries, in comparison to the parabolic path of a projectile, since that is a real-life (and, thanks to Galileo, well-known) path produced by (1) a centripetal force (i.e. heaviness) and (2) innate force (or inertia).

3. Although Galileo considered the force of heaviness to be downward in parallel lines, since the center of the earth is so far away compared to the distances travelled by familiar projectiles, they in fact converge at the center of the earth. So heaviness is a centripetal force.

4. Newton is saying here that the "same forces," i.e. the forces at B and E on our abstractly considered curve, are [ultimately] to the force of heaviness producing the (nearly) parabolic path of a projectile, as the sagittae EX and BY are to the sagittae of the parabolic arcs accomplished in the same time as arcs AC and DF in our curve.

5. In this way we can measure an unknown force by a known one (weight), or compare them. Let ABCDEF be our orbit formed by centripetal force toward S, and let abcdef be our near-parabolic path of some projectile on Earth, formed by centripetal force (called 'weight') toward T, the center of the Earth, so far away that bT and eT are practically parallel and practically perpendicular to fN, the horizontal.

Let arcs AB, BC, DE, EF and ab, bc, de, ef each be accomplished in the same amount of time, t, and join chords AC, DF, ac, df and join the orbital radii BS, ES, bT, eT, forming BU, EW, bu, ew, which ultimately become sagittae, and are ultimately as the forces toward S and T at the locations B, E, b, e.

Newton is saying

$$\frac{f_B/f_E}{f_b/f_e} = \frac{\lim\ as\ \Delta t \to 0[BU/EW]}{\lim\ as\ \Delta t \to 0[bu/ew]}$$

so since the forces (i.e. accelerations) due to weight are known (as is the geometry of the parabola), and therefore so too we know the value of

$$\lim\ as\ \Delta t \to 0\left[\frac{bu}{ew}\right]$$

then if, by the geometry of the orbit about S [which might be ascertained by a telescope!], we can determine

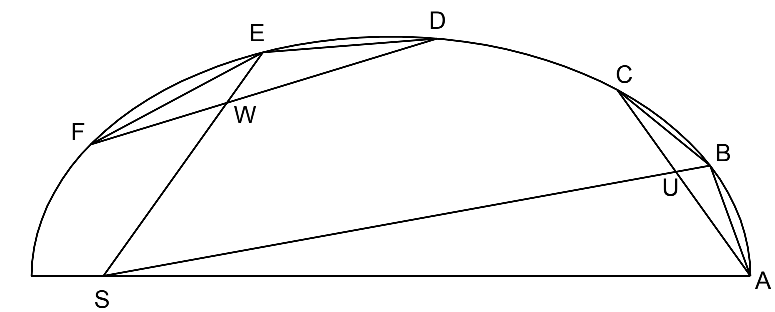
$$\lim as\ \Delta t \to 0 \left[\frac{BU}{EW}\right]$$

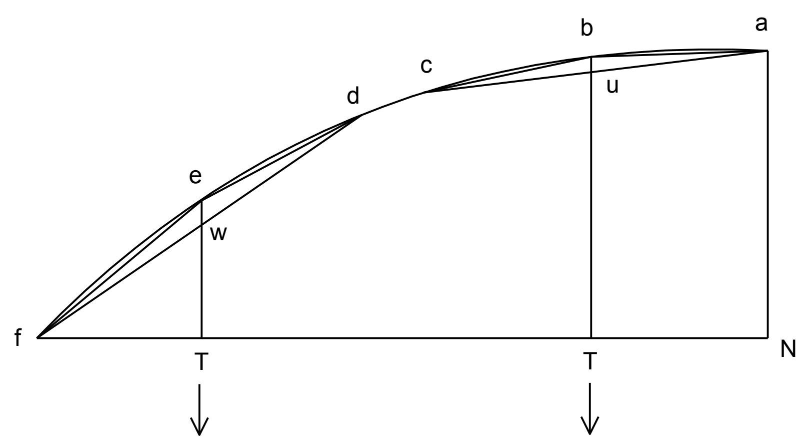
we will also know $\left[\frac{f_B}{f_E}\right]$

and so we will also know $\dfrac{f_B/f_E}{f_b/f_e}$

that is the relative strength of the forces in the S-orbit to those in the projectile path. And so we will have a way of comparing the strength of an unknown centripetal force to that of WEIGHT. Confer the argument to come at the end of the *Principia*, comparing the orbit of the Moon to the falling of a stone.

But this Corollary is more like a commercial than something we will actually use later.





COROLLARY 6

All the same things hold, by Cor. 5 to the Laws, whenever planes in which the bodies move, together with the center of the forces which is located in the same plane, do not rest, but move uniformly in a directed line.

Corollary 5 to the laws said that if a system of bodies is moving uniformly in a straight line, then the movements of the bodies relative to each other are the same as if they were all at rest.

So if the center of forces, S, and all the planes in which various bodies move around S, are all in motion uniformly in a straight line, the behavior of the bodies in the system relative to each other is no different from what it would be were they all at rest.

Think of our solar system. If the Sun (S) is at the focus of the orbits of the planets in their various planes, and it (and their orbital planes) are all drifting through space uniformly in a straight line, we can consider the whole business as though it were at rest in absolute space.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 31

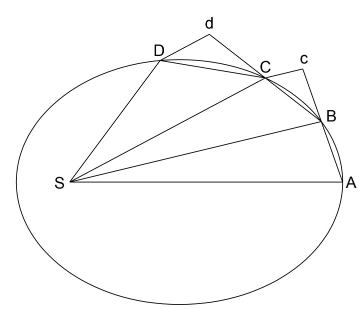
PROPOSITION 2 AND COROLLARIES

Preliminary notes:

1. This is the converse of Proposition 1.
2. "Case 1" is if S is immobile.
3. "Case 2" is if S is moving uniformly in a straight line.
4. Euclid 1.40 says that "Equal triangles on equal (and in-line) bases are also in the same parallels, i.e. of the same height."
5. In the argument he mentions "least triangles" (cf. Galileo, Kepler), i.e. infinitesimals. He warned us (in the Scholium after the Lemmas) that he would occasionally lapse into such language.
6. From Kepler's 2nd Law about planetary motion (which was established by observations and calculations), it now follows that planets are continuously impressed by centripetal forces toward the Sun. Hence the center of forces is always labeled "S" in these propositions (for "Sol", presumably).

PROPOSITION 2

Every body describing a plane curve, and which, by radii drawn to a point S (either fixed, or moving uniformly in a straight line) sweeps out areas proportional to the times, is urged by a centripetal force toward S.

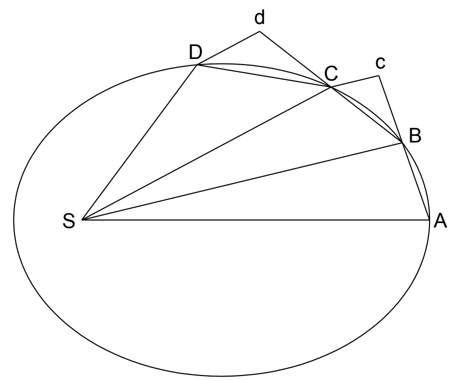


In equal times, let the body move through arcs AB, BC, CD, thus describing equal areas about S (given). And let such arcs be shrunk *in infinitum*. As this happens, the equal areas approach equality to the rectilinear triangles, and the arcs approach the chords. So now if we consider the "least triangles" (Galileo-style) swept out by the body, some force is driving it to describe these, since otherwise it would continue in a straight line (Law 1).

Extend AB to c, so that AB = Bc. Join Cc.

Now, since the times are equal, the body *would have been at* c *were it not for a force acting on it when it was at* B *and turning it aside from its rectilinear path* (Law 1) so that instead it went to C. But in what direction was the force at B acting?

Since $\triangle$SAB = $\triangle$SBc (these are ultimate triangles, now!)
and $\triangle$SAB = $\triangle$SBC (again, "least" triangles, which are ultimately equal to the curvilinear triangles, which are always equal because they are swept out in equal times around S).
Hence $\triangle$SBC = $\triangle$SBc



But then, by Euc. 1.40, it follows that Cc is parallel to BS.

But Cc is representative of the change of velocity in the body at B (Cor. 1 to the Laws; parallelogram of forces), since the difference in speed and direction between being at c and being at C is represented by the line cC (in the direction from c to C, by the way). Hence the force at B, effecting the change in speed and direction, must have been in that direction. But the line BS is parallel to cC, and therefore the force acted in the direction of B to S.

Similarly, if we go through the same steps, we can show that the force at C acts along the line CS, etc. Hence the force effecting the orbit is always acting toward point S, and hence it is a centripetal force toward S that produces such an orbit.

Q.E.D.

Note: We need more than one line of force to establish the Theorem clearly. Just looking at one point B, for instance, we might not see clearly that the force at B acts "toward S," but only that it acts along the line parallel to cC, which *happens* to pass through S. We need to see that every such parallel, from other points, will also pass through S, which we see better just by taking one more case.

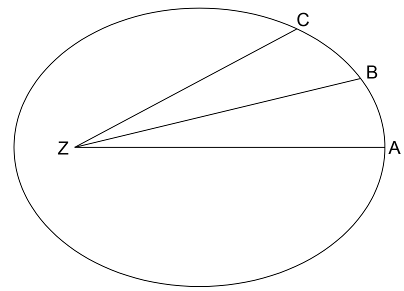
CASE 2

This just applies Cor. 5 to the Laws. If the plane of the orbit, together with its point S, is moving uniformly in a straight line, and the body sweeps out equal areas in equal times around S, it is still true that the forces urging the body out of a uniform rectilinear path all point toward S. That is because when things all move with the same uniform and rectilinear motion, they are at rest relative to each other, unless some other additional motion is given to one but not another.

COROLLARY 1

By Propositions 1 and 2, which show that "centripetal orbits" and "orbits equably describing areas about a point" are convertible, it follows that if one of these conditions be absent, so is the other. That is, *if areas are not proportional to times, then the forces do not tend to one point*, but must deviate.

More specifically, if a body moves through arcs AB and BC in equal times, but area ZBC is greater than area ZBA, so that area-description is *accelerating*, then the force itself is also tending to point more in the progressive direction of the motion, i.e. left of Z.



And if area ZBC is less than area ZBA, so that area-description is *retarded*, then the force itself is tending backward, to the right of Z, in the regressive direction.

COROLLARY 2

If area-description is *accelerating* even in a resisting medium (like air), then all the more is the force itself shifting in the progressive direction of the motion.

Note: Newton cannot say, similarly, that "if area-description is *decelerating* in a resisting medium, then the force moving the body deviates backwardly." After all, the whole cause of the deceleration might be the resistance of the medium.

SCHOLIUM

> *It is possible to urge a body by a centripetal force composed from many forces. . . . Furthermore, if some force act continuously according to a line perpendicular to the described surface, this force will cause the body to be deflected from the plane of its motion; but it will neither augment nor diminish the quantity of the surface described, and for that reason is to be ignored in the composition of the forces.*

The first part is plain enough. The centripetal force toward S need not be the result of a single cause, but can be the composite of many forces, i.e. a net force, composed of two (or more) forces, neither of which is centripetal.

In the second part, Newton is considering a force acting always perpendicularly to the plane of the orbit. That will move the body into other planes, he says, but will not change the quantity of the area described about S in a given time.

(I suppose we should imagine a Planet orbiting the Sun, and a force acting on the planet, always perpendicular to the plane of its orbit. This will have the effect either of moving the whole orbit up, parallel to itself, which will obviously not alter the equable description of areas, or, if the force is always on the Planet, perpendicular to the plane of the orbit it is trying to describe around the Sun, the orbit will rotate, and in that rotating plane itself, the equable description of areas around the Sun will not be affected. But the areas traced out in three-dimensional space by the Planet-Sun line will be affected.)

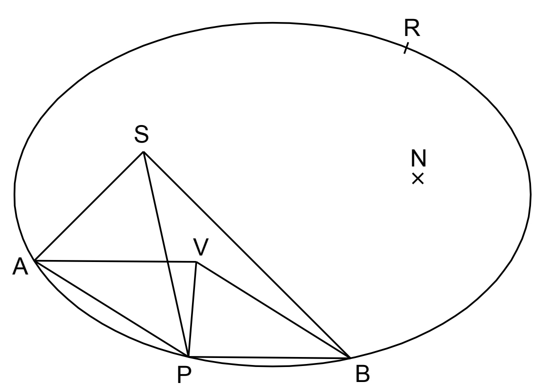
QUESTION: *Is it possible for a body in some curved orbit to be sweeping out equal areas in equal times around **two** distinct points?*

Suppose a body in a curved orbit sweeps out equal areas in equal times around S. Can it also be doing so around N? Choose a random point P on the orbit, and take arcs on either side that are described in equal times, so that the curved triangles SAP and SBP are equal in area.

Complete parallelogram APBV. Therefore PV ultimately points to S as $\triangle t$ goes to zero (by Prop. 2). Therefore PVS is ultimately collinear. But if the body is also describing equal areas in equal times around N, it follows for the same reason that PVN is also ultimately collinear. But P, S, N are fixed points, and the ultimate direction of PV is one only—therefore P, S, N must actually be collinear.

And likewise it follows, if we pick R at random elsewhere on the orbit, that R, S, N are actually collinear. And so S and N are collinear with *any* point picked on the orbit—which can only be if S and N are the same point (or if the orbit is itself a straight line, which is contrary to the given, here).

Therefore it is not possible for a body in some curved orbit to be sweeping out equal areas in equal times around two distinct points.



THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 32

PROPOSITION 3 AND COROLLARIES AND SCHOLIUM

Preliminary notes:

1. Newton has in mind a special case which he is not yet mentioning explicitly, i.e. Moon and Earth (hence L = **L**una, and T = **T**erra).
2. Both Moon and Earth (L & T) are under continuous influence of centripetal force toward the Sun.
3. But the Earth-Moon distance is so little compared to the Earth-Sun distance that the lines drawn from the Sun to the Earth and to the Moon are all effectively parallel.
4. So this is like a preliminary to Prop. 65 later, giving us some inroad into the three-body problem (or even the n-body problem).
5. The claim is: If T is somehow accelerating and L describes areas as times around T, then L is urged by a force composed of the force accelerating T and a centripetal force toward T.
6. We use Cor. 6 (to the Laws) in this Prop. 3.
7. Prop. 3 is like an extension of Prop. 2: Prop. 2 says we can infer a centripetal force to a center of uniform area-description if the center is at rest or moving inertially—Prop. 3 says we can infer this even if the center is moving with an accelerated motion.

PROPOSITION 3

> *Every body L, which by a radius drawn to the center of another body T, however T might be moving, describes around T areas proportional to the times, is urged by a force composed of the centripetal force tending toward T, and every accelerative force by which that other body is urged.*

We are given the behavior of L: it describes equal areas in equal times around the center of T. And we are given an acceleration of T, due to some force.

We argue by **introducing** a new force, equal and contrary to that urging T, which new force acts on both L and T according to parallel lines (these lines being drawn parallel to the line along which T was being acted on by whatever force).

By Cor. 6, the two bodies will be unaffected among themselves by this new force, although the force by which T was urged is now eliminated (or nullified), and hence T is either at rest or moving uniformly in a straight line (Law 1), while L (thanks to Cor. 6) must still be sweeping out equal areas in equal times around T, and therefore is urged by forces tending toward T (Prop. 2). And therefore, in the original, given situation, the net force urging L was composed of a centripetal force toward T *plus* the one that was urging T.

Q.E.D.

I will not reproduce the text of the four corollaries, but will merely paraphrase their content.

COROLLARY 1

This is not much more than a restatement of the Proposition itself: If from the whole force on L we subtract the force urging T, what remains is the force urging L toward T as center.

COROLLARY 2

And if the areas swept out by L around T are *nearly* as the times (note this is a new given), then the "remainder" force urging L (with the force urging T subtracted from the whole force on L) will tend *nearly* to point always to T.

COROLLARY 3

And conversely: If the remainder-force on L always points *nearly* to T, L will sweep out areas *nearly* as the times around the center of T.

COROLLARY 4

By contrast, if L sweeps out areas around T that are nowhere near as the times, even though T is either at rest or moving uniformly in a straight line, then either L is urged by no force tending to T, or that force is insignificant compared to other forces acting on L.

NOTE: Corollaries 2, 3, 4 again are relevant to the Earth and Moon. The Earth-Moon system is affected by other bodies, after all, so we don't get the moon sweeping out exactly equal areas in exactly equal times around the Earth. But it is close. Other bodies are so far away as to have no significant influence, and even the Sun is far enough that it acts on both almost in parallel lines (and without significant "spaghettification"). And the Earth moves almost uniformly in a straight line around the Sun in a given day (since the radius of the orbit is so large relative to one day's arc). Therefore the most significant force on the Moon is a centripetal one towards Earth.

SCHOLIUM

> *Since an equable description of areas is an indication of a center which that force by which the body is most affected and by which it is drawn back from rectilinear motion and retained in its orbit has regard to, why may we not, in the following, employ an equable description of areas as an indication of a center around which all circular motion is accomplished in free spaces?*

This sums up Propositions 1-3: *Nearly equable description of areas indicates a center toward which the force most influencing the body tends.*

The phrase "in free spaces" makes us think of celestial bodies. So he is saying: why not take this new principle, or indicator of centripetal force, to the heavenly bodies? In so doing, we say that those bodies do NOT naturally move in circles (contra Copernicus, Ptolemy, Aristotle), but either rest or move in straight lines when not urged by any forces, and hence move in circles (or other curves) only by impressed forces. More specifically, since they sweep out equal areas in equal times around the Sun (Kepler's 2nd Law, verified by observation and calculation), therefore they are all urged by centripetal forces toward the Sun.

Now, that is not really altogether new. At any rate, Law 1 said that the motion of the heavenly bodies was not simply natural, but due to some force impressed on them. What he adds now is (thanks to Propositions 1–3) a way of determining the center of the forces urging celestial bodies.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 33

PROPOSITION 4 AND ITS 9 COROLLARIES

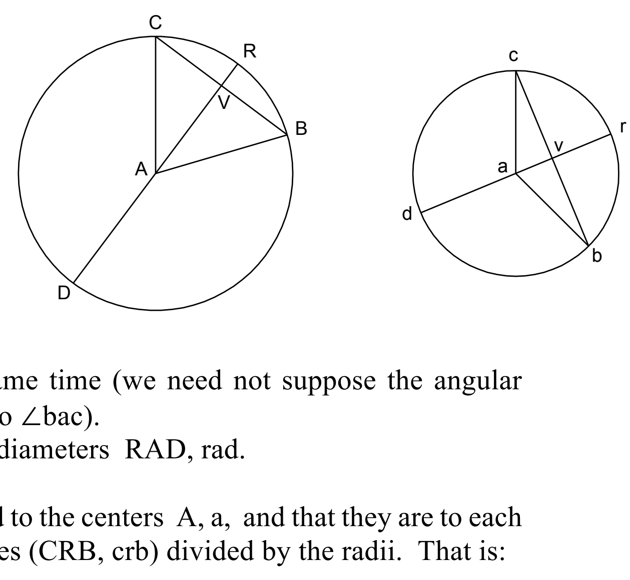
NOTE: This next Proposition and its Corollaries are FOUNDATIONAL to Newton's argument for universal gravitation in Book Three.

PROPOSITION 4

The centripetal forces of bodies which describe diverse circles with equable motion tend to the centers of those circles, and are to each other as are the squares of simultaneously described arcs divided by the radii of the circles.

Newton was not so good as to provide us with a diagram for his argument. He will do the same in Prop. 6. Rather than bother with a diagram and proportion, he prefers to say that "this is *as that*," and simply argue from the prior propositions. Since he is talking about two circles, however, and since he is saying the forces are "as something," he is talking about a proportion. *So it is good to draw two circles, and to advise the students to draw two circles (with agreed-upon letters) when you assign this Proposition.*

So let there be two bodies describing two circles (his "diverse circles") equably—i.e. moving uniformly along the circumferences about the centers (sweeping out equal angles, and hence also equal areas, in equal times around the geometric centers).



Let the centers of these circles be A, a.

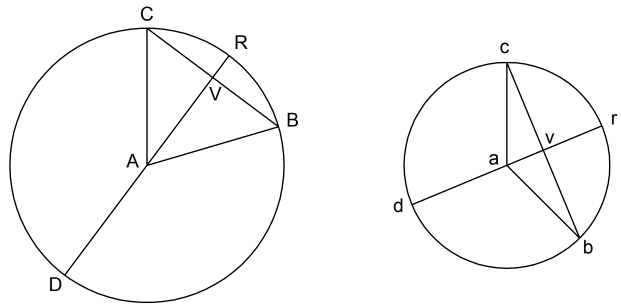
Let the arcs CB, cb be described at the same time (we need not suppose the angular velocities are equal, hence ∠BAC need not be equal to ∠bac).

Bisect the arcs at R, r and draw through the diameters RAD, rad.

Prop. 4 is saying that the centripetal forces at R, r tend to the centers A, a, and that they are to each other as the squares of the arcs described in equal times (CRB, crb) divided by the radii. That is:

$$F_R : F_r = \frac{(arcBC)^2}{AR} : \frac{(arcbc)^2}{ar}$$

That they tend to the centers A, a is obvious from Prop. 2, since the bodies are sweeping out equal areas in equal times around A, a. (He says it also follows from Prop. 1 Cor. 2, since that corollary says the line of force is the ultimate position of RV [or double RV, the diagonal of the parallelogram in ∠CRB], but in a uniform circular motion, RV is already in the ultimate position as arcs CR and RB shrink, i.e. as the time goes to zero.)



The proof of the proportion is as follows. Since $CV^2 = DV \cdot VR$, and $cv^2 = dv \cdot vr$, therefore

$$RV : rv \ = \ \frac{CV^2}{DV} : \frac{cv^2}{dv}$$

But since DV is ultimately equal to DR (as we shrink the arcs described in equal times by the two motions), and dv to dr, hence DV : dv is ultimately as DR : dr, and hence ultimately as the halves of these diameters, i.e. as the radii AR : ar. Therefore:

$$RV : rv \ = \ \frac{CV^2}{AR} : \frac{cv^2}{ar} \qquad \text{(ultimately)}$$

Again, the squares on CV, cv are actually as their quadruples, i.e. as the squares on CB, cb, and these in turn are (by Lemma 7) ultimately as the squares on the arcs, i.e. $(\text{arcCB})^2$ and $(\text{arc cb})^2$. Therefore:

$$RV : rv \ = \ \frac{(arcCB)^2}{AR} : \frac{(arc\ cb)^2}{ar} \qquad \text{(ultimately)}$$

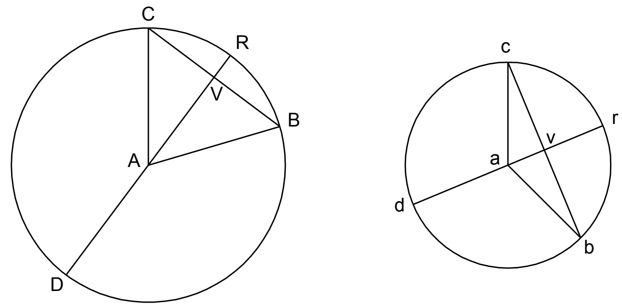
But the forces at R and at r, by Prop. 1 Cor. 4, are ultimately as the sagittae (or, since we are dealing with circles, “versed sines”) of the arcs, as the arcs shrink, i.e. ultimately as RV : rv. Therefore:

$$F_R : F_r = \frac{(arcBC)^2}{AR} : \frac{(arcbc)^2}{ar} \qquad \text{(ultimately)}$$

But the arcs described in equal times in our two circles will always be in a given, fixed ratio (because of the uniformity of the motions), and hence the squares of the arcs will also be in a fixed ratio. Moreover, the radii are in a fixed ratio. Hence the ratio on the right side is a fixed ratio, and must therefore be actually as the two forces. Therefore:

$$F_R : F_r = \frac{(arcBC)^2}{AR} : \frac{(arcbc)^2}{ar}$$

Q.E.D.



Q1. Is $\angle CAB = \angle cab$? (No, there is no necessity in that.)
Q2. Is this Proposition true ultimately, or actually? (Actually, since in the final proportion, all ratios are fixed ratios, not fluid ones. The two forces are two fixed forces, and $AR : ar$ is fixed, and so too is the ratio of the squares on the arcs accomplished in equal times.)
Q3. Is it intuitive that the forces are *inversely* as the radii?
Q4. In what units are we measuring the arcs?

(It is good to note here that **the arcs must be measured in linear units**, not in degrees! If equable motions in unequal circles complete the two circles in the same time, then the angular velocities are the same, but the linear velocity in the greater circle is greater.)

COROLLARY 1.

Given the uniformity of the two motions, the arcs they accomplish in the same time must be as the speeds through them: (arcBC) : (arc bc) = (speed thru arc BC) : (speed thru arc bc). Therefore we may put the squares of the speeds into the proportion:

$$F_R : F_r = \frac{(speed\ thru\ arcBC)^2}{AR} : \frac{(speed\ thru\ arc\ bc)^2}{ar}$$

COROLLARY 2.

But the speeds are both uniform, and hence in a given ratio (regardless of the amount of time). The speed in each circle, for instance, is the whole circumference of the circle divided by the period (or time to go one full circle). So the speed through arc BC is the same as (2πAR) over T, where T is the period for circle A. Likewise, the speed through arc bc is the same as (2πar) over t, where t is the period for circle a. So, plugging these into the proportion of Cor. 1 above, we get:

$$F_R : F_r = \frac{(2\pi AR/T)^2}{AR} : \frac{(2\pi ar/t)^2}{ar}$$

or

$$F_R : F_r = \frac{AR^2/T^2}{AR} : \frac{ar^2/t^2}{ar}$$

thus

$$F_R : F_r = \frac{AR}{T^2} : \frac{ar}{t^2}$$

Q.E.D.

COROLLARY 3.

Therefore, in the special case where the periodic times are equal, i.e. when T = t, then the forces are simply as the radii, and conversely, where the forces are as the radii, the periodic times must be equal.

NOTE: Newton says “AND CONVERSELY.” All these Corollaries, 3–7, are convertible.

NOTE: Newton adds “and for that reason the velocities as the radii,” not as a premise, but as another consequence, or another indicator. This consequence follows from the same given, like this:

$v_1 : v_2 = 2\pi AR/T : 2\pi ar/t$ [by def.]

so $v_1 : v_2 = AR : ar$ [given that T = t]

COROLLARY 4.

And in the special case where the squares of the periods are as the radii, then the forces must be equal. For, by Cor. 2,

$$F_R : F_r = \frac{AR}{T^2} : \frac{ar}{t^2}$$

and where the squares of the periods are as AR : ar, the ratio on the right becomes the ratio of equality. And, conversely, if the forces are equal, and hence the ratio on the right is one of equality, then the squares of the periods must be as the radii AR : ar, or (if you like) the periods themselves must be as the square roots of the radii, i.e. T : t = √AR : √ar.

NOTE: Newton adds "and for that reason the velocities" are "also in the subduplicate ratio of the radii . . ." This consequence follows from the same given. That is,

since $v_1 : v_2 = 2\pi AR/T : 2\pi ar/t$ [by def.]

thus $v_1 : v_2 = 2\pi AR/\sqrt{AR} : 2\pi ar/\sqrt{ar}$ [given that $T : t = \sqrt{AR} : \sqrt{ar}$]

so $v_1 : v_2 = \sqrt{AR} : \sqrt{ar}$ [simplifying]

COROLLARY 5.

And in the special case where the periods themselves are simply as the radii, then the forces are as the reciprocals of the radii. For, by Cor.2,

$$F_R : F_r = \frac{AR}{T^2} : \frac{ar}{t^2}$$

so if we suppose that T : t = AR : ar, we can substitute $AR^2 : ar^2$ in the denominators, and simplify to

$$F_R : F_r = \frac{1}{AR} : \frac{1}{ar}$$

Conversely, if the forces are in this ratio, it must be (by equating this with the ratio in Cor. 2) that the periods are as the radii.

NOTE: "And for that reason the velocities equal . . ." If the periods are as the radii, then, since the radii are as the circumferences (Lemma 5), it follows that the periods are as the circumferences. But then the speeds must be equal, since it is a property of equal uniform speeds that the distances they accomplish are as the times for which they were allowed to go.

COROLLARY 6.

In the special case where the periods are in the sesquiplicate ratio of the radii, it will follow that the forces are as the reciprocals of the squares of the radii. For, by Cor. 2,

$$F_R : F_r = \frac{AR}{T^2} : \frac{ar}{t^2}$$

so if we suppose that $T : t = \sqrt{AR^3} : \sqrt{ar^3}$, i.e. that $T^2 : t^2 = AR^3 : ar^3$, then

$$F_R : F_r = \frac{1}{AR^2} : \frac{1}{ar^2}$$

and conversely, if the forces are reciprocally as the squares of the radii, then, equating that ratio with the ratio of Cor. 2, it follows that the periods are in the sesquiplicate ratio of the radii.

NOTE 1: This is reminiscent of Kepler's Third Law, namely that "The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of their mean distances from the Sun." That is, if r_1 is the mean distance for Planet 1, and t_1 is the period for Planet 1, then $(r_1)^3 : (r_2)^3 = (t_1)^2 : (t_2)^2$. Take the square root of both sides, and you see that the periods are in the sesquiplicate ratio of the radii.

NOTE 2: Here for the first time Newton is showing the connection between Kepler's 3rd Law and an inverse square law of force variation. If two circular orbits happen to be about the same center, and are due to the same cause, and the squares of their periods are as the cubes of their radii, then the force varies according to an inverse square rule.

NOTE 3: Newton here adds that if the periods be in the sesquiplicate ratio of the radii, then also, "for that reason," the velocities will be reciprocally in the subduplicate ratio of the radii. Calling the radii R and r, and the velocities in the two circles V and v, and the periods in them T and t, we can show what he means as follows.

First:

$$V : v = \frac{2\pi R}{T} : \frac{2\pi r}{t}$$

This follows basically from the definitions of the uniform velocities V, v. But we don't need the 2π. And if, as we are here supposing, it happens to be the case that the periods are in the sesquiplicate ratio of the radii, then for $T : t$, we can substitute $\sqrt{R^3} : \sqrt{r^3}$. Thus we have:

$$V : v = \frac{R}{\sqrt{R^3}} : \frac{r}{\sqrt{r^3}}$$

or

$$V : v = \frac{1}{\sqrt{R}} : \frac{1}{\sqrt{r}}$$

as Newton says.

Q.E.D.

COROLLARY 7.

Generally, if T : t = $R^n : r^n$ (where R, r are the radii), then, by Cor. 2,

$$F_R : F_r = \frac{AR}{T^2} : \frac{ar}{t^2}$$

i.e.
$$F_R : F_r = \frac{R}{(R^n)^2} : \frac{r}{(r^n)^2}$$

or
$$F_R : F_r = \frac{R}{R^{2n}} : \frac{r}{r^{2n}}$$

or
$$F_R : F_r = \frac{1}{R^{2n-1}} : \frac{1}{r^{2n-1}}$$

And, conversely, if the forces are in this ratio, the periods must be as $R^n : r^n$.

COROLLARY 8.

This is a fairly crucial and daring step. He says that all the things he has said are just as true about ANY two **similar** curves (i.e. similar parts of them, and with reference to similarly placed centers of force, I suppose), for example, ellipses.

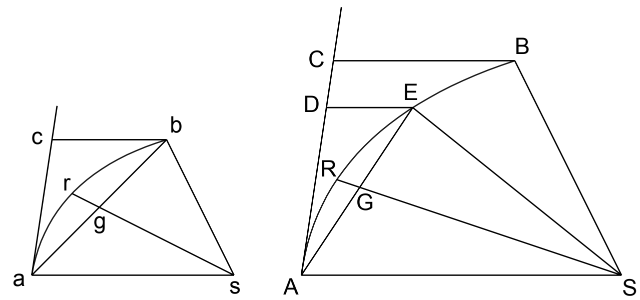
Why? Because the circularity was not important, but only (1) **The uniformity of the motions**, and although in our ellipses (or whatever) the linear velocities will not be uniform, still the ‘area velocities’ will be, so we can just substitute areas for arcs, etc., and also (2) **The similarity of the figures**, since the “radii,” if they are not equal, must at least be in a fixed ratio (for a similar argument to hold), which indeed they are when taken in similar places in similar figures.

The SCHOLIUM before Prop. 4 may have been hinting at this, i.e. “Why may we not . . . ?” That is, “Why may we not use circular motions as a path into discovering what else we might say, more generally, about centripetal orbits, regardless of shape?”

PROOF OF COROLLARY 8.

Given:

- ab & AB are corresponding parts of similar curved orbits with finite curvature, around similarly-placed centers of uniform area-description, s, S.
- ac & AC are tangents to the corresponding points a, A.
- bc is parallel to *as*, BC is parallel to AS.
- Arcs ab & AE are described in equal times.
- ED is parallel to BC.



Prove:

The analog to Prop. 4, i.e. that $$F_a : F_A = \frac{(arc\ ab)^2}{as} : \frac{(arc\ AE)^2}{AS}$$

ultimately, as the arcs of equal time are shrunk to a, A.

Well,	CB : DE = $AB^2 : AE^2$	[ult; Lemma 11]
so	CB : DE = $(AB^2 : ab^2)(ab^2 : AE^2)$	[ult; just compounding]
so	(cb : CB)(CB : DE) = (cb : CB)$(AB^2 : ab^2)(ab^2 : AE^2)$	[ult; comp. with (cb : CB)]
so	cb : DE = (ab : AB)$(AB^2 : ab^2)(ab^2 : AE^2)$	[ult; ab : AB = cb : CB, sim figs]
so	cb : DE = (AB : ab)$(ab^2 : AE^2)$	[ult; simplifying]
so	cb : DE = (AS : as)$(ab^2 : AE^2)$	[ult; AS : as = AB : ab, sim figs]
or	cb : DE = $AS \cdot ab^2 : as \cdot AE^2$	[ult]

i.e. $$cb : DE = \frac{ab^2}{as} : \frac{AE^2}{AS}$$ [ult] $\left[\times \frac{1}{as \cdot AS}\right]$

But $ab^2 : AE^2$ = (arc ab)^2 : (arc AE)^2 [ult, chords as arcs]

and cb : DE, being subtenses, are ultimately as the sagittae gr, GR, [by Lemma 11 Cor. 2] and therefore ultimately as the forces at r, R, [by Prop. 1 Cor. 4] which are ultimately as the forces at a, A.

So $$F_a : F_A = \frac{(arc\ ab)^2}{as} : \frac{(arc\ AE)^2}{AS}$$ [ult]

And unlike in the case of Prop. 4 about the circles, we have to keep the “ultimately,” since the ratio of the arcs is not a fixed ratio, as it was in the case of the circles.

Q.E.D.

QUESTION: Will Corollaries 1-7 apply to such similar figures? Cor. 1 will apply only to instantaneous velocities at corresponding points, since the linear speeds are not uniform in our similar figures (only the area-speeds are uniform).

So $$F_a : F_A = \frac{(speed\ at\ a)^2}{as} : \frac{(speed\ at\ A)^2}{AS}$$

Why? Well, we just proved that

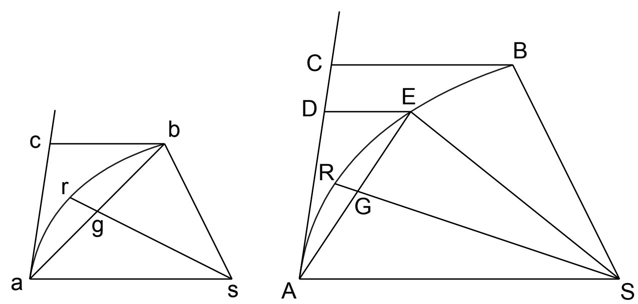
$$F_a : F_A = \frac{(arc\ ab)^2}{as} : \frac{(arc\ AE)^2}{AS} \qquad \text{[ult]}$$

but the times of arcs ab and AE are always equal, and therefore, as we shrink the arcs and the speeds in each become more uniform, the ratio of the average speeds in each is approaching the ratio of the arcs traversed, and so, ultimately, the speeds are as the arcs, and therefore the ratio of the instantaneous velocities at the points a, A is the same as the ultimate ratio of the arcs, and therefore

$$F_a : F_A = \frac{(speed\ at\ a)^2}{as} : \frac{(speed\ at\ A)^2}{AS}$$

And in this case we need not add an "ultimately," since all the quantities are fixed.

Q.E.D.



But COROLLARY 2 requires more argument than that. We wish to say, if p, P are the *periods* (of motions a, A respectively), that

$$F_a : F_A = \frac{as}{p^2} : \frac{AS}{P^2}$$

The argument begins with what we showed above,

i.e. $$F_a : F_A = \frac{(arc\ ab)^2}{as} : \frac{(arc\ AE)^2}{AS} \qquad \text{[ult.]}$$

thus $$F_a : F_A = \frac{ab^2}{as} : \frac{AE^2}{AS} \qquad \text{[ult; chords as arcs]}$$

so $$F_a : F_A = \frac{ab^2}{ab} : \frac{AE^2}{AB} \qquad \text{[ult; as : AS = ab : AB, sim. figs.]}$$

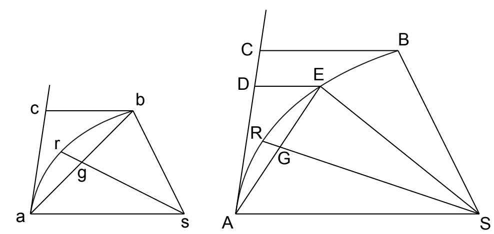
so $F_a : F_A = (ab^2 : AE^2)(AB : ab)$ [ult.]

so $\quad F_a : F_A = (ab^2 : AB^2)(AB^2 : AE^2)(AB : ab)$ $\quad$ [ult; inserting AB^2 by compounding]

so $\quad F_a : F_A = (AB^2 : AE^2)(ab : AB)$ $\quad$ [ult; simplifying]

or $$F_a : F_A = \frac{AB^2}{AB} : \frac{AE^2}{ab}$$ [ult]

Now, just hold on to that result, since we will use it in a minute. But first we need to develop another equation involving the periods, as follows:



$\quad$ p = Period of a = n · (time to go sab)
$\quad$ P = Period of A = m · (time to go SAE)

where $\quad$ n·sab = whole figure a
and $\quad$ m·SAE = whole figure A

Thus $\quad$ whole figure a : whole figure A = n·sab : m·SAE

so $\quad$ sab : SAB = n·sab : m·SAE $\quad$ [whole sim. figs. are as sab : SAB]

so $\quad$ sab : SAB = (n : m)(sab : SAE)

so $\quad$ n : m = (SAE : sab)(sab : SAB)

so $\quad$ n : m = SAE : SAB

but $\quad$ p : P = n(time to go sab) : m(time to go SAE) = (n : m) (time to go sab : time to go SAE)

so $\quad$ p : P = (SAE : SAB) c (time to go sab : time to go SAE)

but the times to go those two sectors are given as equal,

so $\quad$ p : P = SAE : SAB

so $\quad$ p : P = (time arc AE) : (time arc AB)

so $\quad$ p : P = time AE : time AB $\quad$ [ult; arcs as chords]

But as we shrink down to a and A, the speeds through AE, AB become as uniform as we please, and therefore the ratio of the times through them ultimately becomes the ratio of the chords themselves, and so

$\quad$ p : P = AE : AB $\quad$ [ult.]

so $\quad$ $p^2 : P^2 = AE^2 : AB^2$ $\quad$ [ult.]

Now, substituting this ratio in the ultimate proportion we found a minute ago, we have

$$F_a : F_A = \frac{P^2}{AB} : \frac{p^2}{ab}$$

So $$F_a : F_A = \frac{1}{ABp^2} : \frac{1}{abP^2}$$ [ult; mult. both last terms by $1/p^2P^2$]

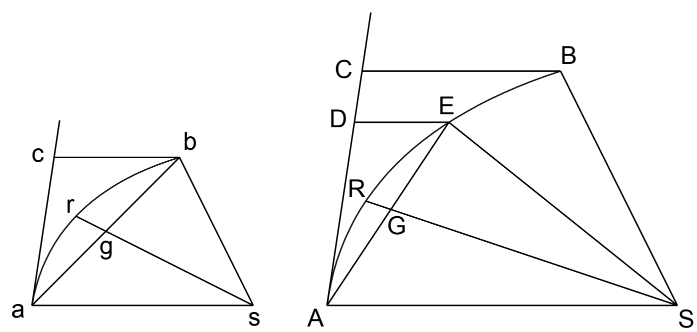
so $$F_a : F_A = \frac{ab}{p^2} : \frac{AB}{P^2}$$ [ult; mult. both last terms by ab·AB]

so $$F_a : F_A = \frac{as}{p^2} : \frac{AS}{P^2}$$ [since ab : AB = as : AS, sim. figs.]

And we don't have to say "ultimately," since all the ratios are fixed.

Q.E.D.

And the rest of Corollaries 3–7 now follow from this.



COROLLARY 9.

From the same demonstration it likewise follows that the arc which a body revolving uniformly in a given circle by a centripetal force describes in any time is a mean proportional between the diameter of the circle and the [height of the] descent of the body brought about by the same given force in the same falling time.

NOTE: This Corollary will be used in the argument for Universal Gravitation in Book. 3, Prop. 4, Theorem 4, about the MOON. Newton himself supplies no proof, so I will attempt to give one here. In doing so, I will invoke "The Mean Speed Theorem," which Galileo was the first to prove (after a fashion). That theorem states that a mobile accelerating uniformly from rest up to final speed S in time t will cover the same distance which another mobile covers in the same time t while traveling at a constant speed ½ S. That is easily proved using Newton's Lemmas and a speed-over-time diagram. The "velocity curve" in the case of uniform acceleration will be a straight line rising up at some angle from the time line. The area proportional to the distance covered by such a motion is that of the triangle, which is half the rectangle contained by the height representing the speed S and the horizontal line representing t. That rectangle, however, is proportional to the distance covered by the mobile moving uniformly at speed S during time t.

And now for the proof of Corollary 9:

GIVEN:
- Body in uniform motion, speed v, around a circle with center O, diameter AG.
- It travels arc AF in time t.
- And it would fall through distance AL in t without an inertial component.
- And we will consider the centripetal force (acceleration) to O as uniform (Galileo-style) on the grounds that the distances AL, and arc AF, are very short compared to the whole diameter and the whole orbit. In other words, this corollary is really an ultimate truth, as t goes to zero.

PROVE: AG : arc AF = arc AF : AL

Since the body falling from rest at A to O is accelerating uniformly, the mean-speed theorem applies, and the distance it goes is equal to half the distance that its final speed (at L, call it S_L) would accomplish in the same time, t,

i.e. $AL = \frac{1}{2} S_L \cdot t$ [mean speed theorem]

And since the acceleration is uniform, the final speed is just the product of that acceleration times the time, t (since a = speed / t),

i.e. $S_L = a \cdot t$

So $AL = \frac{1}{2} [a \cdot t]\, t$

i.e. $AL = \frac{1}{2} a \cdot t^2$

And since the velocity in the circle is uniform, it is as the arc accomplished over the time taken, i.e. v = arc AF / t,

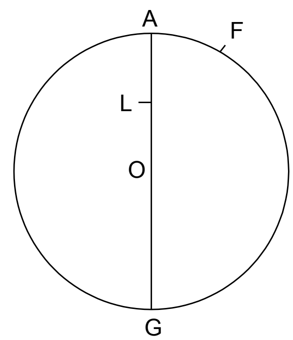
so $t = (\text{arc } AF) / v$

so $$AL = \frac{1}{2} a \left[\frac{arcAF}{v} \right]^2$$

But Prop. 4 Cor. 1 says that for uniform motions in circles, forces (i.e. accelerations) are as the squares of the velocities over the radii,

i.e. $$a = \frac{v^2}{r}$$

so $$AL = \frac{1}{2} \left[\frac{v^2}{r} \right] \frac{(arcAF)^2}{v^2}$$



so $$AL = \frac{(arcAF)^2}{2r}$$

so $$AL = \frac{(arcAF)^2}{AG}$$

so AG : arc AF = arc AF : AL

Q.E.D.

Is Corollary 9 an actual proportion, or an ultimate proportion?
It is an ultimate proportion, as t goes to zero.

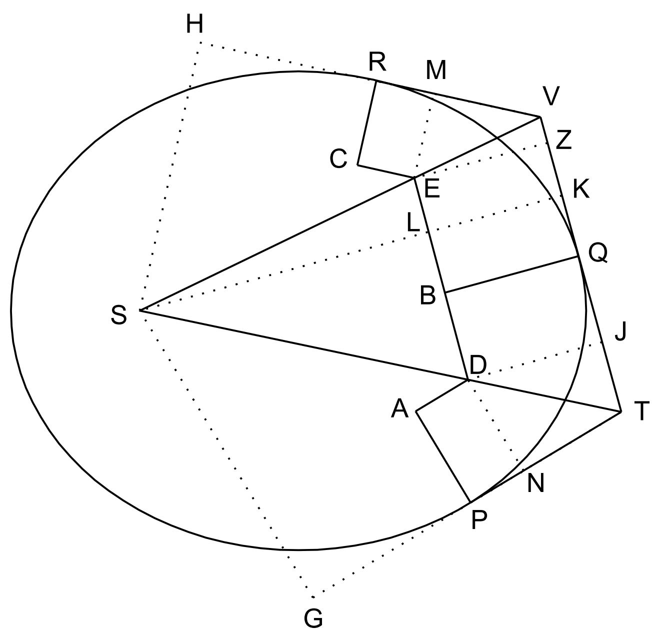
THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 34

PROPOSITION 5

Here we have the first "Problem" in Newton's *Principia*.

Given in any places whatever the velocity with which a body, by forces tending to some common center, describes a given figure, to find that center.



GIVEN: Velocities at P, Q, R on a figure described by the movement of a body continuously urged by centripetal forces toward a fixed, common center S,

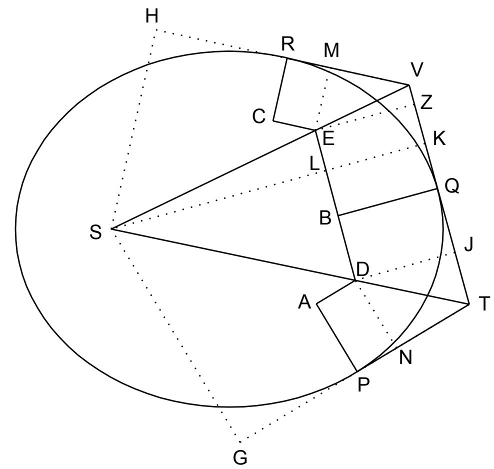
FIND: The location of that center (i.e. how to construct it from the givens).

Construction:

At P, Q, R draw tangents intersecting at T and V.
Draw perpendiculars PA, QB, RC of such lengths that

$\text{Velocity}_P : \text{Velocity}_Q = QB : PA$

and $\text{Velocity}_Q : \text{Velocity}_R = RC : QB$



Draw perpendiculars through C, B, A intersecting at E and D.
Join VE and TD.

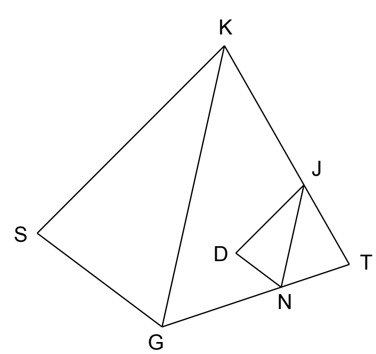
I say that S is the intersection of VE and TD.
Join SE, SD.
Drop perpendiculars: SH to VR, SK to VT, SG to PT.
Drop perpendiculars: EM to VR, DN to PT, DJ to VT.

Proof:

Since S is the center of forces, therefore the perpendiculars dropped from it to any tangents to the resulting orbit are reciprocally as the velocities at the points of tangency (Prop. 1 Cor. 1).

So $SG : SK = Vel_Q : Vel_P$
but $AP : BQ = Vel_Q : Vel_P$ [construction]
so $SG : SK = AP : BQ$

But $AP = DN$ [rectangle APND]
and $BQ = DJ$ [rectangle BDJQ]
so $SG : SK = DN : DJ$
while SK and DJ are parallel, as are SG and DN
so $\triangle$SGK is similar to $\triangle$DNJ
so GK is parallel to NJ
so $\triangle$GKT is similar to $\triangle$NJT
so fig. SGTK is similar to fig. DNTJ [composed of sim. triangles]
so $TJ : JD = TK : KS$
while JD and KS are parallel,



And from this it follows that S D T is a straight line.
Likewise S E V is a straight line.

Therefore the center of forces lies at the intersection of the two constructed lines VE and TD.

Q.E.I.

Q1. What can we now do? (Given an orbit produced by centripetal force, we can find the center; the odd presentation of Prop. 5 can obscure this fact.)

Q2. Does S have to be a special point geometrically? (No.)

Q3. What exactly do we need in order to construct it?

(3 instantaneous velocities and a way to draw tangents.)

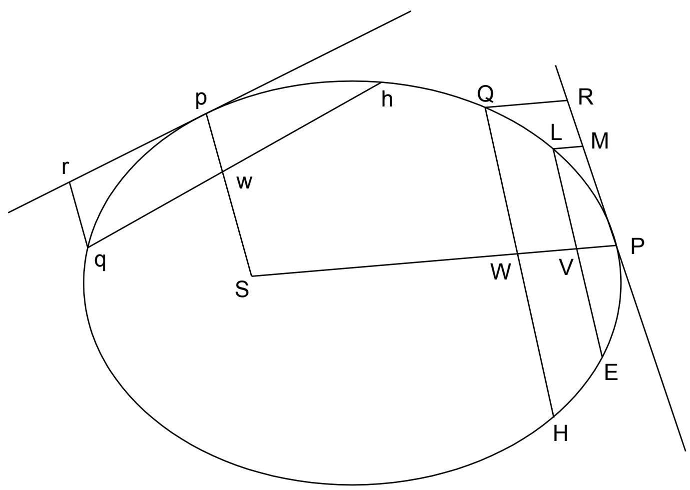
And given only 2 we can only find a line along which S must lie.

And given only 1 we can find nothing.

CLASS 35

PROPOSITION 6

If a body in non-resisting space revolves around an immovable center in any orbit whatever, and describes in the least time any arc just then nascent, and the sagitta of the arc is understood to be drawn which bisects the chord and, having been produced, passes through the center of the forces, the centripetal force in the middle of the arc will be as the sagitta directly and doubly as the times inversely.



Preliminary notes:

1. There is no diagram for this Proposition (the one Newton provides is really just for the corollaries). It is not really intelligible (for normal people) as he presents it—a diagram and some discussion are needed.

2. It is about "least times" and "arcs just nascent," and therefore the proportion being established is *ultimately* true.

3. If in an orbit produced by centripetal force the times of two arcs are *equal*, then, as we shrink them to their midpoints (keeping the times always equal),

Force at midpoint 1 : Force at midpoint 2 = sagitta 1 : sagitta 2 (*ultimately*)

by Prop. 1 Cor. 4.

4. So then what happens if the times are *unequal*, but always in a given ratio? Or at least *ultimately* in some definite ratio? (This latter condition is guaranteed just by the fact that the arcs vanish together. Newton therefore makes no explicit assumption that the times are in a fixed ratio as they shrink, whether here or in subsequent applications of Prop. 6.) That is what Prop. 6 is about: showing that the ratio of the forces at two points is also expressible as the limit of (1) the ratio of the sagittae of the two arcs accomplished in the *unequal* times compounded with (2) the inverse ratio of the squares of the times, provided the ratio of the times is always the same as we shrink the arcs.

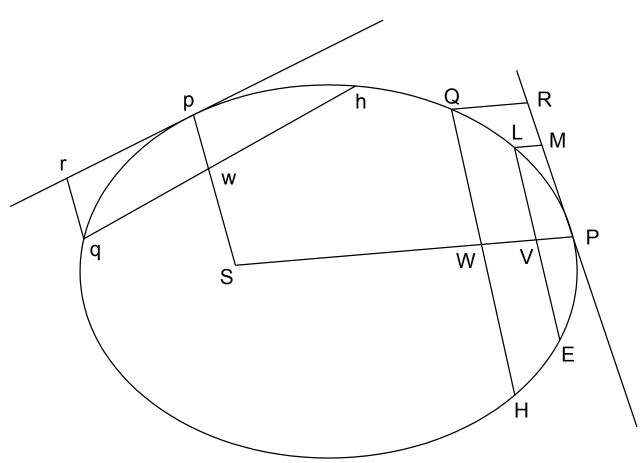
5. Note that in the figure we will use, QW is *ultimately* parallel to PR, and qw to pr, and also QW to LV.

GIVEN: The path of motion around center of forces S;
Points P, p at which to find the ratio of forces (accelerations);
Arcs HQ and hq, accomplished in unequal times in a fixed ratio (or at least a fluid ratio in which the time of arc QH is always greater than the time of arc qh, and the arcs vanish together as we diminish the times to nothing), and "temporally bisected" at points P, p (i.e. arcs HP, PQ take equal time, as do arcs hp, pq);
Chord QH cuts SP at W;
Chord qh cuts Sp at w.

PROVE: $F_p : F_P$ = (pw : PW) comp. (time thru arc PQ)^2 : (time thru arc pq)^2 [**ult.**]

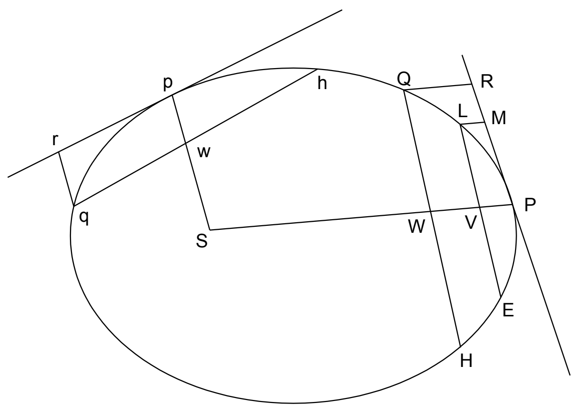
Since the time of arc QH has to that of qh a fixed ratio, one is always greater; let it be that the time of QH is the greater.
Let arc EL, also temporally bisected at P, be accomplished in the *same* time as arc qh, and let this always be so throughout the shrinking of arcs toward p and P.

Draw the chord LE, cutting the line SP at V.
Draw LM and QR always parallel to SP.
Draw qr always parallel to sp.



Note that since the arcs HQ and EL are temporally bisected at P, they are not necessarily distance-bisected there, so the arcs are not necessarily equal. Nor is there any necessity, of course, for the chords HQ and EL to be bisected at W and V (hence pw, PW, PV are not necessarily sagittae, but are at least ultimately so), or for them to be parallel to each other or to the tangent PR. But, as we shrink the times, the speeds through the arcs EP and PL become more uniform and more like the instantaneous speed at P, and hence the lengths of the arcs EP, PL, done always in equal times, become ultimately equal. So too EV, VL, ultimately equal to those arcs, are themselves ultimately equal, and so too HW, WQ are ultimately equal. Moreover, chords EL and QH, approaching the tangent, are ultimately parallel to each other.

Therefore PW is ultimately the sagitta bisecting the chord HQ,
pw is ultimately the sagitta bisecting the chord hq,
PV is ultimately the sagitta bisecting the chord EL



And since the times of arcs EL and hq are always equal,

thus $F_P : F_p = PV : pw$ [ultimately; Prop. 1, Cor. 4]

and those sagittae are ultimately as the subtenses ML : rq,

so $F_P : F_p = ML : rq$ [ultimately]

so $F_P : F_p = (ML : RQ) \text{ c } (RQ : rq)$ [def. of compounding]

And since arcs HQ and EL are both shrinking to the same point P, their velocities are ultimately equal (i.e. the instantaneous velocity at P), and therefore we can apply Lemma 11, Cor. 3 (the one about "with a given velocity"):

$ML : RQ = (\text{time arc } EL)^2 : (\text{time arc } HQ)^2$ [ultimately] but the times through the half-arcs are always as the times through the wholes, if we assume a fixed ratio between the times as they diminish—at the very least, they are approaching a fixed ratio as they shrink to nothing together,

so	ML : RQ = (time arc PL)^2 : (time arc PQ)^2	[ultimately]
so	$F_P : F_p$ = [(time arc PL)^2 : (time arc PQ)^2] c (RQ : rq)	
But	time arc PL = time arc pq	[constructed]
and	rq = pw	[ultimately]
and	RQ = PW	[ultimately]
so	$F_P : F_p$ = [(time arc pq)^2 : (time arc PQ)^2] c (PW : pw)	[ultimately]

Q.E.D.

Q1. Newton invokes Lemma 11, Cor. 3, which will apply only to curves of finite curvature. Does this mean our curve must be a conic section?

Not necessarily; it is enough if the curve has finite curvature at P.

Q2. Do we have to say "ultimately" in the conclusion?

Yes, since PW : pw is not a fixed ratio, but is approaching a fixed ratio.

NOTE: Lemma 11 Cor. 3 gives us insight into the subtenses and arcs shrinking to P, while Proposition 1 Cor. 4 gives us insight into the forces at P and p as pq and PL shrink. Proposition 6 puts this all together.

NOTE: Lemma 10 Cor. 4 says practically the same thing as Prop. 6 (as Newton notes), except that it is not about "sagittae." Lemma 10 Cor. 4 says that "in the beginning of the motion, the forces are directly as the spaces described and inversely as the squares of the times":

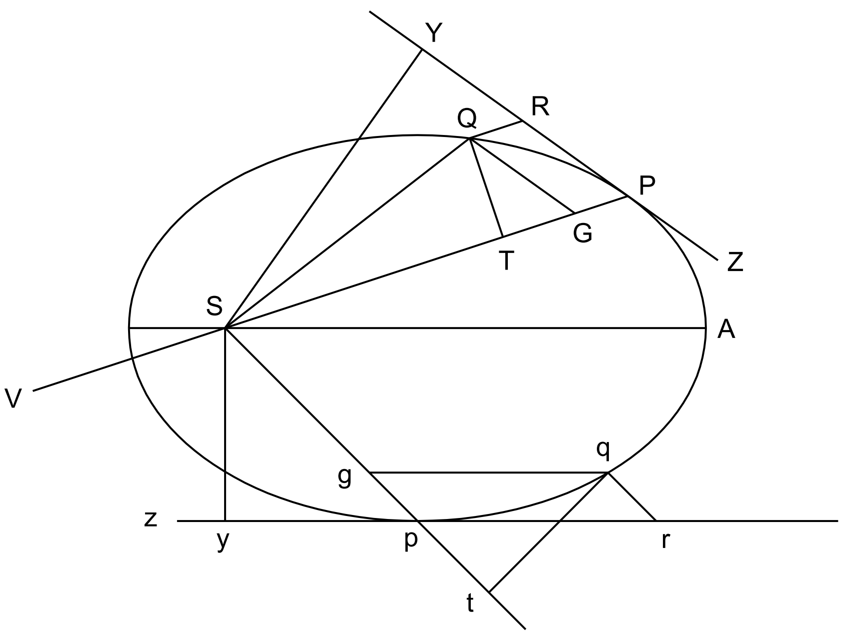
$$F_P : F_p = (t_p^2 : t_P^2)\,(d_P : d_p)$$

But these "spaces" or distances in Lemma 10 Cor. 4 mean those which are accomplished just by the forces in question, not including any other inertial (and hence distance-producing) motion. And such distances, ultimately, are as the sagittae.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 36

PROPOSITION 6 COROLLARIES



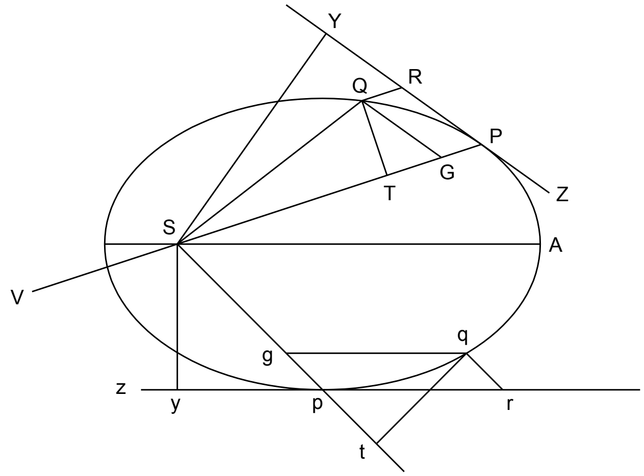
COROLLARY 1.

Note: The body is "P" for Planet, and the center is "S" for Sun (or Sol).

Given: Body P orbiting a center of forces, S, describing curve APQ.
ZPR tangent to curve APQ.
Q a random point on the curve.
QR parallel to SP.
QT perpendicular to SP.

Prove: F_P is (ultimately) inversely as $\dfrac{SP^2 \cdot QT^2}{QR}$

i.e. $$F_P : F_p = \frac{sp^2 \cdot qt^2}{qr} : \frac{SP^2 \cdot QT^2}{QR}$$ [ultimately]



For, if we draw QG parallel to RP, and qg parallel to rp,
then GP and pg are ultimately the sagittae of the arcs about P and p which take double the times of arcs PQ and pq.

And	time in arc PQ : time in arc pq = area SPQ : area spq	[Prop.1]
so	time PQ : time pq = $\triangle$SPQ : $\triangle$spq	[ultimately]
so	time PQ : time pq $\underline{\text{U}}$ 2$\triangle$SPQ : 2$\triangle$spq	[doubling]
so	time PQ : time pq $\underline{\text{U}}$ SP·QT : sp·qt	[rectangles]
so	$(\text{time PQ})^2 : (\text{time pq})^2$ $\underline{\text{U}}$ $SP^2 \cdot QT^2 : sp^2 \cdot qt^2$	[squaring]
but	$F_P : F_p$ $\underline{\text{U}}$ (GP : gp) c $(\text{time pq})^2 : (\text{time PQ})^2$	[Prop.6]
so	$F_P : F_p$ $\underline{\text{U}}$ (GP : gp) c ($sp^2 \cdot qt^2$: $SP^2 \cdot QT^2$)	
but	GP : gp = QR : qr	[parallelograms]

so $$GP : gp = \left[\frac{1}{QR \cdot qr}\right] QR : \left[\frac{1}{QR \cdot qr}\right] qr \qquad [\times \text{ same}]$$

i.e. $$GP : gp = \frac{1}{qr} : \frac{1}{QR} \qquad [\text{simplifying}]$$

Therefore we may substitute the ratio of the inverses of QR, qr for the ratio of (GP : gp) in the ultimate proportion above, and then multiply these, or compound them, with the other ratio ($SP^2 \cdot QT^2$: $sp^2 \cdot qt^2$), and we have:

$$F_P : F_p = \frac{sp^2 \cdot qt^2}{qr} : \frac{SP^2 \cdot QT^2}{QR} \qquad [\text{ultimately}]$$

Q.E.D.

COROLLARY 2.

Drop SY perpendicular to the tangent. Now consider △SQP. Its area is ½ SP·QT.
But as arc QP shrinks, QP approaches the tangent and therefore approaches being the base of △SQP with height SY.

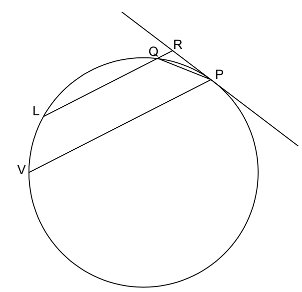
So △SQP = ½ SY·QP [ultimately]
so SP·QT = SY·QP [ultimately]

Therefore, substituting this expression into Corollary 1, we have:

$$F_P : F_p = \frac{sy^2 \cdot qp^2}{qr} : \frac{SY^2 \cdot QP^2}{QR} \qquad \text{[ultimately]}$$

NOTE: △QPT is ultimately similar to △SYP.

COROLLARY 3.



Draw a circle tangent to ZPR at P, cutting the orbit at Q—as we shrink arc PQ, then, this circle will approach the circle of curvature at P. Draw PV as a chord in this circle which is always parallel to RQ (and so PV passes through S), and thus chord PV will be approaching a limit length, namely that cut off in the circle of curvature at P. (We are assuming finite curvature.) Note: when Newton says "concentric circle," he means one having S as its center of forces, not necessarily as its geometric center.

Newton says that, ultimately

$$PV = \frac{QP^2}{QR}$$

Why is that true? Extend RQ to L.

So $RP^2 = QR \cdot RL$ [by Euclid 3.36]

but RP^2 U QP² [Lemma 7]

and RL U PV

so QP^2 U QR·PV

so $$PV = \frac{QP^2}{QR}$$ [ultimately]

And from this follows Cor. 3, since we can now substitute PV in for the expression QP^2/QR in the equation from Cor. 2, giving us:

$$F_P : F_p = sy^2 \cdot pv : SY^2 \cdot PV$$ [ultimately]

Q.E.D. [NOTE: If pv and PV are chords in circles of curvature, this is no longer "ultimately," but simply, true]

COROLLARY 4.

With the same givens,

$$F_P : F_p = (vel_P)^2 : (vel_p)^2 \quad c \quad (pv : PV)$$

For $(vel_P) : (vel_p) = sy : SY$ [Prop. 1, Cor. 1]

so $sy^2 : SY^2 = (vel_P)^2 : (vel_p)^2$

but $F_P : F_p = sy^2 \cdot pv : SY^2 \cdot PV$ [Cor. 3 above]

so $F_P : F_p = (vel_P)^2 \cdot pv : (vel_p)^2 \cdot PV$

Q.E.D.

COROLLARY 5. (General)

Therefore we can now discover laws by which a centripetal force, producing a generically known figure, varies. For the force is reciprocally as $\dfrac{SP^2 \cdot QT^2}{QR}$ by Cor. 1, or reciprocally as $SY^2 \cdot PV$ by Cor. 3.

But all these terms (SP, QT, QR, SY, PV) are determined purely by the geometry of the figure. Examples are now to follow.

So in Prop. 7, we have a CIRCLE, point S any point within it.
In Prop. 8, we have a SEMICIRCLE, point S infinitely distant.
In Prop. 9, we have a SPIRAL, point S the angular center.
In Prop. 10, we have an ELLIPSE, point S the center of the ellipse.

He assumes in all these that the body moves on the given figure, sweeping out equal areas in equal times around point S, so that he may apply the Prop. 6 Corollaries in order to determine the specific rule of force variation. He uses Corollaries 1 and 5 of Prop. 6 in all these upcoming Propositions.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 37

PROPOSITION 7 (CIRCLE) AND COROLLARIES

Preliminary notes:

1. Here Newton begins to show the power of the preceding theorems in determining laws of force in given figures with centers of force in given locations. This first case is that of a circle, with the center of force at some place within, which need not be the geometric center.

2. Prop. 4 already covered the case when the center of force is at the center of the circle (equable motion around the circle means sweeping out equal areas in equal times around the geometric center).

3. Question: In Ptolemy, if an epicycle sweeps out equal angles in equal times around an equant-point, is the center of the deferent the center of forces? Is the equant-point the center of forces? (No and no.)

Not the geometric center, since the epicycle sweeps out equal angles in equal times around the equant-point, therefore not around the geometric center, and therefore does not sweep out equal areas in equal times around the geometric center.

Not the equant-point, either, since the epicycle sweeps out equal *angles* around the equant-point, and therefore, since that point is not the geometric center, the epicycle must be sweeping out *unequal areas* around the equant-point in equal times.

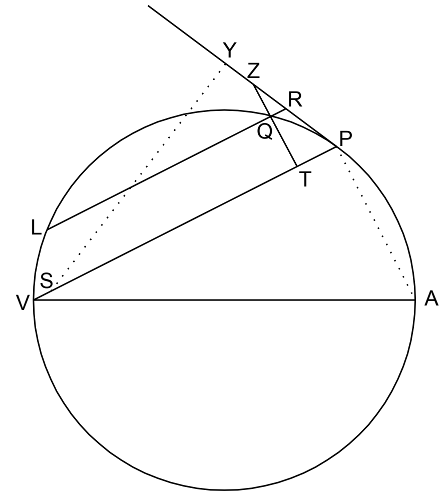
4. The Proposition finds a "LAW OF FORCE," which in this context means a rule for how a centripetal force fluctuates in relation to the distance from the center of force.

5. The Law of Force derived in this Proposition shows that if the center of forces, for circular motion, does not lie at the center of the circle, then the forces (and hence the speeds) of P (the "Planet") will not be uniform around the center of forces, since the ratio derived is continually changing with the motion of P. In fact, I think it is right to say that the forces, and the speeds, will not be uniform around *any point*—so that the Ptolemaic, Copernican idea is IMPOSSIBLE! That is, if we insist on there being a center of forces other than the center of the circle, then we cannot also have the planet sweep out equal angles in equal times. Of course, their physics was all wet, so they did not think about a "center of forces," but only about a "center of uniform motion" to which the spirit of the planet was looking. For them, circular motion was natural. There is nothing mathematically impossible about sweeping out equal angles in equal times around some point other than the geometric center of the circle; but then that point will not be the center of forces.

6. Propositions 7, 8, 9 consider scenarios which do not actually occur in nature.

PROPOSITION 7

If a body rotate on the circumference of a circle, the law of centripetal force tending to any given point whatever is required.



Note: "Any given point" i.e. any given center of forces, around which we are given that the body is sweeping out equal areas in equal times.

So let the center of forces be S, let the body be P, moving toward Q, on circle APQLV. Let PRZ be tangent at P, let PS be joined and drawn through to the circle at V. Let diameter VA be drawn, and AP joined. Draw QT perpendicular to SP, and let QT be extended to Z on the tangent. Draw QR parallel to SP, cutting the tangent at R, and the circle at L.

Now $\angle ZPT = \angle PAV$ [Euc. 3.32]

So $\triangle ZQR$ and $\triangle ZTP$ and $\triangle VPA$ are all similar.

so $RP : QT = ZP : ZT$

and $\underline{AV : PV = ZP : ZT}$

so $RP : QT = AV : PV$

so $RP^2 : QT^2 = AV^2 : PV^2$ [squaring]

But $RP^2 = QR \cdot RL$ [Euc. 3.36]

So $QR \cdot RL : QT^2 = AV^2 : PV^2$

Thus $QR \cdot RL \cdot PV^2 = AV^2 \cdot QT^2$

so $$\frac{QR \cdot RL \cdot PV^2}{AV^2} = QT^2 \qquad [\div AV^2]$$

so $$\frac{QR \cdot RL \cdot PV^2}{AV^2} \cdot \frac{SP^2}{QR} = QT^2 \cdot \frac{SP^2}{QR} \qquad \left[\times \frac{SP^2}{QR}\right]$$

But as Q goes to P, RL becomes ultimately equal to PV, hence

$$\frac{QR \cdot PV \cdot PV^2}{AV^2} \cdot \frac{SP^2}{QR} = QT^2 \cdot \frac{SP^2}{QR} \qquad \text{[ultimately]}$$

And the QRs CANCEL OUT on the left side, leaving us with

$$\frac{PV^3 \cdot SP^2}{AV^2} = QT^2 \cdot \frac{SP^2}{QR} \qquad \text{[ultimately]}$$

But (by Cors. 1 and 5 to Prop. 6), the centripetal force is reciprocally as the ultimate ratio

$$QT^2 \frac{SP^2}{QR}$$

Therefore, in this particular case, for our circle, the centripetal force is reciprocally as the ultimate ratio

$$\frac{PV^3 \cdot SP^2}{AV^2}$$

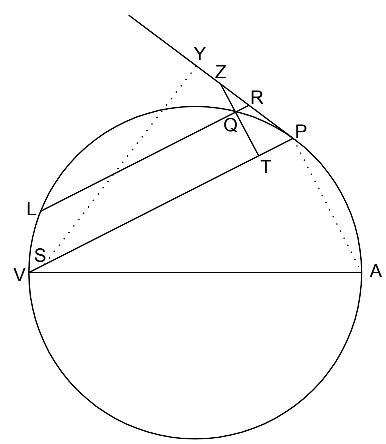
But all those terms are constants for any given point P, and hence the centripetal force at P is not just *ultimately* as the reciprocal of that ratio, but simply as the reciprocal of that ratio. And since AV is the same for all points, we can say that at any point P, the force is as

$$\frac{1}{SP^2 \cdot PV^3}$$

Q1: "Actually" or "ultimately"? (Actually.)
Q2: Why can we replace RL with PV, but leave QR in there? Why don't we have to put in a "zero" for QR? (Really we are replacing the ratio RL/AV with its ultimate form, i.e. ratio PV/AV ; but there is no ultimate value of QR/AV, since QR goes to zero and AV is fixed. So we cannot replace that ratio, but must find others to replace. The ultimate ratio is not the same, necessarily, as the ratio of the ultimate magnitudes, as we have seen many times.)
Q3: How does Prop. 7 differ from 6 and its corollaries? (It is circle-specific.)

"IDEM ALITER"

Newton proves "the same thing another way." Drop SY perpendicular to the tangent PZ. Now



$\angle$ZPT = $\angle$PAV [Euc. 3.32]
so $\angle$YPS = $\angle$PAV
so $\triangle$SYP and $\triangle$VPA are similar
so AV : PV = SP : SY
so SP·PV = AV·SY

so $$\frac{SP \cdot PV}{AV} = SY$$

so $$\frac{SP^2 \cdot PV^2}{AV^2} = SY^2$$

so $$\frac{SP^2 \cdot PV^3}{AV^2} = SY^2 \cdot PV$$

But (by Cors. 3 and 5 to Prop. 6), the centripetal force is reciprocally as ($SY^2 \cdot PV$), and therefore (as before), in our present case, it is reciprocally as

$$\frac{PV^3 \cdot SP^2}{AV^2}$$

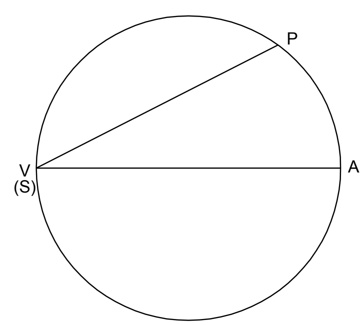
And since AV never changes, once again the force is as $$\frac{1}{SP^2 \cdot PV^3}$$

i.e. $$F_P : F_p = \frac{1}{SP^2 \cdot PV^3} : \frac{1}{Sp^2 \cdot pv^3}$$

Q.E.I.

COROLLARY 1.

So, if S happens to be on the circumference, for example at V, then since



$$F_P \propto \frac{1}{SP^2 \cdot PV^3}$$

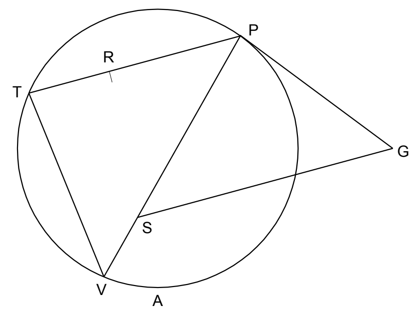
and since SP = PV, thus

$$F_P \propto \frac{1}{SP^2 \cdot SP^3}$$

or $$F_P \propto \frac{1}{SP^5}$$

Newton calls SP the "altitude," i.e. the height above the center, S.
NOTE: What happens when the body approaches V (S)? Then SP gets as small as you please, hence the inverse of SP^5 gets as large as you please, so the force at S, it seems, would be infinite. So perhaps this motion cannot be completed, and putting S out on the circumference is a limit case.

COROLLARY 2.



Let a body revolve on circle APTV around center of forces S (that is, sweeping out equal areas in equal times around S), and then let it revolve on the same circle in the **same periodic time** around center of forces R. What is the ratio of the forces to these two centers for a given point P?

Newton says if we draw PG tangent, and SG parallel to RP, then

$$F_{PR} : F_{PS} = SG^3 : SP \cdot RP^2$$

For $\quad F_{PR} \propto \dfrac{1}{RP^2 \cdot PT^3} \quad$ [Prop. 7]

and $\quad F_{PS} \propto \dfrac{1}{SP^2 \cdot PV^3} \quad$ [Prop. 7]

So $\quad F_{PR} : F_{PS} = SP^2PV^3 : RP^2PT^3 \quad$ [watching reciprocals]

So $\quad F_{PR} : F_{PS} = SP^3PV^3 : SP \cdot RP^2PT^3 \quad$ [$\times SP$]

Thus $\quad F_{PR} : F_{PS} = \dfrac{SP^3PV^3}{PT^3} : SP \cdot RP^2 \quad$ [$\div PT^3$]

But since $\quad \angle SPG = \angle PTV \quad$ [Euc. 3.32]

thus $\quad \triangle PSG$ and $\triangle TPV$ are similar, therefore

$$SP : SG = PT : PV$$

so $\quad SP^3 : SG^3 = PT^3 : PV^3$

so $\quad \dfrac{SP^3PV^3}{PT^3} = SG^3$

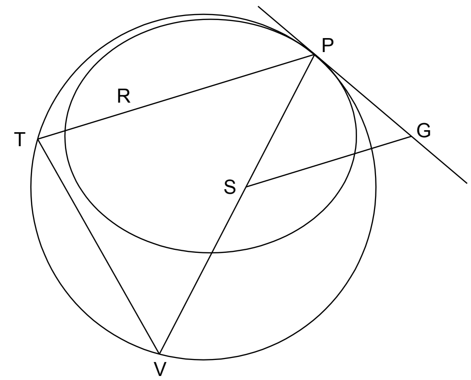
Hence $\quad F_{PR} : F_{PS} = SG^3 : SP \cdot RP^2$

Q.E.D.

Q1. Is this just one motion? No, it is two, with equal periods (given). It is not possible for the same body to sweep out areas as times around both R and S, as we saw after Prop. 2 in these notes.

Q2. How do the equal periods come into play? Prop. 7 shows the force is proportional to the reciprocal of RP^2PT^3 (or SP^2PV^3 for the other motion)—but what if P moves about R as center, now with one period, now again with a very much shorter one? Surely in the motion of shorter period, the force at P is greater. So, to compare the forces in any two motions around the same circle, we have to keep the period the same (or take it into account in some other way, introducing an adjusting ratio).

COROLLARY 3.

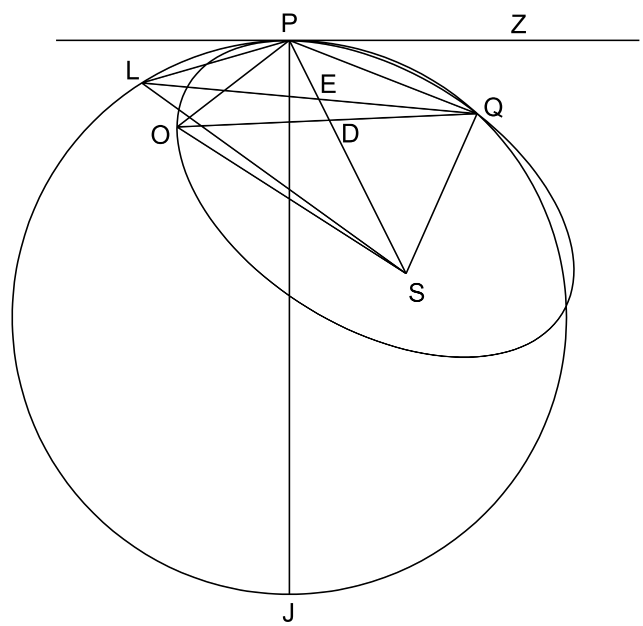


Newton generalizes Cor. 2 to ANY curve (of finite curvature, for example, any conic):

$$F_{PS} : F_{PR} = SP \cdot RP^2 : SG^3$$

For the forces in that orbit-shape at P are the same as in a circle of the same curvature, with the same centers of force—since the subtenses, hence sagittae, hence forces, are ultimately equal as we go to P in the circle and in the curve.

NOTE: This is used for the *Idem Aliter* in Prop. 11.



Q1: What defines the circle of curvature again? From P draw a perpendicular (PJ) to the orbit's tangent (PZ), take any point Q (near P) on the orbit, join PQ, and draw QJ perpendicular to PQ. Thus the circle on PJ as diameter is tangent to PZ at P, and cuts the orbit at Q. Let Q go to P. The final PJ is the diameter of the circle of curvature.

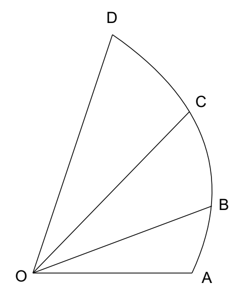
Q2: Why will the force at P in our circle of curvature (with the same period as in our orbit), toward S, be the same as in our orbit at P? Because in either case, the forces are ultimately as the sagittae. Let the time through arc PQ be the same as that through arc PL, for the circle, and time through arc PQ be the same as that through arc PO for the orbit. Let QO cut PS at D, and QL cut PS at E. Evidently, as Q goes to P, the "bend" QPL approaches the "bend" QPO, and hence the sagittae, too, which are a function of these "bends," approach equality. But the sagittae are as the forces at P. Hence, in the limit, when Q is at P, and we have the circle of curvature, the instantaneous "bend" is the same, hence the sagittae, hence the forces.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 38

PROPOSITION 9 (SPIRAL)

*Let a body rotate in a **spiral** PQS cutting all radii SP, SQ, etc. in a given angle; the law of centripetal force tending to the center of the spiral is required.*



An **arithmetic spiral** is one in which the radius grows arithmetically as it revolves, that is, if O is the center, and OA, OB, OC, OD are four radii in order, and ∠AOB = ∠COD, then OD – OC = OB – OA. But Newton is talking about a **geometric spiral**, in which, S being the center and SP, SQ, sp, sq are four radii in order, and ∠PSQ = ∠pSq, then SP : SQ = Sp : Sq. We know that he is talking about this sort of spiral because it is a property of such a curve that the angle formed by radius and tangent is everywhere the same (and this is how he describes his curve). To see this property, draw pr and PR tangent, and join PQ, pq.

If we move q to p and Q to P, always keeping ∠PSQ = ∠pSq,
then ∠Spq = ∠SPQ always, since △Spq is similar to △SPQ.
And so the limiting angles which these approach,
namely ∠Spr and ∠SPR
must also be equal. Hence the tangent always makes the same angle with the radius. (Note that a circle also has this property, being a degenerate case of arithmetic spiral.)

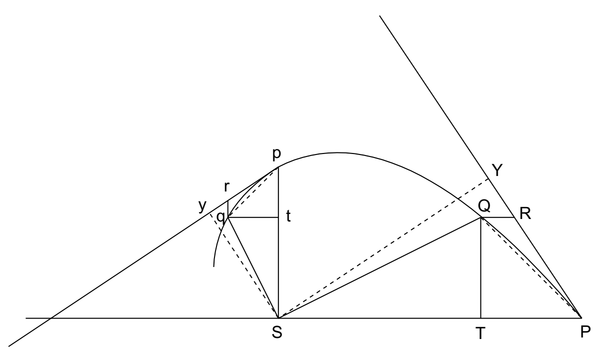
Now, to derive the force-law for this spiral toward its own center S as center of forces:

Since △SQP is similar to △Sqp always,

thus QT : qt = SP : Sp
and QT : QR = qt : qr

so $$\frac{QT}{QR} = \frac{qt}{qr}$$

so $$\frac{QT}{QR}QT \ : \ \frac{qt}{qr}qt \ = \ SP \ : \ Sp$$
[1st proportion, and equal coefficients]



so $$\frac{QT^2}{QR} : \frac{qt^2}{qr} = SP : Sp$$

so $$\frac{QR}{QT^2} : \frac{qr}{qt^2} = \frac{1}{SP} : \frac{1}{Sp}$$ [inverting]

so $$\frac{QR}{QT^2 SP^2} : \frac{qr}{qt^2 Sp^2} = \frac{1}{SP^3} : \frac{1}{Sp^3}$$ [× antecedents & consequents by inversions of SP^2, Sp^2]

so $$F_P : F_p = \frac{1}{SP^3} : \frac{1}{Sp^3}$$

Q.E.D.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 39

LEMMA 12

Here Newton adds another Lemma to the preceding 11 Lemmas. It is again a purely mathematical truth, about conjugate diameters of an ellipse or of an hyperbola. Lemma 12 states:

> *Parallelograms described about any conjugate diameters of a single ellipse (or a single hyperbola) are all equal among themselves.*

It is easy to think this means that "All parallelograms circumscribed about a single ellipse are equal." But that is false. The truth is that, for a given ellipse, we can circumscribe a parallelogram about it with an area as large as you like. To see this, just pick a point P extremely far away from your ellipse, and from P draw the two tangents to the ellipse, forming a very large tangential triangle. Draw the other pair of tangents parallel to these two you have drawn, and you will thus complete a parallelogram, and you have a whole lot of area, there. If you now draw P even further away, you contain that original triangle and add still more area. So it is important to point out that Lemma 12 is about the tangential parallelograms which are circumscribed about *conjugate diameters.*

Perhaps a brief review of some basics about central conics is in order.

WHAT are conjugate diameters? Recall that in any central conic (a conic section that has a center, namely either an ellipse or a hyperbola), any straight line through the center is a diameter, that is, it bisects ordinates drawn in the section parallel to the tangents at the ends of the diameter. (And those lines are said to be drawn *ordinatewise* to the diameter when it bisects them, and they are parallel to the tangent; also, a diameter bisects *only* those straight lines in the section which are ordinatewise to that specific diameter.) But each diameter has a special diameter as its partner, namely the one drawn ordinatewise to itself.

The kind of parallelogram we are concerned with in Lemma 12 is one whose opposite sides are not only tangent to the ellipse (or hyperbola), but which are also drawn parallel to a pair of conjugate diameters.

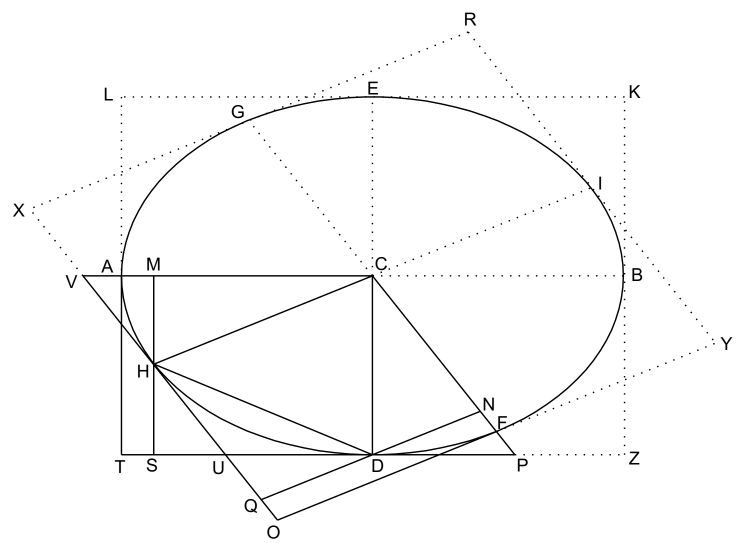
Can such parallelograms about conjugate diameters in an ellipse get as skinny as you like? (No, there is a limit, a most-skinny parallelogram.) In a hyperbola? (Yes, they can get as skinny as you like.)

Are conjugate diameters ever equal in an ellipse? (Yes, but they are not at right angles, but are the portions of the diagonals of the axial rectangle which are cut off within the ellipse.) In an hyperbola?

(They are the asymptotes! Although in a right hyperbola all conjugate diameters are equal, that is, AB = ED, and HI = GF, in the hyperbola figure below.)

The claim, now, is that the area of all such parallelograms about a given ellipse is the same.

Since Newton offers no proof for this amazing theorem, I will present one here. I should credit Robert S. Bart, once a teacher at St. John's College, for the proof, since I first learned it from his excellent notes on Newton's *Principia*, and also since I will be using his lettering in my figure. The proof itself, however, goes back to Apollonius I believe, who proved the theorem in his *Conics*, Book 7, Proposition 31.



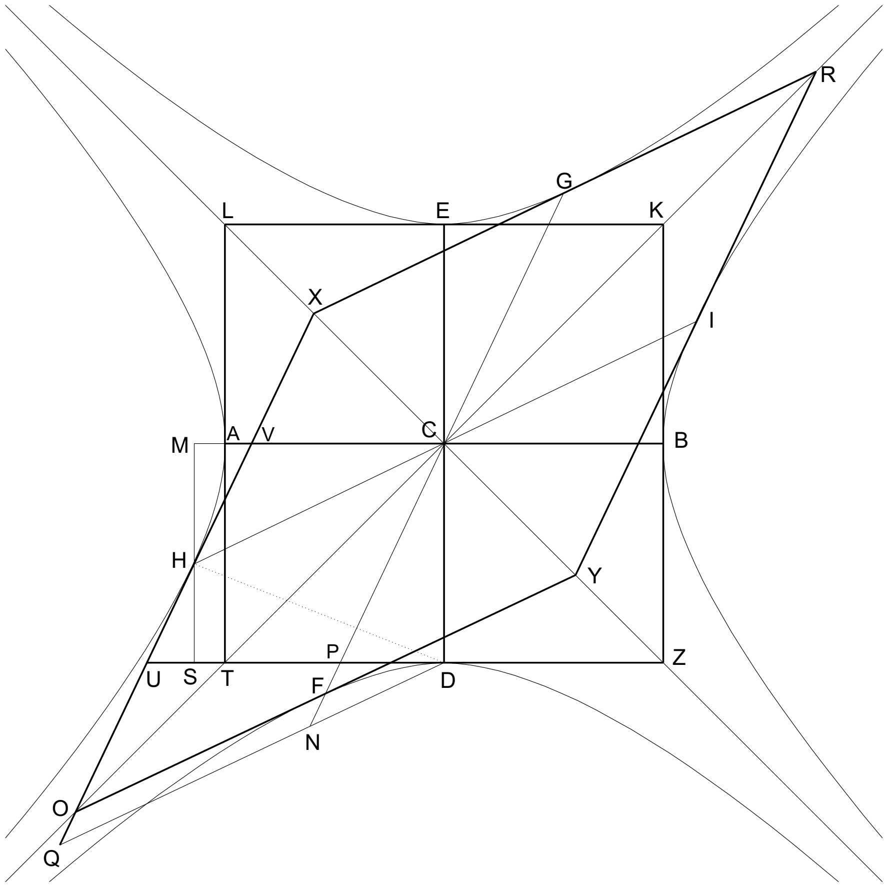
Given any ellipse (or hyperbola), center C, choose any two pairs of conjugate diameters, HI and GF, AB and DE, complete the parallelograms described about these, LKZT and RYOX.

I say that LKZT and RYOX are equal.

Draw DN parallel to HC, through to Q on XO (on XO extended, for the hyperbola).
Draw HM parallel to CD, through to S on ZT (on ZT extended, for the hyperbola).

1. Now	CV : CA = CA : CM	[Ap. 1.37; HV tangent, HM ordinate, CA ½ diameter]
2. So	CVUP : CATD = CATD : CMSD	[Euc. 6.1, parallelograms as bases]
3. or	**CVUP : CATD = CATD : CNQH**	[CMSD = 2$\triangle$CHD = CNQH]
4. again	CP : CF = CF : CN	[Ap. 1.37; DP tangent, DN ordinate, CF ½ diameter]
5. thus	**CPUV : CFOH = CFOH : CNQH**	[Euc. 6.1, parallelograms as bases]
6. therefore	CATD = CFOH	[means between same extremes; Steps 3 & 5]
7. thus	LKZT = RYOX	[i.e. 4*CATD* = 4*CFOH*]

Q.E.D.



Here I repeat the very same argument for the hyperbola, just to accompany the diagram:

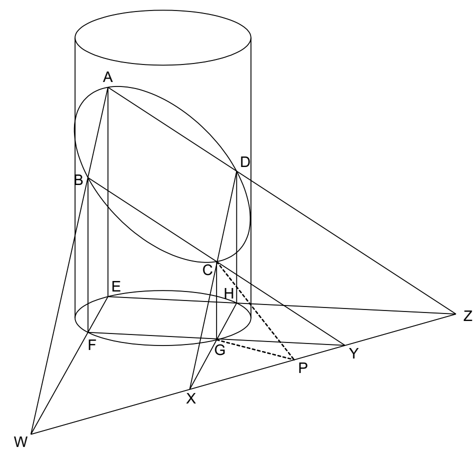
1. CV : CA = CA : CM [Ap. 1.37; HV tangent, HM ordinate, CA ½ diameter]
2. CVUP : CATD = CATD : CMSD [Euc. 6.1, parallelograms as bases]
3. **CVUP : CATD = CATD : CNQH** [CMSD = 2△CHD = CNQH]
4. CP : CF = CF : CN [Ap. 1.37; DP tangent, DN ordinate, CF ½ diameter]
5. **CPUV : CFOH = CFOH : CNQH** [Euc. 6.1, parallelograms as bases]
6. CATD = CFOH [means between same extremes; Steps 3 & 5]
7. LKZT = RYOX [i.e. 4CATD = 4CFOH]

Q.E.D.

Here I will add an alternative method of proof for the case of the ellipse which I developed when someone complained to me that the proof above is too complex. I'm not sure this one is any simpler, but it is quite different, and is in some ways rather interesting.

The conjugate parallelograms inscribed in an ellipse all have the same area.

Let the ellipse be in a right cylinder, and let any random point C be taken on it. Let AC be the diameter through C, and BD the conjugate diameter. I say that the area of parallelogram ABCD is a constant for all points C chosen along the ellipse.



On the cylindrical surface, let the straight lines AE, BF, CG, DH be drawn down to the base circle, forming quadrilateral EFGH. And let WXYZ be the straight line in which the plane of the ellipse intersects the base plane; and let AB and EF meet at W, DC and GH at X, BC and FG at Y, AD and EH at Z.

Draw GP at right angles to XY; join CP. Since CG is at right angles to the base plane, therefore CP is also at right angles to XY.

Also, since AC is a diameter, and passes through the cylinder's axis, therefore EG does as well, and so EG is a diameter of the circle. Likewise FH is a diameter of the circle. And since BD is parallel to the ordinates to AC, so too FH is parallel to the ordinates to EG; which means that EG and FH are at right angles to each other, and hence EFGH is the square inscribed in the base circle.

Now,	$\triangle ABC : \triangle WBC = AB : BW$	[Euc. 6.1]
and	$\triangle EFG : \triangle WFG = EF : FW$	[Euc. 6.1]
but	$\underline{AB : BW = EF : FW}$	
so	$\triangle ABC : \triangle WBC = \triangle EFG : \triangle WFG$	
or	$\triangle ABC : \triangle EFG = \triangle WBC : \triangle WFG$	[alt.]
so	$ABCD : EFGH = \triangle WBY : \triangle WFY$	[doubles; BC : BY = FG : FY]
so	$ABCD : EFGH = \triangle XCY : \triangle XGY$	
so	$ABCD : EFGH = PC : PG$	[triangles on same base as heights]

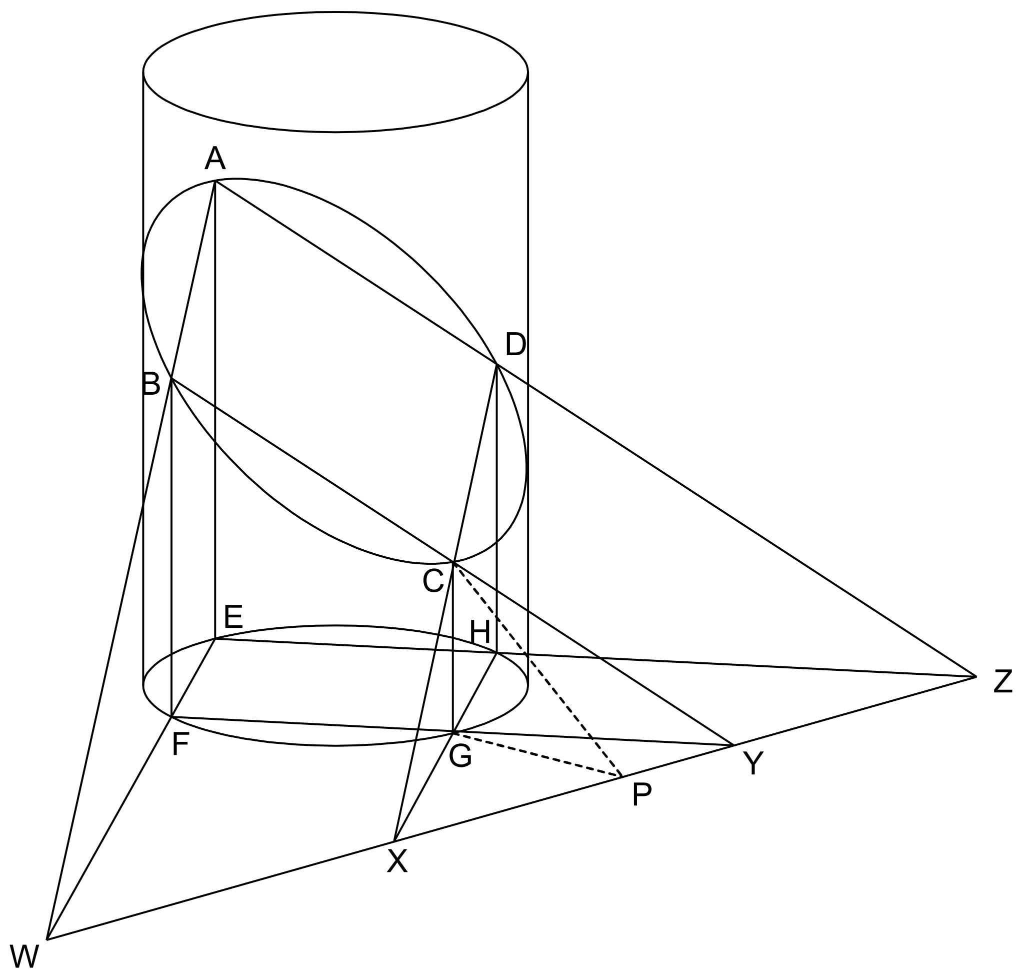
But the ratio PC : PG is fixed regardless of where C is taken on the ellipse, since it is based on the inclination of the cutting plane to the base plane.

Hence the ratio ABCD : EFGH is fixed for all C.

But the area of EFGH is also fixed for all C.

Therefore the area of ABCD is fixed for all C.

Q.E.D.


A
D
B
C
E
H
Z
F
G
Y
P
X
W

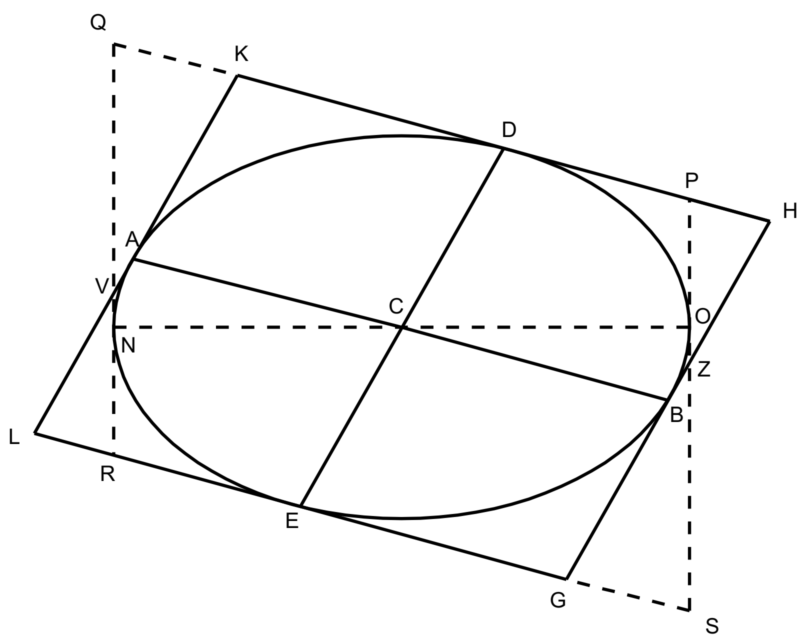
One more point about Lemma 12 before we move on. If we call the parallelograms we have been describing the "parallelograms described about conjugate diameters," we can add another geometrical theorem: *The area of parallelograms described about conjugate diameters of an ellipse is the least of all parallelograms drawn tangentially about it.* The proof is easy.

Given: Ellipse, center C, conjugate diameters AB, DE
and parallelogram GHKL circumscribed tangent to it at A, E, B, D
and NCO any other diameter of the ellipse
and parallelogram PQRS circumscribed tangent at O, D, N, E

Prove: Parallelogram PQRS is greater than parallelogram GHKL

Well,	GB = BH
so	GZ > ZH
so	△GZS > △ZPH
Similarly	△QVK > △VRL
So	△GZS + △QVK + VRGZPK > △ZPH + △VRL + VRGZPK
i.e.	prlgm. PQRS > prlgrm. GHKL

Q.E.D.



THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 40

PROPOSITION 10

Let a body rotate on an ellipse. The law of centripetal force tending to the center of the ellipse is required.

Preliminary questions:

(1) What does "law of centripetal force" mean? (A formula for computing the strength of the force at any point on the ellipse, in terms of quantities determined by the geometry of the ellipse and the geometry of the location of the center of forces. That is, a way of determining the strength of the force in terms of distance from the center. NOTE: This gives us only *relative* quantity of force so far.)

(2) Does this Proposition apply to all motion on an ellipse? That is, is it true to say, "If a body rotate on an ellipse, the body is drawn to the center by a centripetal force, and that force is directly as the distance from the center"? (NO. Only if C is the center of forces, i.e. only if the body sweeps out equal areas in equal times around C.)

(3) Does any such motion happen in nature? (Well, there is Hooke's Law about springs, and the laws about transverse waves; and there is also Props. 70-73 at the end of Book 1 of the *Principia*, where Newton shows that if the matter of a sphere attracts according to an inverse-square law, then the matter inside is attracted to the center according to a DIRECT-FORCE law! But that is derivative, i.e. a net-effect of another, more basic force, and not itself a basic, natural force rule. So probably there is no natural and elementary direct-force law. But some non-elementary forces increase with distance generally, e.g. a rubber-band being stretched from a fixed point [and the forces bonding quarks, too, apparently]. So maybe this motion can be produced by a fellow with a bungee cord and roller skates. But we have a kind of COROLLARY to Prop. 10 thus: Any force which **decreases** with distance from its source cannot produce this kind of motion, i.e. elliptical with the center of forces at the geometrical center. For instance, if a magnet were at C, that could not do it.)

(4) Then what is the point? (To show the power of his method and principles—but more specifically, this case is interesting, since it is the simplest law of centripetal force, i.e. where the Force is directly as the Distance from the center. This becomes clear in Cor. 1, i.e. that we have, in a way, been looking to answer the question *What sort of orbit do we get by the simplest conceivable law for centripetal force?* Answer: an ellipse, whose center is the center of forces. Also, it is used in Prop. 11, the *Idem Aliter*. Also, it has a simple n-body solution, as we find in Prop. 64. And we do need it for Props. 70–73, where we will forestall objections about moving from points to spheres.)

(5) Does the proposition make intuitive sense? Where is the **velocity** greatest? Least? (Greatest at B, least at A, by taking the perpendiculars from C to the tangents at B and A inversely, which gives the ratio of the velocities.) Since the velocity is greatest at B,

shouldn't the **force** be greatest there? (Forces are not as velocities; they are as changes in velocity. Forces are not causes of velocities—confer Law 1; you can move very fast, without any force—but forces are causes of **changes** in velocity. And the sharper curvature at A shows that the change in velocity is more dramatic there. If we draw equal areas from C around points B and A, and shrink these, we have a COROLLARY: The sagittae will be ultimately in the same ratio as BC and CA, since these are as the forces, and the sagittae are ultimately as the forces. This is really a purely geometrical fact, it seems: the equal areas, and the sagittae being more and more in the ratio of BC and CA themselves, is all geometry.)

(6) Do you get a specific ellipse by this law of force? (No.) How do you get different ellipses if the law of force is the same for all ellipses? (From different initial velocities in the body, and different initial distances from the center C.)

(7) How would you get a circle? (See Prop. 4.)

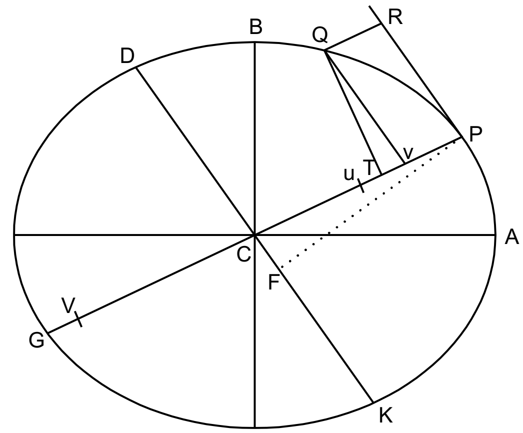
(8) How would you get similar ellipses?

(9) Would mass affect the shape of the ellipse?

(10) Is everyone aware of the "ordinate-squares are as diameter-rectangles" property in central conics? Are the ord-squares ever equal to the diam-rectangles in an ellipse? (Yes, in the special case where the conjugate diameters are equal, and hence are not at right angles.)

(11) What do you think the law of force would be for a hyperbola, if the center of force were at its geometric center? (According to Prop. 12, Idem Aliter, the centripetal force *away from* the center would again be directly as the distance, so the rule by which the force magnitude varies is the same, but the force has been changed from a centripetal one to a centrifugal one.)

PROOF OF PROP. 10.



P is the point on the orbit we will examine.
CA, CB are the semi-axes.
GP is a diameter, DK its conjugate.
PF is perpendicular to DK.
QT is perpendicular to GP.
Qv is ordinatewise to GP.
Parallelogram QvPR is completed (so PR is tangent).

Now, the ordinate-squares are as the diameter-rectangles,

thus	$Pv \cdot vG : Qv^2 = PC^2 : CD^2$	[PC^2 is the diam-rect PCxCG]
but	$\underline{Qv^2 : QT^2 = PC^2 : PF^2}$	[△QvT is similar to △PCF]
so	$Pv \cdot vG : QT^2 = (PC^2 : CD^2)$ c $(PC^2 : PF^2)$	

which we get just by multiplying the ratios on the left, and also those on the right.

So $\qquad Pv \cdot vG : QT^2 = PC^2 \cdot PC^2 : CD^2 \cdot PF^2$

that is, dividing the left ratio by Pv, and dividing the right ratio by PC^2,

$$vG : \frac{QT^2}{Pv} = PC^2 : \frac{CD^2 \cdot PF^2}{PC^2}$$

But $\qquad QR = Pv$

and $\qquad BC \cdot CA = CD \cdot PF \qquad$ [Lemma 12]

hence $\qquad CD^2 \cdot PF^2 = BC^2 \cdot CA^2$

and $\qquad vG \ \underline{U}\ 2PC \qquad$ [as Q goes to P]

so $$2PC : \frac{QT^2}{QR} = PC^2 : \frac{BC^2 \cdot CA^2}{PC^2} \quad \text{[ultimately]}$$

Therefore, cross-multiplying,

$$\frac{QT^2 \cdot PC^2}{QR} = \frac{2BC^2 \cdot CA^2}{PC} \qquad \text{[ultimately]}$$

But (by Prop. 6, Cor. 5), centripetal force is as the ultimate form of the reciprocal of $\dfrac{QT^2 \cdot PC^2}{QR}$

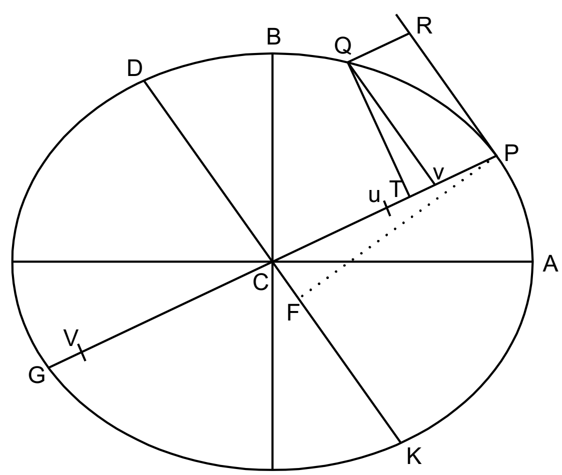
And therefore, in the present case, the centripetal force is ultimately as $\dfrac{1}{2BC^2 \cdot CA^2/PC}$

But $2BC^2 \cdot CA^2$ is fixed, so that the centripetal force is ultimately as $\dfrac{1}{1/PC}$

i.e. the centripetal force is ultimately as PC.

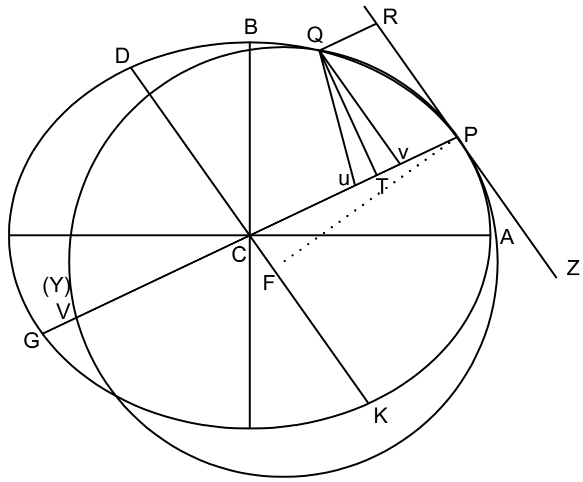
And since PC is itself fixed, it follows that the centripetal force at P is actually as PC.

Q.E.I.



Newton now gives us an “idem aliter,” that is, “the same thing another way.” This “idem aliter” or alternative proof is rather tedious, and not strictly necessary, but it is nice to see that the same result is reached by other means. That adds to our certainty.

IDEM ALITER



Make $Tu = Tv$, on the other side of T, and take uV so that

	$uV : vG = DC^2 : PC^2$	[construction]
but	$\underline{Qv^2 : Pv \cdot vG = DC^2 : PC^2}$	[alt. of ord-squares and diam-rects]
so	$Qv^2 : Pv \cdot vG = uV : vG$	

Cross-multiplying, and then dividing both sides by vG, we have

$$Qv^2 = Pv \cdot uV$$

Now we add the rectangle $uP \cdot Pv$ to both sides, and we have

$$Qv^2 + uP \cdot Pv = Pv\,[uV + uP]$$

But since	$uT = Tv$	
thus	$uP \cdot PV = PT^2 - vT^2$	[Euc. 2.6]
thus	$Qv^2 + (PT^2 - vT^2) = Pv\,[uV + uP]$	
or	$Qv^2 - vT^2 + PT^2 = VP \cdot Pv$	
i.e.	$QT^2 + PT^2 = VP \cdot Pv$	[right triangle QvT]
i.e.	$QP^2 = VP \cdot Pv$	[right triangle QPT]

Now construct the circle through P and Q tangent at P to the ellipse and its tangent ZPR; let its intersection with PC be Y.

Now	$\angle vQP = \angle QPR$	[parallels]
and	$\underline{\angle QPR = \angle QYP}$	[Euc. 3.32]
so	$\angle vQP = \angle QYP$	
and	$\angle QPY$ is common to $\triangle vQP$ and $\triangle QPY$	

hence these triangles are similar,

thus	$Pv : PQ = PQ : PY$	
so	$QP^2 = PY \cdot Pv$	

but	$\underline{QP^2 = VP \cdot Pv}$	[just proved]
so	$PY = PV$	

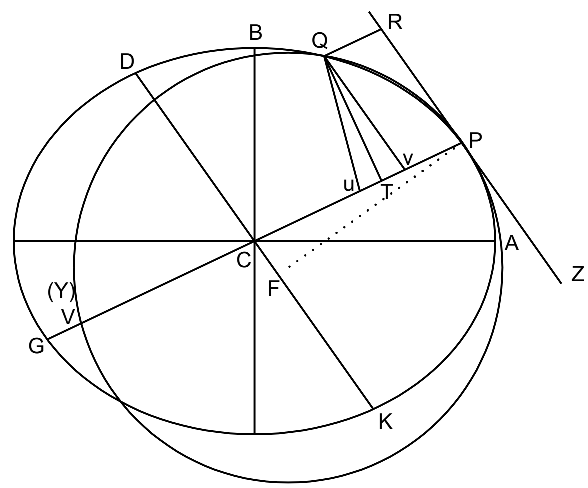
i.e. our circle, drawn tangent to ZPR and the ellipse at P, and passing through Q, also passes always through point V.

But as Q goes to P,

	$Pv = QR$	[always]
and	$PQ = \text{arc}PQ$	[ultimately]
and	$QP^2 = VP \cdot Pv$	[proved]
thus	$(\text{arc}PQ)^2 = VP \cdot QR$	[ultimately]
i.e.	$PV = (\text{arc}PQ)^2/QR$	[ultimately]

i.e. the ultimate length of PV is the chord of the circle of curvature at P through C, and hence we may now apply Prop. 6 Cor. 3. (Think back to Lemma 11, where we kept drawing the cutting circle until the two points coincided at the point of tangency, and AG became AJ, the diameter of the circle of curvature; the only difference is that there we were using the property of the circle at its diameter, whereas here we are using similar triangles which are not right triangles.)

Now, as Q goes to P, u also goes to P, hence uV becomes PV ultimately.
And as Q goes to P, v also goes to P, hence vG becomes PG ultimately.



So	$uV : vG = PV : PG$ [ultimately]	
i.e.	$uV : vG = PV : 2PC$	[ultimately]
But	$\underline{uV : vG = DC^2 : PC^2}$	[construction]
thus	$DC^2 : PC^2 = PV : 2PC$	[ultimately]
So	$PV \cdot PC^2 = 2DC^2 \cdot PC$	[ultimately]
so	$PV = \dfrac{2DC^2}{PC}$	[ult]
so	$PV \cdot PF^2 = \dfrac{2DC^2}{PC} PF^2$	[ult]

But, since PV is a chord in the circle of curvature for point P, and that chord is drawn through the center of forces, therefore, by Prop. 6 Cor. 3,

$$F_P \propto \frac{1}{PV \cdot PF^2}$$

(Note: PF is the distance from C to the tangent RP)

Hence
$$F_P \propto \frac{1}{\frac{2DC^2}{PC} \cdot PF^2}$$

But since $2DC^2 \cdot PF^2$ is a constant for all points P (by Lemma 12, $2DC \cdot PF$ is the same for all points, i.e. the conjugate parallelogram is always the same area, hence $2DC \cdot PF^2$ is also a constant),

thus
$$F_P \propto \frac{1}{\frac{1}{PC}}$$

So that the force at P is directly as PC.

Q.E.I.

NOTE: Why bother with an *idem aliter* if it is so much more tedious? Recall that calculus is new, and it looks fishy to many people at first—as though we can get any result we like. So he wants to drive home that, no matter what fluid quantities we consider the ratio of forces to be a limit of, we get the same results.

NOTE: Newton ends this proposition ***Q.E.I.***, since it is a type of problem, namely to find something given certain conditions. The initials stand for *Quod Erat Inveniendum*, or "What was to be found."

THE NEWTONIAN ART OF CLASSICAL PHYSICS

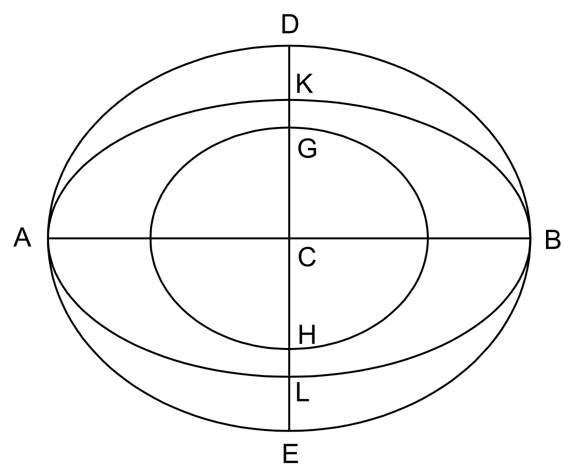
CLASS 41

PROPOSITION 10 COROLLARIES AND SCHOLIUM

COROLLARY 1.

Prop. 10 proved that the force in our ellipse, if its geometric center is the center of forces, is directly as the distance from the center of the ellipse. Corollary 1 states that *conversely* if the orbit is produced by a centripetal force toward C which varies in the same ratio as the distance from C, the orbit is an ellipse and C is its geometric center. Prop. 10 proceeded by convertible properties of ellipses. The specific ellipse you get will depend upon the magnitude of the centripetal force and also the initial position and velocity of the body.

COROLLARY 2.



This corollary states (I'm paraphrasing, here) that *If the same point is both the geometric center and the center of forces for TWO ellipses, then the two elliptical orbits will have the same periodic times.*

QUESTION: Newton states it thus, i.e. absolutely, as though there were no other condition than that we have two ellipses sharing the same point C as geometric center and force center. Does it really follow that their periods must be equal under those givens alone? If so, there is only ONE TIME, one period, for all bodies that move on concentric ellipses (sweeping out areas equably around the geometric center), entirely determined by the geometry. But watch the motion once—now use a slow-motion camera to stretch out the time to double. In the slow-motion film, the body is still moving on the same ellipse, and sweeping out equal areas in equal times around C, only slower, with double the period. So can't we have bodies moving on ellipses around C, and sweeping out equal areas in equal times around it, but at unequal paces, i.e. with unequal periods? So isn't Newton wrong?

Of course not—he is just speaking tersely. We are also given (as his use of Prop. 4 Cor. 3 makes clear) that the forces in our *two* ellipses, at any two points, are as their distances from the common center. Not only is there an as-the-distance force law at work in each ellipse, but it works between ellipses, too. Put otherwise, we are given that *the two elliptical orbits are produced by the same cause (or equal causes) of centripetal force toward their common geometric center*.

The main argument can be outlined like this:

(STEP 1)

If the concentric ellipses be SIMILAR, then they will have the same periodic times.

(STEP 2)

If the concentric ellipses SHARE A MAJOR AXIS, then they will have the same periodic times.

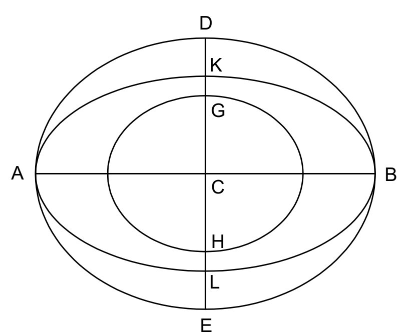
(STEP 3)

But then, given ANY two (physically and geometrically) concentric ellipses, their periodic times will be equal. Let the minor axes be GH, KL, let the center be C. Let AB be the major axis of KL. On AB as major axis, describe the ellipse DE which is similar to ellipse GH. Now

	Period (GH) = Period (DE)	[by Step 1, since they're similar]
	Period (KL) = Period (DE)	[by Step 2, due to the common major]
so	Period (KL) = Period (GH)	

And therefore any two elliptical orbits which are both geometrically and physically concentric have equal periods.

To see the force of the argument, then, it only remains to prove Step 1 and Step 2.



STEP 1

Given two similar ellipses which are both physically and geometrically concentric, their periods must be equal.

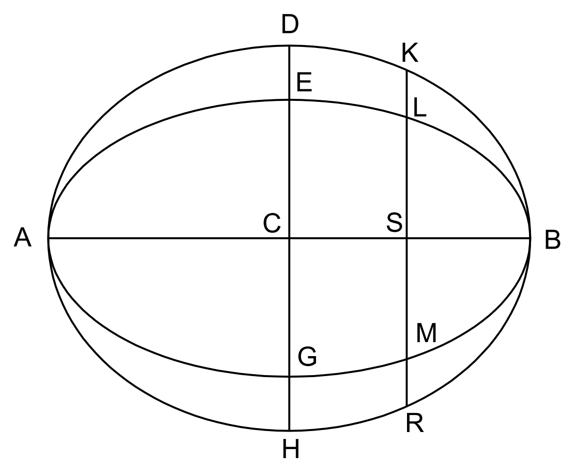
For, by Prop. 10 itself, the forces in one ellipse are as the distances from the center of force. But by their similarity, the distances in one are as the corresponding distances in the other. Therefore the force in one ellipse is to the force in the other (at a similar point) as the distance to the distance. But by Prop. 4 Cor. 8, all things said about circles apply to similar figures with similarly placed centers of force, so long as we realize the bodies are sweeping out AREAS uniformly, not arcs (and so long as we take corresponding distances in place of circular radii). Hence the “Prop. 4 Cor. 8 version of Prop. 4 Cor. 3” applies here, and that Cor. said that when the forces in our two figures are as the radii (i.e., in this case, as the similar distances), the periodic times are equal. But the forces are as the similar distances. Therefore the periodic times are equal.

Q.E.D.

NOTE: Nothing in this argument requires the similar ellipses to have coincident axes. They could be oriented quite differently about C.

STEP 2

Given two ellipses sharing a common major axis and which are both physically and geometrically concentric, their periods must be equal.



For, let AB be the common major, C the common center, let EG, DH be the minor axes, KLSMR another ordinate parallel to the minors.

Now	$CG^2 : SM^2 = AC \cdot CB : AS \cdot SB$
but	$\underline{CH^2 : SR^2 = AC \cdot CB : AS \cdot SB}$
so	$CG : SM = CH : SR$
or	$CG : CH = SM : SR$
or	$2CG : 2CH = 2SM : 2SR$
i.e.	$EG : DH = LM : KR$

So all such ordinates cutting through the ellipses at a common point on the major axis are as the minors.

Drawing in skinny little rectangles on such ordinates, it is manifest that the areas of the two ellipses must be in the same ratio as the minors (cf. Lemma 4):

$$EG : DH = \text{area of ellipse } EG : \text{area of ellipse } DH$$

Now, the “period” of each ellipse is the time it takes to sweep out its entire area. And just as a uniform velocity is defined by distance per time, *and thus the time is equal to the distance divided by the velocity*, so here the periods (times) will be as the areas covered, divided by the area-velocities (i.e. areas-per-time):

$$P_{EG} : P_{DH} = \frac{area\ EG}{area\ velocity\ EG} : \frac{area\ DH}{area\ velocity\ DH}$$

And just as for uniform length-velocities the velocities are as the distances covered in an equal time, so here our uniform area-velocities are as the areas covered in an equal time T. Thus

$$P_{EG} : P_{DH} = \frac{area\ EG}{part\ of\ EG\ swept\ out\ in\ T} : \frac{area\ DH}{part\ of\ DH\ swept\ out\ in\ T}$$

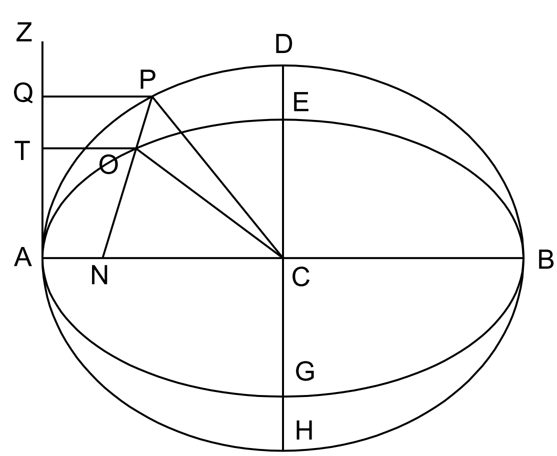
But we have seen that the areas are as EG : DH. Therefore:

$$P_{EG} : P_{DH} = \frac{EG}{part\ of\ EG\ swept\ out\ in\ T} : \frac{DH}{part\ of\ DH\ swept\ out\ in\ T}$$

But the parts of their areas swept out in time T are as the instantaneous arc-velocities of the bodies at the principal vertices. To see this, let sector ACO be described in ellipse EG during time T, and in that same time, from the same principal vertex A, let sector ACP be described in ellipse DH. Join PO, and extend it to N on major axis AB. Draw AZ tangent, and drop subtenses OT and PQ perpendicular to it. Now, these subtenses are ultimately *as the forces at A*, since these are as the sagittae of the arcs (Prop. 1, Cor. 4). But since the distance CA is the same in each orbit, and the rule of centripetal force is the same, therefore the forces are equal at that point. Therefore the subtenses PQ, OT are ultimately equal—from which it follows that PON is ultimately parallel to the tangent, and is therefore ultimately an ordinate. Therefore,

$$ON : PN \ \underline{U}\ EC : DC$$

or $$ON : PN \ \underline{U}\ EG : DH$$



But sector ACO : sector ACP $\underline{U}$ $\triangle$ACO : $\triangle$ACP
and $\triangle$ACO : $\triangle$ACP $\underline{U}$ AO : AP

since those two triangles approach having a common height AC.

So sector ACO : sector ACP $\underline{U}$ AO : AP
but AO : AP $\underline{U}$ ON : PN

since the chords are ultimately equal to the sines or tangents (Lemma 7).

So sector ACO : sector ACP $\underline{U}$ ON : PN
But $\underline{\text{ON : PN U EG : DH}}$
so sector ACO : sector ACP $\underline{U}$ EG : DH

But the sectors have a constant ratio, since the times are equal (and the area-description is uniform in each figure), and EG : DH is a fixed ratio. Therefore the sectors are *actually* in that ratio. That is, the areas swept out in each figure in the same time T are as the minors EG : DH.

But we said

$$P_{EG} : P_{DH} = \frac{EG}{\textit{part of EG swept out in } T} : \frac{DH}{\textit{part of DH swept out in } T}$$

Therefore

$$P_{EG} : P_{DH} = \frac{EG}{EG} : \frac{DH}{DH}$$

i.e. the periods are equal.

Q.E.D.

SCHOLIUM

PART 1.

An ellipse, if we stretch its center away toward infinity (picture it tilting in a cone), is becoming a parabola. And therefore if a body sweeps out equal areas in equal times around C, and C becomes infinitely distant, in the limit, the force, which is always as the distance, becomes constant. So the limit is a parabola on which the body is always being acted on by the same force in the same direction—the case of Galileo (who treated gravity as a constant force). But if we keep tilting and get a **hyperbola**, the body continues to sweep out equal areas in equal times, and have the forces as the distances, but the force is obviously **centrifugal**. (Not a natural motion, perhaps, but a natural question.)

NOTE: Newton therefore considers Prop. 10 not only because it is the simplest rule of force in itself (although we know of no such case in nature), but also because, by this Scholium, we can see how to understand Galileo as a limit-case of Newton.

PART 2.

By Corollaries 1 and 2 (or perhaps by their converses) we saw that all elliptical orbits (and circular ones, of course) with a common geometric center as their center of forces and having equal periods must also have forces as the distances from the center.

And the basis for this is that they differ from each other by having ordinates, on the same abscissas, which are in a given ratio (or even the same ordinates, but at a different angle to the same diameter). Therefore Newton generalizes, now: In *any* figures with these specifications, i.e. where they differ solely by having ordinates augmented or diminished in a given ratio (or the angle of the ordinates changed, but kept in a given ratio, as we can take the ordinates of a circle, and put them not at 90°, but at 60° to the diameter, and get an ellipse), and have equal periods, the forces in them toward a common center of forces will be as the distances from the center.

THE NEWTONIAN ART OF CLASSICAL PHYSICS

CLASS 42

SECTION 3
On the Motion of Bodies in Eccentric Conic Sections.

PROPOSITION 11

TITLE. "Section Three: The Motions of Bodies in ECCENTRIC Conic Sections." We already did the ellipse and circle in Props 4, 7, 10, right? So what is new? We are putting the center of force off the geometric center and at the focus. Although he has no justification for saying so yet, it might even say "On the *natural* motions of bodies in conic sections," because that is what is special about these, i.e. they occur in nature.

FOCI. These have other physical properties that are very interesting, e.g. in optics and acoustics, and wave theory in general. Note: Newton's Latin word for "focus" is UMBILICUS, which means "navel." Very cute.

APOLLONIUS. In his book *On Conic Sections*, Book 1, Apollonius derives the principal properties of the conics, and names them, in the following order:

Prop 11 = PARABOLA
Prop 12 = HYPERBOLA
Prop 13 = ELLIPSE

NEWTON. In this book *Principia Mathematica Philosophiae Naturalis*, Newton here derives the Laws of Force for bodies moving on conics and sweeping out equal areas in equal times around the focus, in the following order:

Prop 11 = ELLIPSE
Prop 12 = HYPERBOLA
Prop 13 = PARABOLA

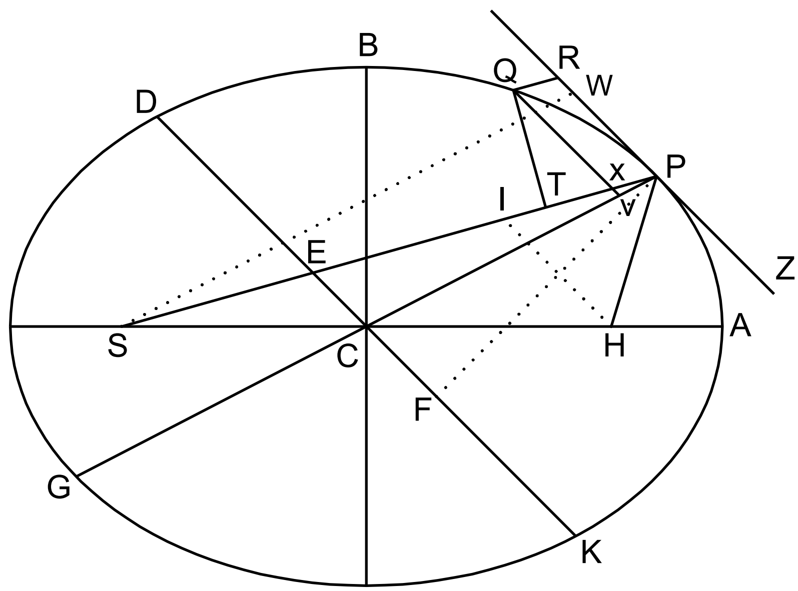
Surely we are meant to see the same numbering. Why the reverse order? Because in mathematics, the parabola is the simplest, having the simplest rule—and there is only one kind, as with the circle. The others are like "excess" (hyperbola) and "defect" (the ellipse), vs. "equal," and so they vary infinitely in kind—and so we define "parabola" first, then "hyperbola," then "ellipse," just as we define "right angle" first (which is defined by equality), then "obtuse" (excess), then "acute" (defect).

But in natural philosophy, the ellipse is the most natural and complete of the conics—the open curves are motions never finished, and therefore, before long, they encounter other bodies and get deflected from such figures. So we start with the ellipse. Also, the ellipse was the first non-circular celestial motion discovered, namely by Kepler. Also the parabola is a "lucky" case, like the special case of the circle, which is rare, or non-existent. So the parabola is almost like a special case of the hyperbola, and in natural philosophy we follow the order of the general to the particular. (A parabola is also like a special case of ellipse; one whose center is infinitely distant.)

PROPOSITION 11

Let a body revolve on an ellipse. The law of centripetal force tending to a focus of the ellipse is required.

NOTE: We are given, then, that the body sweeps out equal areas in equal times around one focus, S, (the Sun).



Let H be the other focus.
Let C be the center.
AC is the semi-major, CB the semi-minor, L the principal parameter (i.e. for the major axis).
P is the place of the body (the Planet) on the orbit which we will consider.
Let SP, PH be joined.
Let PR be tangent.
Draw HI parallel to PR.
Draw RQ parallel to SP.
Draw QV parallel to PR, and so ordinatewise to CP, and cutting SP at x.
Draw DCK parallel to PR, and thus conjugate to PCG, and cutting SP at E.
Draw QT perpendicular to SP.
Draw PF perpendicular to DCK.

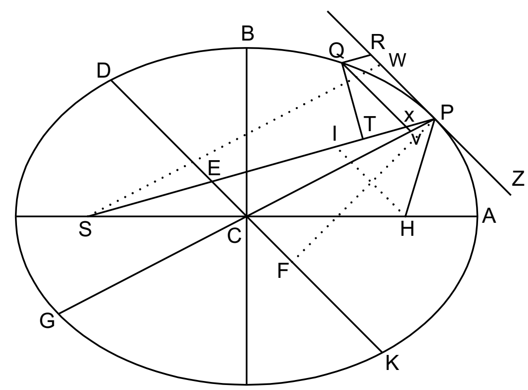
Newton assumes four LEMMAS in the course of his proof. Let's prove them first.

[LEMMA A] EP = AC

1) PI = PH, since △IPH is isosceles, since ∠RPS = ∠ZPH, since S, H are foci, and RPZ is tangent (Apoll. 3.48).
2) SE = EI, since HI is parallel to CE and SC = CH (foci are equidistant from the center)
3) SP = SE + EI + IP (whole is equal to sum of its parts)
4) SP + PH = SE + EI + IP + PH (+PH to both sides of Step 3)
5) SP + PH = EI + EI + IP + IP (since SE = EI, PI = PH, Steps 1 & 2)
6) SP + PH = 2EI + 2IP
7) SP + PH = 2EP
8) SP + PH = 2CA (focal property)
9) **EP = AC** (Steps 7 & 8)

[LEMMA B] Qx : QT = PE : PF

1) ∠FEP = ∠QxT (Qx parallel to PR, FE parallel to PR, so Qx parallel to FE)
2) ∠EFP = ∠xTQ (both 90°)
3) so △xQT is similar to △EPF (equiangular)
4) so **Qx : QT = PE : PF** (corresponding sides)

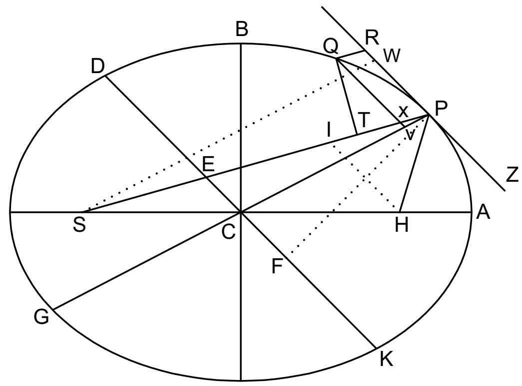


[LEMMA C] AC : PF = CD : CB

1) 2AC·2CB = 2CD·2PF (Lemma 12)
2) so AC·CB = CD·PF
3) so **AC : PF = CD : CB**

[LEMMA D] AC·L = 2CB^2

1) 2AC : 2CB = 2CB : L (because the minor axis is a mean proportional between the major axis and its upright side, by Apoll. 1.15)
2) 2AC·L = 4CB^2
3) **AC·L = 2CB^2**



THE PROOF ITSELF.

1) $$\frac{L \cdot QR}{L \cdot Pv} = \frac{QR}{Pv}$$ [obviously]

2) $$\frac{L \cdot QR}{L \cdot Pv} = \frac{Px}{Pv}$$ [QR = Px, since RPxQ is a parallelogram]

3) $$\frac{L \cdot QR}{L \cdot Pv} = \frac{PE}{PC}$$ [Px : Pv = PE : PC, since △Pvx similar to △PCE]

4) $$\frac{L \cdot QR}{L \cdot Pv} = \frac{AC}{PC}$$ [since EP = AC, Lemma A above]

5) now $$\frac{L \cdot Pv}{Gv \cdot vP} = \frac{L}{Gv}$$ [obviously]

6) but $$\frac{Gv \cdot vP}{Qv^2} = \frac{GC \cdot CP}{CD^2}$$ [ord-squares are as diam-rects]

7) or $$\frac{Gv \cdot vP}{Qv^2} = \frac{CP^2}{CD^2}$$ [$GC \cdot CP = CP^2$ since GC = CP]

8) now $$\frac{Qx^2}{QT^2} = \frac{PE^2}{PF^2}$$ [since Qx : QT = PE : PF, Lemma B]

9) so $$\frac{Qx^2}{QT^2} = \frac{AC^2}{PF^2}$$ [since EP = AC, Lemma A]

10) so $$\frac{Qx^2}{QT^2} = \frac{CD^2}{CB^2}$$ [since AC : PF = CD : CB, Lemma C]

11) Now, putting together Steps 4, 7, 10, by multiplying the left sides and the right sides, we get:

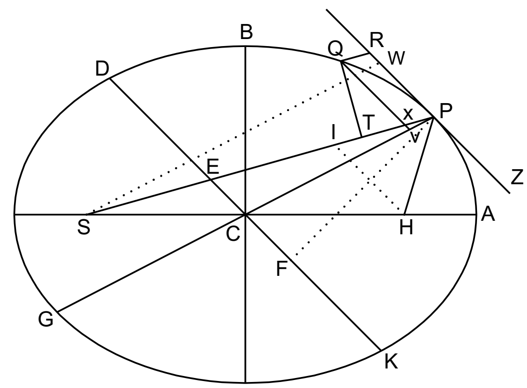
$$\frac{L \cdot QR}{L \cdot Pv} \cdot \frac{Gv \cdot vP}{Qv^2} \cdot \frac{Qx^2}{QT^2} = \frac{AC}{PC} \cdot \frac{CP^2}{CD^2} \cdot \frac{CD^2}{CB^2}$$

12) So, rearranging terms a little,

$$\frac{L \cdot QR}{QT^2} \cdot \frac{Gv \cdot vP}{L \cdot Pv} \cdot \frac{Qx^2}{Qv^2} = \frac{AC}{PC} \cdot \frac{CP^2}{CD^2} \cdot \frac{CD^2}{CB^2}$$

13) Now, multiplying both sides by (L·Pv) / (Gv·vp), we have:

$$\frac{L \cdot QR}{QT^2} \cdot \frac{Qx^2}{Qv^2} = \frac{L \cdot Pv}{Gv \cdot vP} \cdot \frac{AC}{PC} \cdot \frac{CP^2}{CD^2} \cdot \frac{CD^2}{CB^2}$$



14) And now cancelling the superfluous Pv/Pv and CD^2/CD^2, we have

$$\frac{L \cdot QR}{QT^2} \cdot \frac{Qx^2}{Qv^2} = \frac{L}{Gv} \cdot \frac{AC}{PC} \cdot \frac{CP^2}{CB^2}$$

15) Cancelling out PC/PC, and rearranging a bit:

$$\frac{L \cdot QR}{QT^2} \cdot \frac{Qx^2}{Qv^2} = \frac{AC \cdot L}{BC^2} \cdot \frac{PC}{Gv}$$

16) But AC·L = $2BC^2$ (by Lemma D), hence:

$$\frac{L \cdot QR}{QT^2} \cdot \frac{Qx^2}{Qv^2} = \frac{2BC^2}{BC^2} \cdot \frac{PC}{Gv}$$

17) or $$\frac{L \cdot QR}{QT^2} \cdot \frac{Qx^2}{Qv^2} = \frac{2PC}{Gv}$$

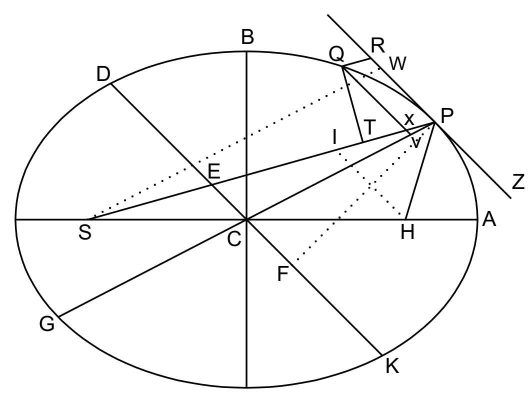
18) inverting everything,

$$\frac{QT^2}{QR \cdot L} \cdot \frac{Qv^2}{Qx^2} = \frac{Gv}{2PC}$$

19) multiplying both sides by $[Qx^2 / Qv^2] \cdot L$,

$$\frac{QT^2}{QR} = \frac{Gv}{2PC} \cdot \frac{Qx^2}{Qv^2} \cdot L$$

20) so $$\frac{SP^2 QT^2}{QR} = \frac{Gv}{2PC} \cdot \frac{Qx^2}{Qv^2} \cdot L \cdot SP^2$$



21) Now, as Q goes to P, ultimately

$$\frac{Gv}{2PC} = 1$$

since Gv goes to 2PC. Also, ultimately

$$\frac{Qx}{Qv} = 1$$

by Lemma 7, Cor. 3. And therefore, ultimately

$$\frac{SP^2 QT^2}{QR} = L \cdot SP^2$$

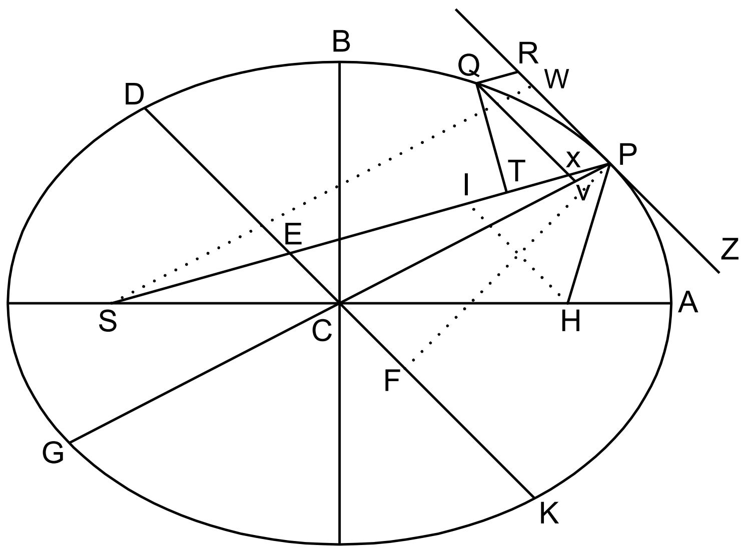
22) But, by Prop.4 Cor.1, the force is inversely as the ultimate form of the left side of this equation. And therefore:

$$F_P : F_p = L \cdot Sp^2 : L \cdot SP^2$$ (where no "ultimate" is necessary anymore)

i.e. $$F_P : F_p = Sp^2 : SP^2$$

and so the force is inversely as the square of the distance from the focus, S.

Q.E.I.



NOTE: S = Sun, P = Planet.

QUESTIONS:

Q1. Does this work with either focus?—that is, does the force also vary inversely as PH^2 ? (No, only the one around which the body sweeps out equal areas in equal times.)
Q2. Can force decrease with the square of the distance from the focus (Prop.11) but at the same time increase with distance from the center (Prop. 10)? (Not without introducing two separate forces, which would interfere with each other so that the body would no longer be sweeping out areas as times around either focus or center.)
Q3. Does an inverse square law make intuitive, physical sense? (If a natural force resides in a body that acts through space, it is like a showerhead or a dandelion; the closer you are, you get wetter squared, or more tickled squared.)
Q4. What do you get if the force is simply as the inverse of the distance (vs. as the inverse of its square)?
Q5. Does the argument degenerate into Prop. 7 if the foci coincide?

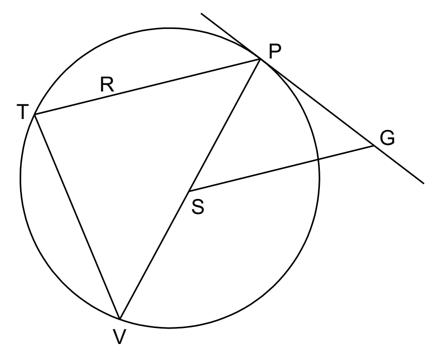
CLASS 43

PROPOSITION 11
Idem Aliter

QUESTION: Again, why so many "idem aliters"? Not to show off, surely. But the method of ultimates (the calculus) is new with Newton, since he discovered it. So he is assuring his readers, to whom the method will be new, that it does in fact produce the same result when one finds the limit of different processes approaching a given ratio (e.g., a fixed ratio of forces). If we got different answers by approaching now this way, now that way, we could not trust the method.

The Idem Aliter here uses Prop. 7 Cor. 3, so here is a quick refresher on Cors. 2–3:

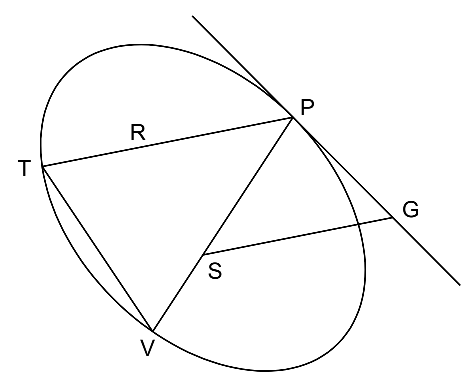
PROPOSITION 7 COROLLARY 2



If a body moves around a circle, sweeping out equal areas in equal times around R, and then moves about the same circle, but sweeping out equal areas in equal times around S, then, if the periods of these different motions be the same,

$F_{PR} : F_{PS} = SG^3 : SP \cdot RP^2$

PROPOSITION 7 COROLLARY 3



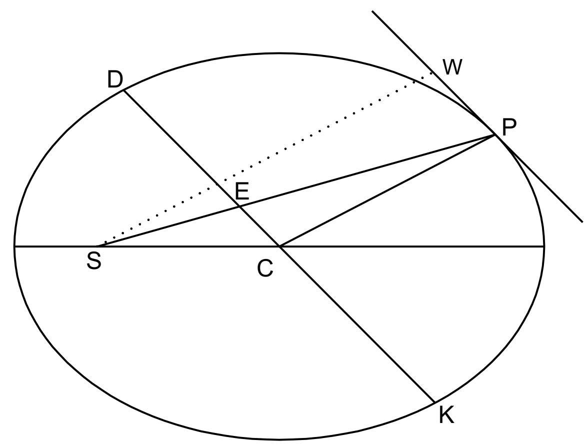
With the same givens, but in any conic you please (any curve of finite curvature), the same rule applies, since the rule of the forces will be the same as for that of the circle of curvature at P.

PROPOSITION 11
Idem Aliter

Draw SW parallel to CP.

Hence $$\frac{F_{PC}}{F_{PS}} = \frac{SW^3}{SP \cdot CP^2}$$ [Prop. 7 Cor. 3]

so $$\frac{F_{PC}}{F_{PS}} = \frac{SW}{SP} \cdot \frac{SW}{CP} \cdot \frac{SW}{CP}$$



But $\triangle$SWP is similar to $\triangle$PCE

so $$\frac{SW}{SP} = \frac{CP}{PE}$$

and $$\frac{SW}{CP} = \frac{SP}{PE}$$

so $$\frac{F_{PC}}{F_{PS}} = \frac{CP}{PE} \cdot \frac{SP}{PE} \cdot \frac{SP}{PE}$$

so $$\frac{F_{PS}}{F_{PC}} = \frac{PE^3}{CP \cdot SP^2}$$

But $$F_{PC} \propto PC$$ [Prop. 10]

So $$F_{PS} \propto \frac{PE^3}{CP \cdot SP^2} \cdot \frac{PC}{1}$$

i.e. $$F_{PS} \propto \frac{PE^3}{SP^2}$$

so $$F_{PS} \propto \frac{AC^3}{SP^2}$$ [since PE = AC, the semi-major, Lemma A]

i.e. $$F_{PS} \propto \frac{1}{SP^2}$$ [since AC^3 is constant]

Q.E.I.