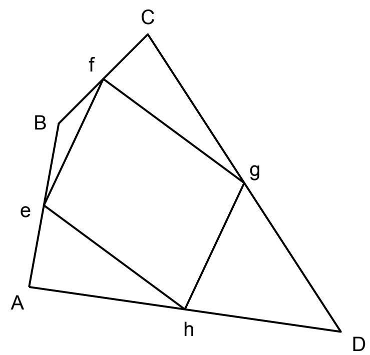
INTRODUCTION

Those who assert that the mathematical sciences say nothing of the beautiful or the good are in error.

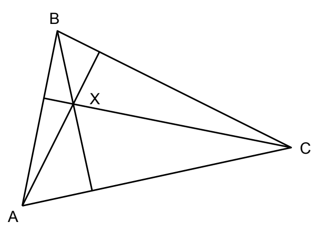
– Aristotle

There are at least four reasons to study geometry.

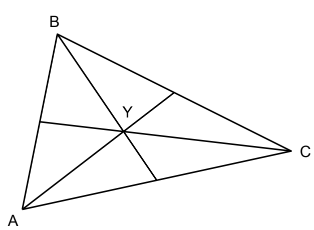
(1) GEOMETRY IS FULL OF WONDERS. At every level of this science, from the most elementary to the most advanced, we are confronted with the unexpected. Often, the seemingly possible proves impossible, and conversely what at first seemed impossible turns out to be possible. An example from the most elementary level is the possibility of one rectangle having *more* area than another one, and yet *less* total length around its sides. At first, it sounds impossible to enclose a bigger yard with less fence, or a smaller yard with a longer fence. (Incidentally, that is why swindlers in ancient times used to sell land by perimeter instead of by area.)



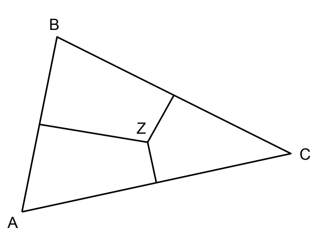
Now a more advanced example. Take any quadrilateral *ABCD*, as ugly and random-looking as you please. Cut each side in half at *e*, *f*, *g*, *h*, and – surprise!– the quadrilateral *e f g h* is a perfect parallelogram.



And a slightly more advanced example. If you take any triangle *ABC* and drop perpendiculars from each vertex to its opposite side, these three perpendiculars all meet at one point, *X* ...



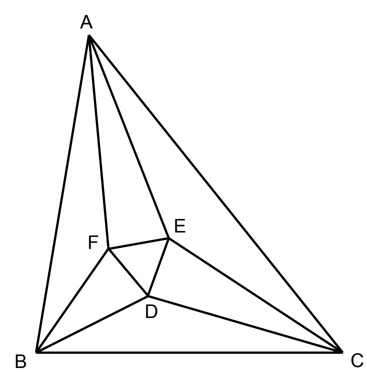
And if you join each vertex of triangle *ABC* to the midpoint of the opposite side, these three lines all meet at one point, *Y* ...



Finally, if you draw perpendiculars from the midpoints of the sides of *ABC*, they also all meet at one point, *Z*. Now the surprise: in any triangle *ABC*, these three points *X*, *Y*, *Z* are in a perfectly straight line. (Not only that, but *XY* is exactly double the length of *YZ*!)

There are many more surprises than these in geometry, but to get into them would take us out of this introduction, and into the science itself.

(2) GEOMETRY IS BEAUTIFUL. There are at least three sources of beauty in geometry. First, there are the figures themselves: perfect things of their kind, without bump or wrinkle, such as a perfect circle, or the five Platonic Solids. Symmetry and proportion, which are universal principles of beauty in nature, architecture, poetry and music, abound in geometric diagrams. There is also a beauty in the truths of geometry themselves. For example, if you take any triangle you like (Δ *ABC*), and cut each of its angles into three equal parts, the six trisecting lines you have drawn will meet each other inside the triangle at three points (*D*, *E*, *F*). The beautiful thing is that the three points *D*, *E*, *F* will be the vertices of a perfectly equilateral triangle. Such revelations are not only surprising, but pleasing in their simplicity and symmetry. In geometry, order pops up unlooked for; a beauty that we do not make, but only discover.



The proofs of geometry can also be beautiful. The best geometric proofs are adorned with a brilliance all their own in virtue of their ingenuity, clarity, universality, and rigor. Geometry, properly presented, yields an experience of intelligible beauty, introducing minds to the special pleasures attending insight and understanding.

(3) GEOMETRY IS FULL OF FUNDAMENTALS. Over the entrance to Plato's Academy there hung a sign which read *Let no one ignorant of mathematics enter here*. Why did a school of philosophy designate mathematics as a prerequisite for admission? Plato saw that many universal principles are most readily accessible to us through mathematics.

The geometrical science of proportion, for example, shows in a concrete way how some things can be known by proportions or analogies. We can come to know an unknown quantity x if we see it in proportion to other terms already known to us, say if x has to 4 the same ratio that 3 has to 2. Knowing 4, 3, and 2, and knowing the relationship between 3 and 2, we can come to know the mysterious x. This is a useful way of getting at something to which we have no direct access, say if x were a length we could not measure directly, like the height of an Egyptian pyramid. There is no way to drop a plumb line from the peak of a pyramid straight down to its base, but three other lengths that we can measure might form a proportion with the inaccessible height. Philosophers and scientists, too, must sometimes find ways to investigate things not directly observable or imaginable, and one tool for this purpose is proportion or analogy, the most fundamental use of which we find in geometry.

Geometry is also fundamental in another way. It is the science most easily acquired by the human mind with rigor and exactness. In geometry, one can settle disagreements. One can draw inescapable conclusions. This makes geometry an ideal entryway into the whole life of the mind.

(4) GEOMETRY EXERCISES THE MIND. People exercise their bodies to maintain their strength and health, and also because it feels

good. There is such a thing as mental exercise, too, which both strengthens and exhilarates the mind. Studying geometry is among the best of mental workouts, simultaneously exercising one's imagination, memory, and reason. In the course of a proof, the imagination must follow a line of reasoning from one part of a diagram to another; it must flip, rotate, and otherwise manipulate geometrical objects; it must interpret two-dimensional diagrams of three-dimensional things; it must picture how the other parts of a diagram are affected if one part is moved or changed. Memory also gets a workout, since geometry is cumulative. Each conclusion must be understood, and then used to establish later results, which in turn help to establish still more advanced results. And geometry obviously exercises reason. There is no reasoning more exact than a mathematical argument. Geometrical objects are perfect subject matter for forming definitions and proofs, proposing difficulties and finding resolutions, drawing distinctions, finding examples … in short, for doing all the best things that human reason can do. Thus geometry builds people's confidence that reason can find satisfying answers to serious questions.

For the above reasons geometry is justly recognized as an essential element in the formation of every educated person and is worthy of lifelong study. Current books written on the premise that geometry is interesting in itself are largely intended for advanced students or professional mathematicians. They presuppose a mastery of elementary theorems. On the other hand, geometry books which begin at the very beginning are generally not written for enthusiastic readers, but for students who need to pass an exam. Such introductions gloss over proofs (or skip them entirely), emphasizing instead various formulas, exercises, and problem-solving techniques.

This course is written for anyone motivated to study geometry for the wonder and beauty of it, for readers disposed to contemplate theorems as if they were works of art. And yet it begins at the very beginning. To master it, you need no prior training in mathematics. In consequence, this course represents a unique introduction to geometry. Readers interested in learning mathematics will find it better suited to their needs than study manuals or high school geometry books because of its scope, its purity, and its rigor.

THE SCOPE OF THIS COURSE by far surpasses that of the typical introduction. This course covers most of the content of the thirteen books of Euclid's *Elements*, whereas typical introductions do not cover material much beyond the first three or four books of Euclid. Written most often for the high school level, they do not go deep enough into geometry to reach the most beautiful and exciting material accessible to recreational mathematicians. Yet this course is not longer than the average high school textbook, but actually shorter, since it does not multiply exercises.

THE PURITY OF THIS COURSE should be refreshing to anyone who loves geometry. Other introductions to the science, written so readers can "get the right answer," employ algebra, trigonometry, number lines, a system of coordinate axes, and a host of other devices. Such devices and techniques, though useful (elsewhere) and important to study (elsewhere), have no place in a formal introduction to geometry intended for those who wish to begin at the beginning and understand the reasons for things. The impression is given that there is no geometry without these extras. The truth is that geometrical things can be known geometrically, without recourse to algebra or trigonometry.

The proof of the Pythagorean Theorem given in this book, for example, makes no use of algebraic operations. The theorem is demonstrable on purely geometrical grounds. The proof given for this theorem in many introductory books is an algebraic one that quickly leaves behind the diagram altogether. The result is a very abstract and unmemorable proof, the steps of which are not explicitly correlated with the right triangle and the squares that the geometrical theorem is about. The purpose of teaching the Pythagorean Theorem algebraically is to encourage proficiency in applying it to problems. This denies students any real understanding of the theorem, however, and reinforces the idea that geometry has no intrinsic worth or beauty.

RIGOR. Many introductory books use theorems they do not prove, such as the theorem that if a cone and a cylinder stand on the same circle and have the same height, the volume of the cone is one third that of the cylinder. Current high school textbooks including this theorem or a formula based on it do not attempt even a sketchy proof for it. In this course a complete proof is given for this theorem and for every other theorem covered. Once again, the implicit message of the textbook is that understanding the theorem is not important, but only the use of a formula which one should be willing to take on faith. This presumes an audience uninterested in the reasons for things, or incapable of understanding them.

Like a novel, it is essential to read this book in the order in which it is written, but unlike a novel, you can stop after any chapter or theorem and come away with something completely understood. But enough of introductions. On to the adventure of geometry.